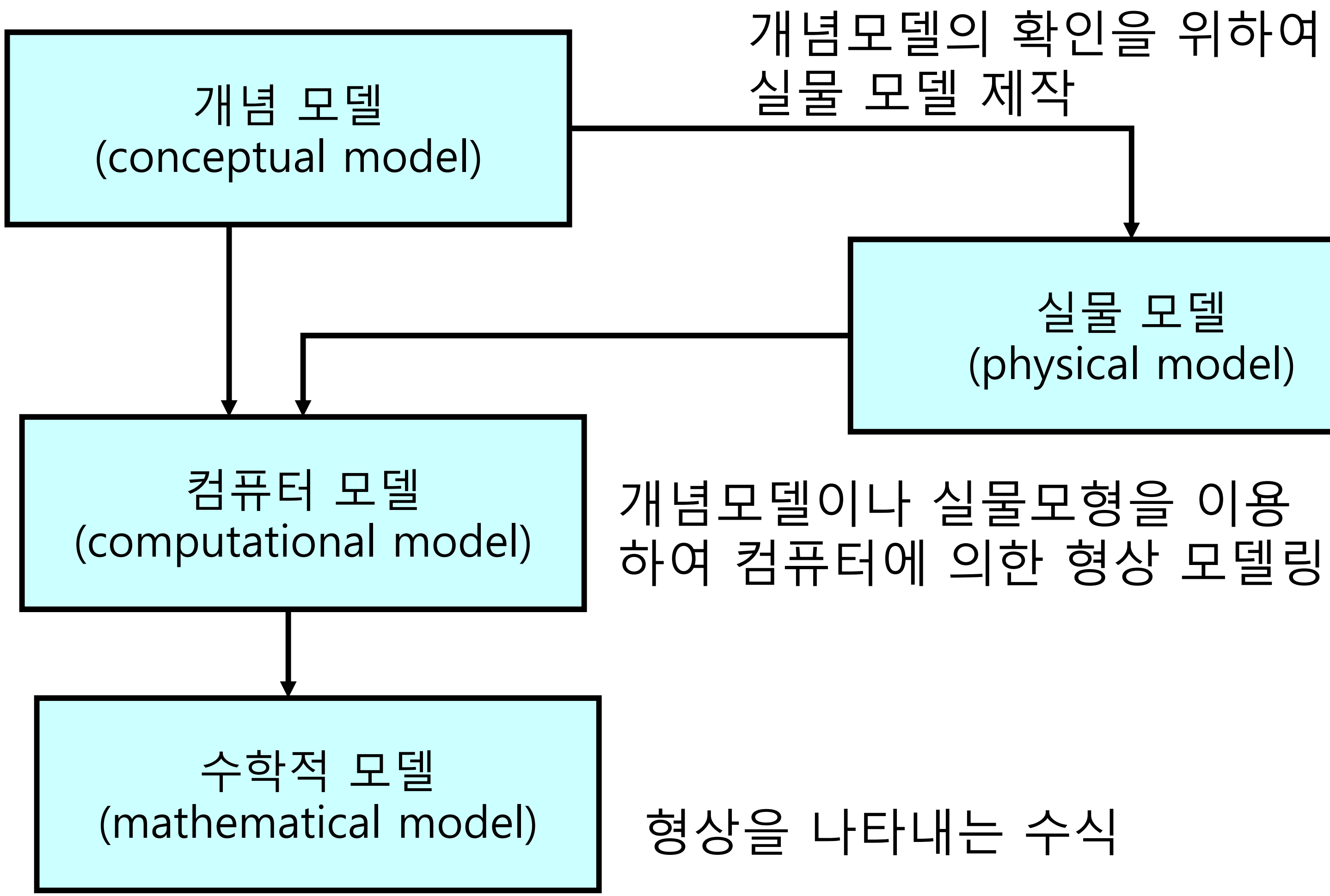
Geometric Model



Computational Model의 종류

- Point Model : 물체를 점으로 근사화
- Curve Model : 물체를 곡선으로 표현
- Surface Model : 물체를 면으로 정의
- Solid Model : 물체의 입체형상을 정의
- Hybrid Model : 하나의 모델로 점, 곡선, 곡면, 입체를 모두 이용

수학적 표현식의 분류

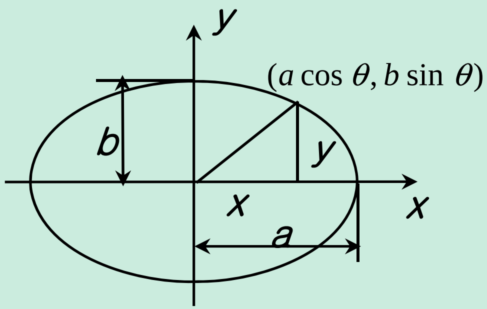
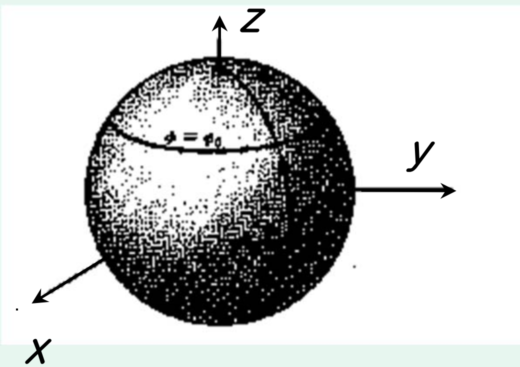
- 비매개 변수식(non-parametric equation)
  - 음 함수 (implicit form): $f(x,y) = 0$
  - 양 함수 (explicit form): $y = f(x)$

- 매개 변수식 (parametric equation)
  - $P(u) = [\ x(u), y(u), z(u)\ ]$

수학적 모델의 표현 방법

대상	양 함수식 표현 (explicit form)	음 함수식 표현 (implicit form)	매개 변수식 표현 (parametric form)
이차원 평면 곡선	$y = f(x)$	$f(x, y) = 0$	$\mathbf{r}(t) = (x(t), y(t))$
삼차원 공간 곡선	$\begin{cases} z = f(x, y) \\ z = g(x, y) \end{cases}$	$\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases}$	$\mathbf{r}(t) = (x(t), y(t), z(t))$
곡면	$z = f(x, y)$	$f(x, y, z) = 0$	$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$

Curve, Surface의 표현 방법

	양 함수식 표현	음 함수식 표현	매개 변수식 표현
	$y = b\sqrt{1-x^2/a^2}$ $y = -b\sqrt{1-x^2/a^2}$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$	$\mathbf{r}(\theta) = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$ $0<\theta<2\pi$
	$z = \sqrt{r^2-x^2-y^2}$ $z = -\sqrt{r^2-x^2-y^2}$	$x^2+y^2+z^2-r^2=0$	$\mathbf{r}(\theta,\phi) = \begin{pmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{pmatrix}$

Curve Equations

$$\text{circle}\begin{cases}\text{parametric form: } x = R\cos\theta,\ y = R\sin\theta,\ z = 0\left(0 \le \theta \le 2\pi\right) \\ \text{nonparametric}\begin{cases}\text{implicit form: } x^2 + y^2 - R^2 = 0,\ z = 0 \\ \text{explicit form: } y = \pm\sqrt{R^2 - x^2},\ z = 0\end{cases}\end{cases}$$

- Parametric form
  - Points can be generated in sequence along the curve by substituting small increments of the parametric values
- Nonparametric implicit form
  - Not sure which variable to choose as the independent variable and to increment at each evaluation
  - Two values (+/-) for the dependent variable, which one?
- Nonparametric explicit form
  - Same inherent problem even though it differentiates the independent variable from the dependent variable

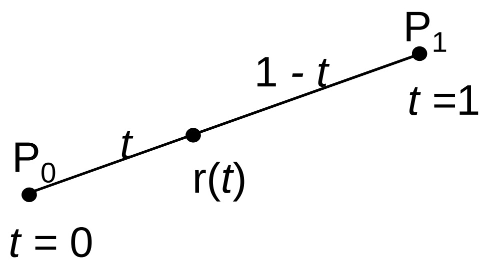
비 매개변수 식의 장단점

- 장점
  - 특별한 경우, 직관적 해석이 편리하다
- 단점
  - 하나의 형상 식이 좌표계에 의하여 변화되거나 표현 할 수 없는 경우가 생긴다.
  - 좌표계가 달라지면 형상 표현에 현실적인 어려움이 있다.
  - 곡선이나 곡면이 평면에 있지 않거나 경계가 주어진 경우에는 그 표현이 어렵거나 불가능 하다.

Parametric Curve

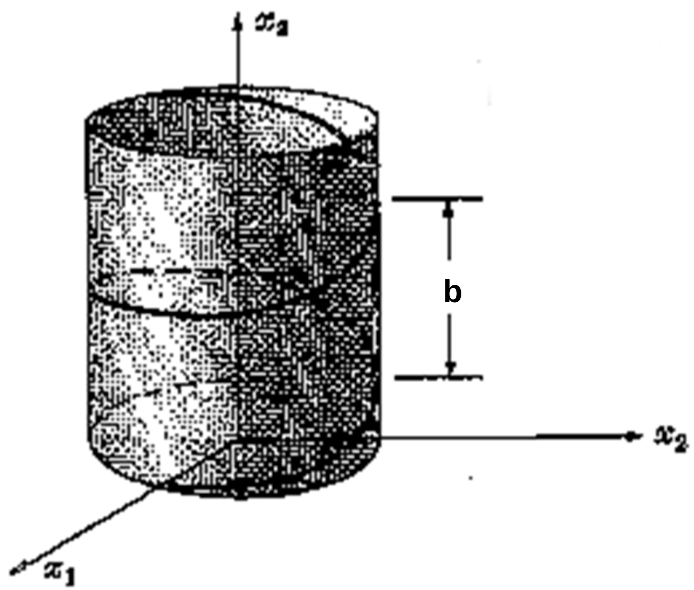
직선: $\begin{cases} x(t) = (1-t)x_0 + tx_1 \\ y(t) = (1-t)y_0 + ty_1 \end{cases}$

$$\Rightarrow \mathbf{r}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$$



나선(helix):

$$\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, \frac{b}{2\pi}\theta)$$



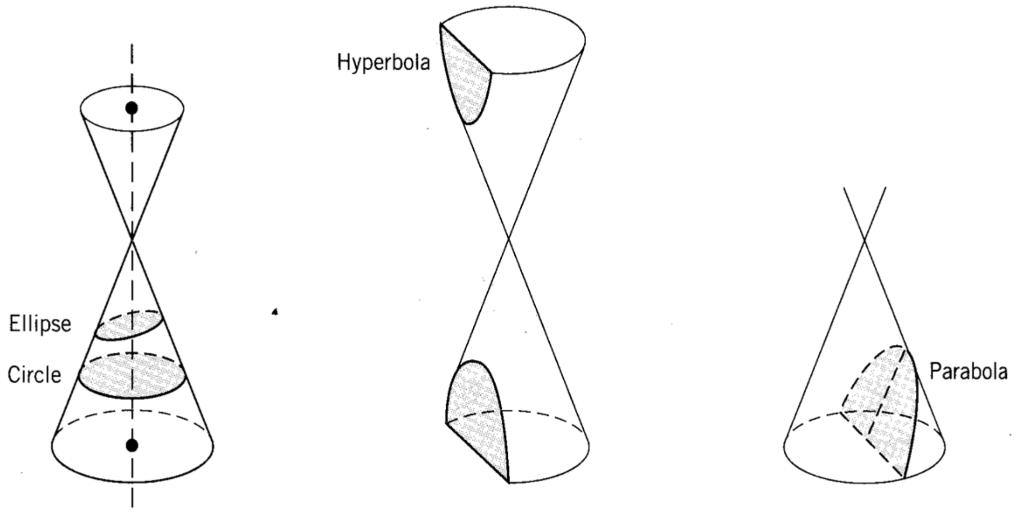
매개 변수식의 장점

- 순차적으로 표현하기 쉽다.
  – Computer Graphics, NC Tool Path Generation에 편리
- 2D/3D 곡선, 곡면의 표현 형태가 비슷하다.
- 자유곡선/ 곡면의 표현이 용이하다.
- 이동, 회전, Scaling과 같은 변환이 쉽다.
- 범위가 지정된 형상을 표현하기 쉽다.
- 형상을 벡터와 행렬에 의하여 쉽게 표현할 수 있다.

- Computer를 이용한 처리 용이

Conic Section Curves (1)

- 원추곡선: 원추면을 한 개의 평면으로 잘랐을 때 발생하는 교차선을 통칭
- 원 또는 원호, 타원 또는 그 일부, 쌍곡선, 포물선
- 축대칭 형태인 많은 기계부품 모델링에 자주 사용

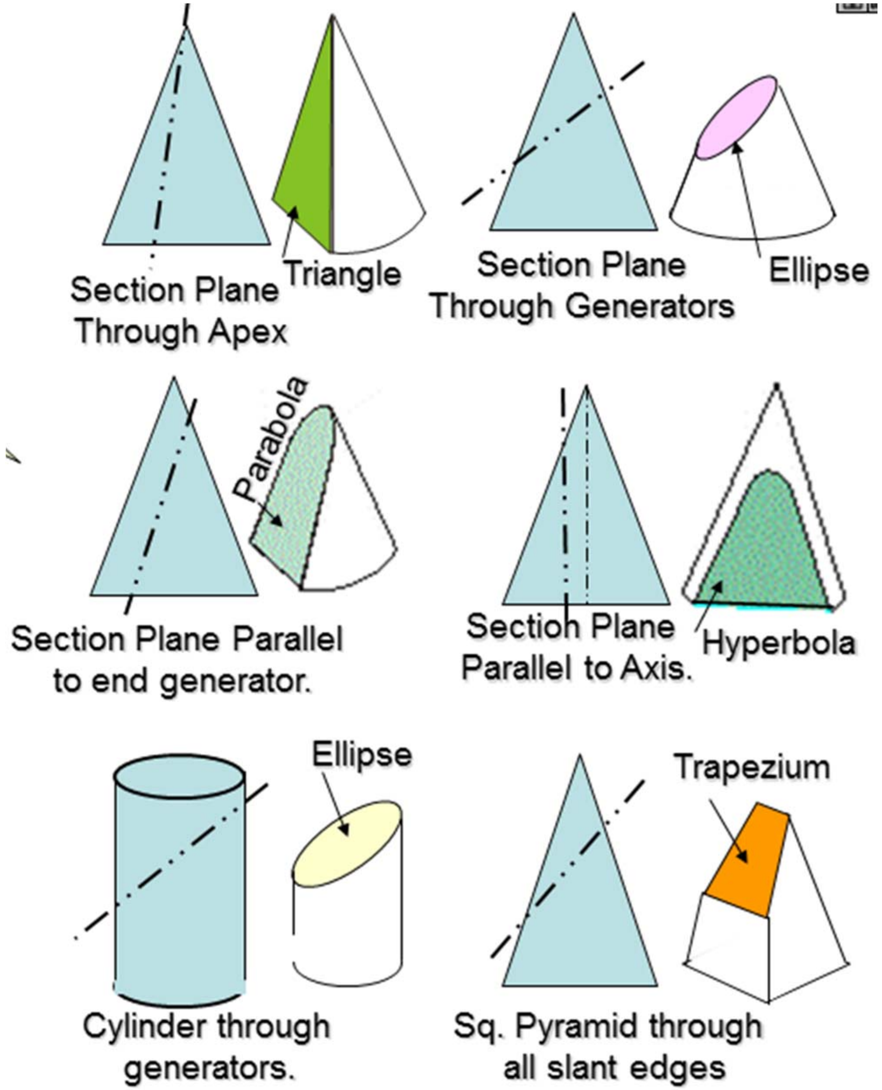


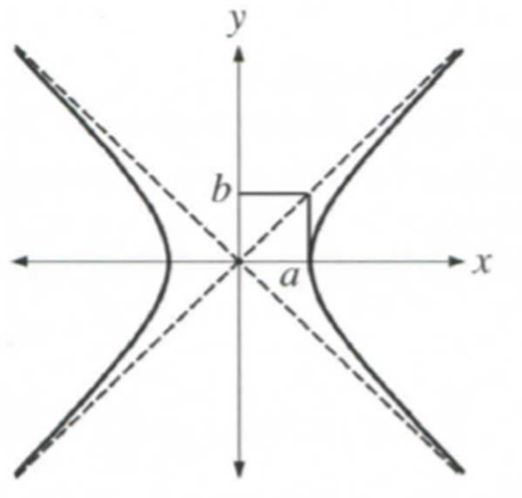
$$ax^2 + by^2 + c + 2fx + 2gy + 2hxy = 0$$

$$\begin{cases} h^2 - ab < 0 : ellipse \\ h^2 - ab = 0 : parabola \\ h^2 - ab > 0 : hyperbola \end{cases}$$

원추곡선	생성 단면
원	밑면에 평행면
타원	밑면에 경사면
포물선	모선에 평행면
쌍곡선	밑면에 수직면

Conic Section Curves (2)





Conic Section Curves (3)

	해석적 표현	매개변수 표현
원/원호 (circle/ circular arc)	$(x-x_c)^2+(y-y_c)^2=r^2$	$\begin{cases} x=r\cos\theta+x_c \\ y=r\sin\theta+y_c \end{cases}$
타원/타원일부 (ellipse/ elliptic arc)	$\frac{(x-x_c)^2}{a^2}+\frac{(y-y_c)^2}{b^2}=1$	$\begin{cases} x=a\cos\theta+x_c \\ y=b\sin\theta+y_c \end{cases}$
쌍곡선 (hyperbola)	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\begin{cases} x=a\cosh u \\ y=b\sinh u \end{cases}$
포물선 (parabola)	$x=cy^2$	$\begin{cases} x=cu^2 \\ y=u \end{cases}$

Conic Section Curves (4)

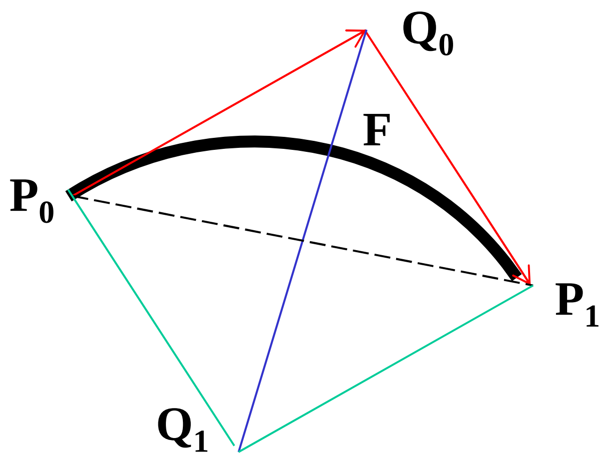
$$ax^2 + by^2 + c + 2fx + 2gy + 2hxy = 0$$

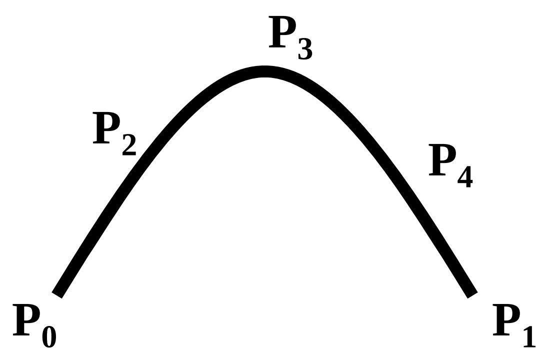
$$\rightarrow x^2 + b'y^2 + c' + f'x + g'y + h'xy = 0 \ \ (5 \text{ unknowns})$$

- 3PTS (P0, P1, Q0)
  - Two end points + Two tangents
  - Fullness factor ρ
- 5PTS (P0, P1, P2, P3, P4)
  - Two end points + Three intermediate points
  - Is it a hyperbola, parabola, or ellipse ?

$$\rho = \frac{|F - Q_1|}{|Q_0 - Q_1|}$$

$$\begin{cases} \rho > 0.75 : \text{hyperbola} \\ \rho = 0.75 : \text{parabola} \\ \rho < 0.75 : \text{ellipse} \end{cases}$$

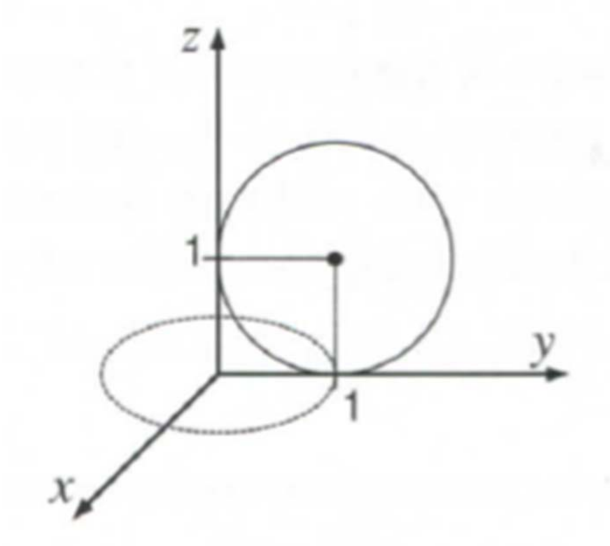


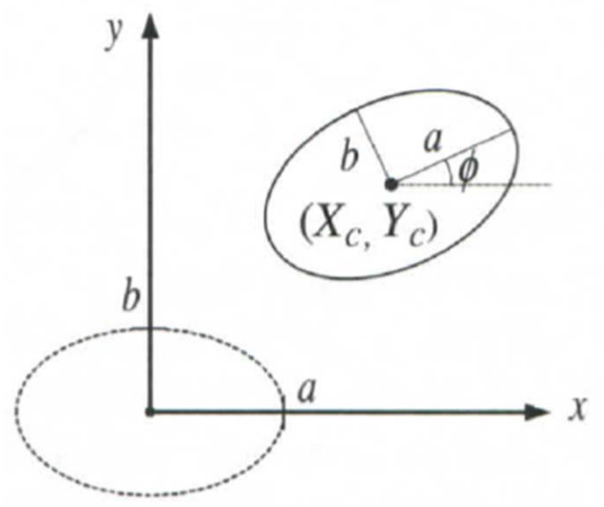


Examples

– A circle of unit radius is centered at (0, 1, 1) and locate on the yz-plane as illustrated in the figure. Derive the parametric equation of the circle by applying the proper transformation matrices.

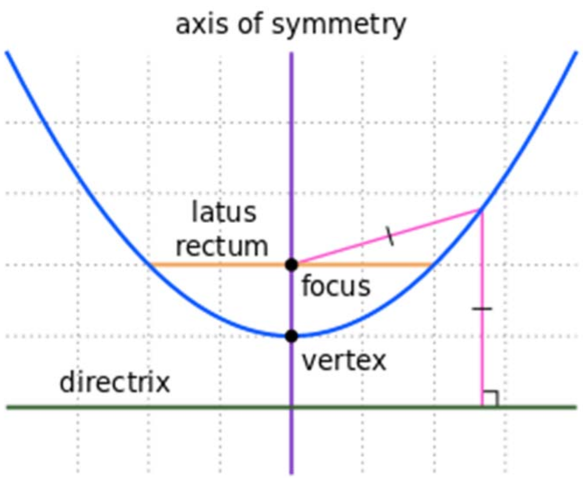
– Derive the parametric equation of an ellipse in the xy-plane, which has a center at (Xc, Yc) and the major and the minor axes illustrated in the figure.





CATIA V5: Sketcher Profile / Conic (1)

- Ellipse
  - Center→Major Semi-Axis Endpoint [Radius(+Angle)]→Minor Semi-Axis Endpoint
- Parabola by Focus
  - Focus→Apex→Start Point→End Point



$$\text{directrix}\left(y=-p\right),\ \text{focus}\left(0,p\right)$$

$$y+p=\sqrt{x^2+\left(y-p\right)^2}\rightarrow x^2=4py$$

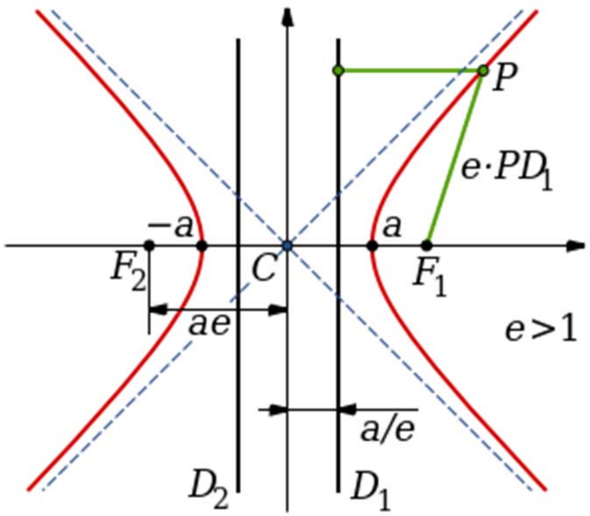
$$\text{origin}\left(h,k\right)\rightarrow\left(x-h\right)^2=4p\left(y-k\right)$$

$$\rightarrow y=ax^2+bx+c$$

$$y=ax^2\rightarrow a=\frac{y}{x^2},\ F=\frac{1}{4a}$$

CATIA V5: Sketcher Profile / Conic (2)

- Hyperbola by Focus
  - Focus→Center→Apex→Start Point→End Point
  - Eccentricity e = (center to focus) / (center to apex)



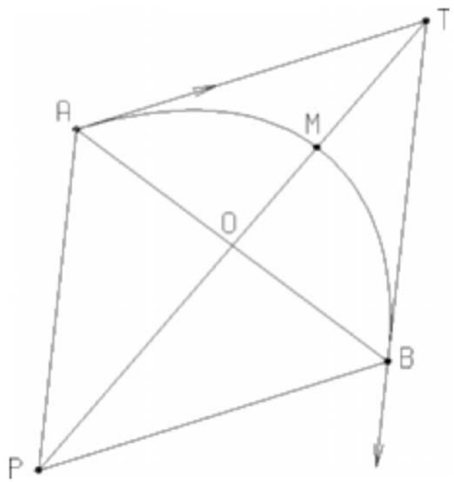
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \xrightarrow[\substack{\text{another point}(X_P, Y_P) \\ \text{focus}(X_F, 0)}]{\substack{\text{origin}(0,0) \\ \text{apex}(X_A, 0)}}$$

$$X_F = \sqrt{X_A^2 + \frac{X_A^2 Y_P^2}{X_P^2 - X_A^2}}$$

CATIA V5: Sketcher Profile / Conic (3)

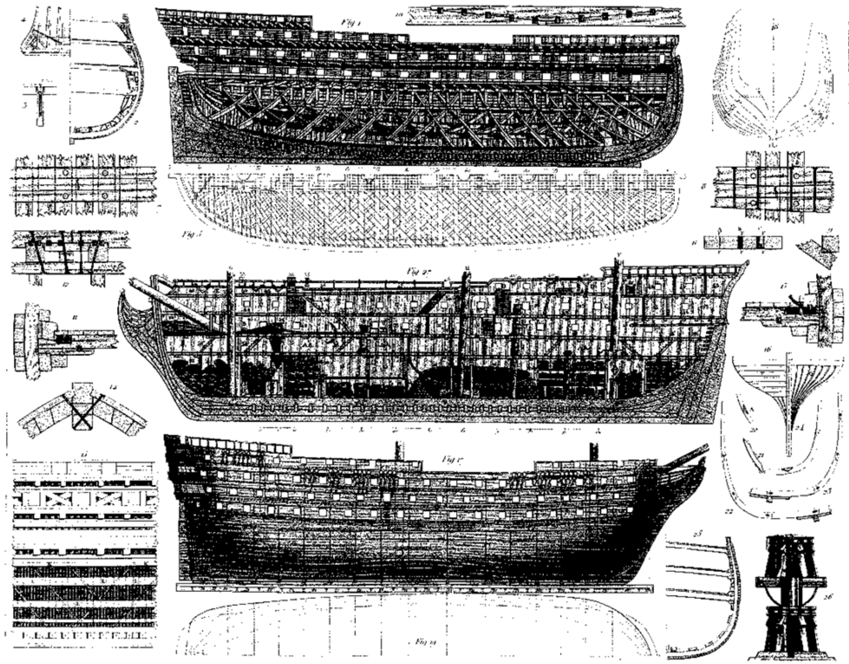
- Conic
  - Two Points (parameter)
    - Start and End Tangent
    - Tangent Intersection Point
  - Four Points
    - Tangent at Passing Point
  - Five Points

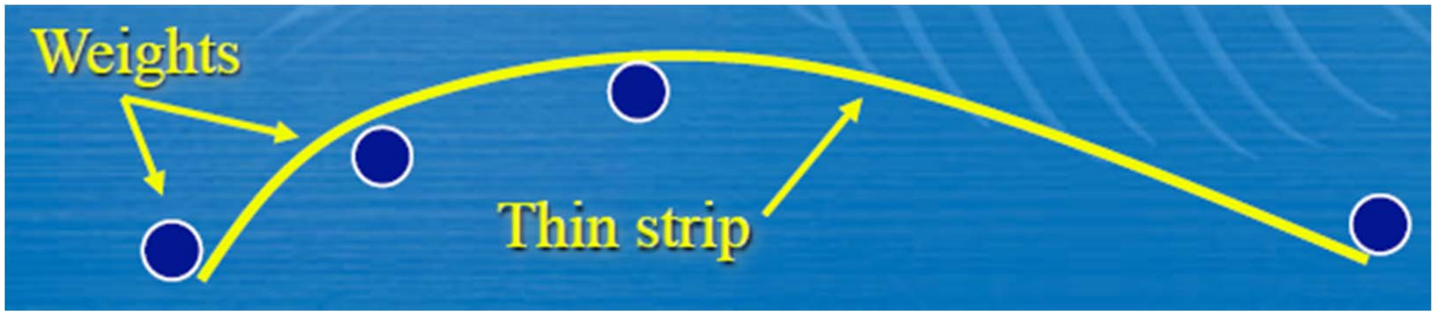
$$\begin{cases} 0 < \text{parameter} < 0.5\text{: ellipse} \\ 0.5 < \text{parameter} < 1.0\text{: hyperbola} \\ \text{parameter} = 0.5\text{: parabola} \end{cases}$$



$$Parameter = \frac{OM}{OT}$$

Splines: Background (1)





Splines: Background (2)

- Flexure:

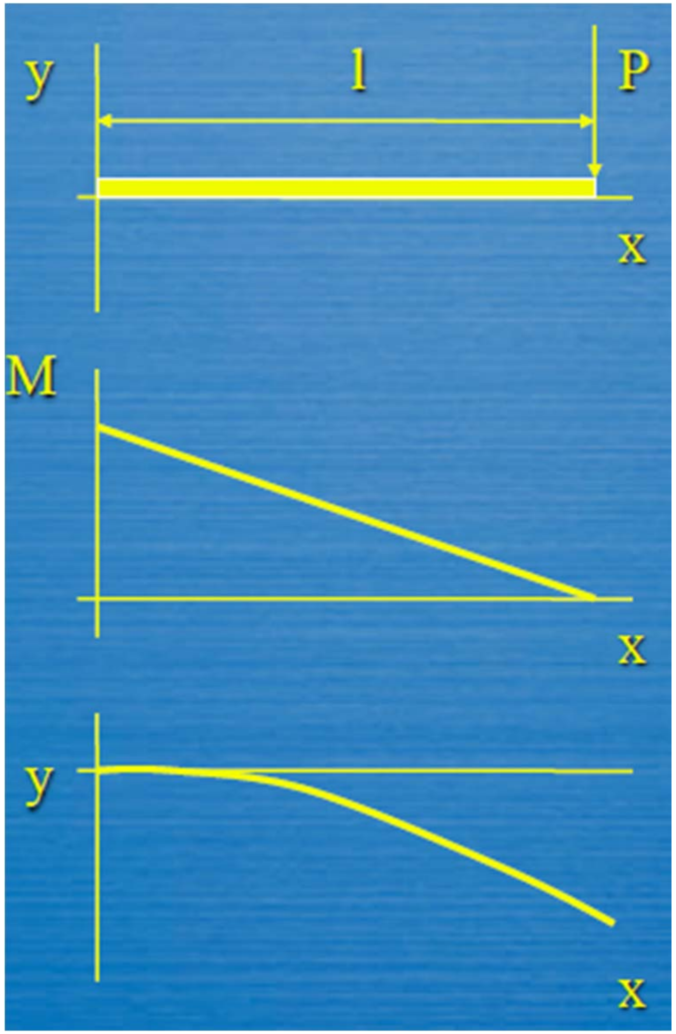
$$M = EI\frac{dy^2}{dx^2}$$

- Moment:

$$M\left(x\right) = P\left(l - x\right)$$

- Displacement:
  - Cubic polynomial in x

$$y\left(x\right) = \frac{P}{EI}\left(\frac{l}{2}x^2 - \frac{1}{6}x^3\right)$$

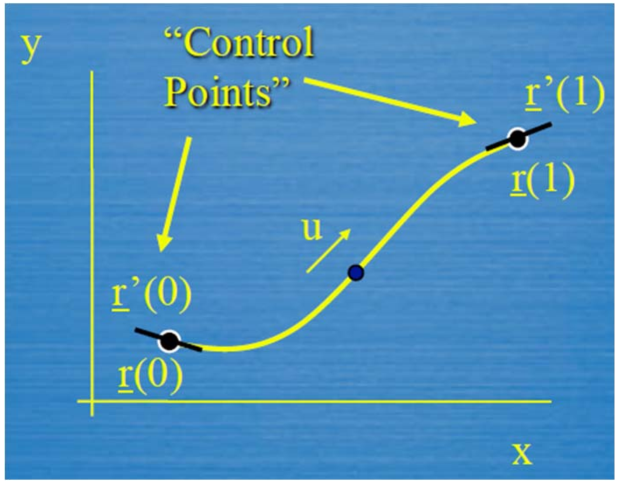


Spline (1)

- 실제 물체의 자유곡선 표현방법
  - 많은 점들을 측정하여 직선으로 연결 ?
  - 적당한 원추곡선의 조합으로 구성 ?
    - 2차 곡선에 의한 한계점
  - 고차곡선의 필요성
- ARC: N개의 점을 지나는 (N-1)차 곡선
  - 고차이므로 많은 계산량 필요, oscillation 발생 ☹
  - 자유곡선을 한 조각으로 표현하는 것은 부적절 ☹
  - CATIA 에서는 15차 곡선까지 지원?

Spline (2)

- SPLINE : 일련의 점들을 지나는 곡선
  - 각 점 사이를 여러 개의 곡선조각으로 이은 것
  - 각 곡선조각 (ARC) : 3차 (or 5차) 곡선
  - 미지수 : 4개 (or 6개)
  - two points, two tangents, (two 2nd order derivatives)
  - 곡선조각에 4차, 6차 등이 사용되지 않는 이유 ?
- Cubic splines
- Refinements
  - B-spines
  - Non-Uniform Rational B-splines (NURBS)



- Spline
  - Control points: solid square points
  - Large number of control points?
    - Create construction points that are constrained first
    - Modify? Recreate better than edit
- Connect with a spline
  - Continuity in Point (C0)
  - Continuity in Tangent (C1): tension at each endpoint
  - Continuity in Curvature (C2): tension at each endpoint

CATIA: Sketcher→Constraint

Modification of geometric models through variational geometry

Robert Light and David Gossard*

Systems for computer-aided mechanical design use geometric models for drafting, analysis and programming of NC machines. Because design is iterative in nature, the topology, geometry or dimensioning of a geometric model must be modified many times during the design cycle. The effectiveness of future CAD systems will depend in large part upon the ease with which geometric models can be created and modified.

This paper presents the results of a research effort to develop flexible procedures for the definition and modification of geometric models. A central idea of this effort is that dimensions, such as appear on a mechanical drawing, are a natural descriptor of geometry and provide the most appropriate means for altering a geometric model.

A procedure is described by which geometry is determined from a set of dimensions. The geometry corresponding to an altered dimension is found through the simultaneous solution of the set of constraint equations. Presented in this paper are the basic approach to modifications of geometric models, a procedure for significant reduction of the number of constraint equations to be solved, and the effect of sparse matrix methods in reducing the time required to solve the equations.

computer-aided design, geometric model, variational geometry

In this paper a procedure is described by which geometry is determined from a set of dimensions. A geometric model is defined with respect to a set of characteristic points in 3-space. Dimensions are treated as constraints limiting the permissible locations of these characteristic points. Each dimensional constraint is described by a nonlinear equation involving the coordinates of associated characteristic points. A given dimensioning scheme is represented by a set of such equations. The geometry corresponding to an altered dimension is found through the simultaneous solution of the set of constraint equations.

These techniques were incorporated into a prototype 2D CAD system. This system allows freehand sketch input to define the general shape of the part. The user interactively defines the dimensioning scheme and the desired values of the dimensions. By solving the system of constraint equations, the geometry of the part is updated to be consistent with the new set of dimensional values.

Presented in this paper are the basic approach to modification of geometric models, a procedure for significant reduction of the number of constraint equations to be solved and the effect of sparse matrix methods in reducing the time required to solve the equations.

BACKGROUND

The use of constraints to define geometry can be seen in Sutherland's early work.[1] Sutherland made use of constraints as a design aid in the creation of a part, but did not use geometric constraints for definition and modification of part geometry. Requicha[2] used directed graphs to represent the relationship between dimensions and geometry. Three graphs were constructed to show the dimensional relations in the x, y and z directions. Hence, this approach is limited to geometries consisting of rectilinear segments. Gopin[3] made the first use of directed-graphs to define and modify geometry in an interactive CAD system. This CAD system restricted the part geometries to rectilinear segments and dimensions.

More recently, Hillyard and Braid[4,5] showed that small variations in geometry could be related to variations in dimensions by a 'rigidity matrix'. Light[6] and Lin et al[7] showed that generalized dimensional constraints could be used to modify geometry through large shape variations, and that geometric properties could be used to constrain geometry. This was demonstrated on prototype 2D and 3D CAD systems.

GEOMETRY

As shown in Figure 1, the geometry of an object is defined by a set of N characteristic points. This set includes points

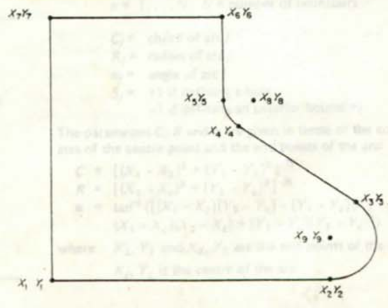


Figure 1. Shape definition points

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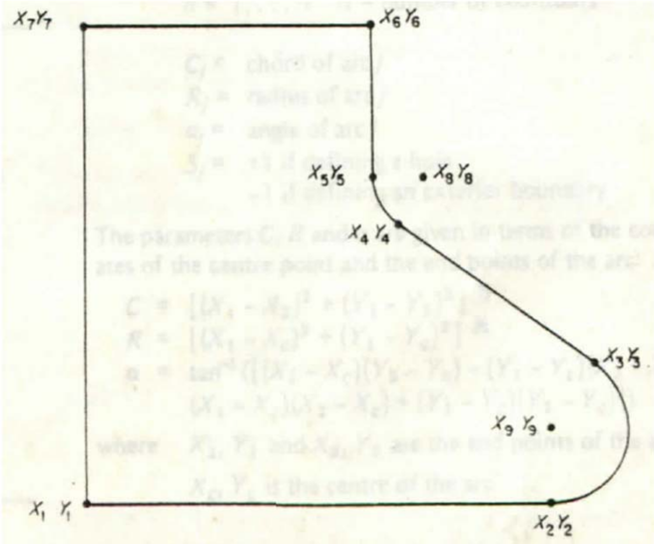
Geometry

- Defined by a set of N characteristic points

$$\mathbf{x} = \{X_1 \quad Y_1 \quad Z_1 \quad \dots \quad X_N \quad Y_N \quad Z_N\}^T$$

$$\mathbf{x} = \{x_1 \quad x_2 \quad x_3 \quad \dots \quad x_{n-2} \quad x_{n-1} \quad x_n\}^T$$

where $n = 3N$



- Generalized dimensional constraints
  - Implicit: right angles
  - Explicit distance between two points

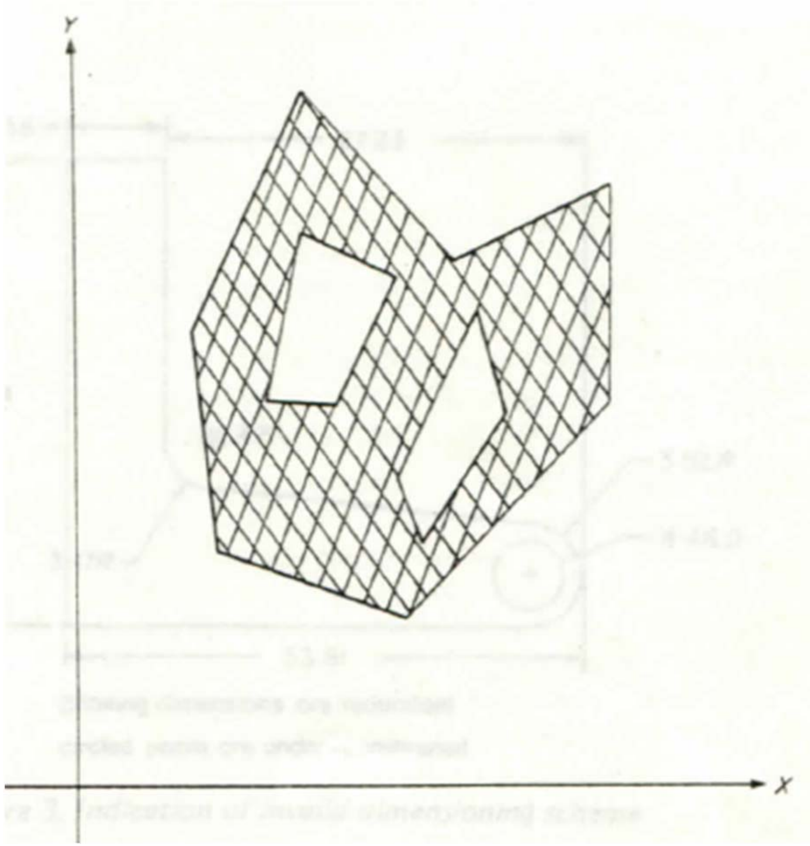
$$F_i(\mathbf{x}, d) = 0 \quad i = 1,\ 2, \dots,\ m \quad \text{where} \begin{cases} d: \text{vector of dimensional values} \\ \mathbf{x}: \text{geometry vector} \\ m: \text{number of constraints} \end{cases}$$

$$\text{constraint equations} \begin{cases} 6: \text{to prevent 3D rigid body translation and rotation} \\ (m-6): \text{designer's specific choice of dimensions} \end{cases}$$

Geometric Constraints

Dimension name	Entities constrained	Equation
Horizontal distance	$P_1,\ P_2$	$X_1 - X_2 - D = 0$
Vertical distance	$P_1,\ P_2$	$Y_1 - Y_2 - D = 0$
Linear distance	$P_1,\ P_2$	$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 - D^2 = 0$
Distance from point to line	$P_1,\ \overrightarrow{P_2P_3}$	$\dot{U} \times V - D = 0 \quad \begin{cases} \dot{U} = \dfrac{X_3 - X_2}{\lvert\overrightarrow{P_2P_3}\rvert} i + \dfrac{Y_3 - Y_2}{\lvert\overrightarrow{P_2P_3}\rvert} j \\ V = (X_2 - X_1) i + (Y_2 - Y_1) j \end{cases}$
Angular dimension	$\overrightarrow{P_1P_2},\ \overrightarrow{P_3P_4}$	$\dfrac{\lvert\overrightarrow{P_1P_2} \times \overrightarrow{P_3P_4}\rvert}{\overrightarrow{P_1P_2} \bullet \overrightarrow{P_3P_4}} - \tan(A) = 0$

Area of an N-sided 2D polygon



$$A = \frac{1}{2}\sum_{j}\left[\sum_{i}\left(X_i Y_{i+1} - Y_i X_{i+1}\right)\right]$$

$i = 1, 2, \ldots,$ number of points in contour

$j = 1, 2, \ldots,$ number of contours

$$A = \frac{1}{2}\sum_{n}^{N}\left[\sum_{i}^{p}\left(X_iY_{i+1} - Y_iX_{i+1}\right) + \sum_{j}^{a} S_j R_j^{\,2}\left(\alpha_j - \frac{C_j}{2R_j}\right)\right]$$

$i = 1, 2, \ldots, a \quad a$: number of points, point indices are sequential around contour

$j = 1, 2, \ldots, p \quad p$: number of arcs in contour

$n = 1, 2, \ldots, N \quad N$: number of contours

C_j : chord of arc j, $C = \sqrt{\left(X_1 - X_2\right)^2 + \left(Y_1 - Y_2\right)^2}$

R_j : radius of arc j, $R = \sqrt{\left(X_1 - X_c\right)^2 + \left(Y_1 - Y_c\right)^2}$

α_j : angle of arc j, $\alpha = \tan^{-1}\dfrac{\left(X_1 - X_c\right)\left(Y_2 - Y_c\right) - \left(Y_1 - Y_c\right)\left(X_2 - X_c\right)}{\left(X_1 - X_c\right)\left(X_2 - X_c\right) + \left(Y_1 - Y_c\right)\left(Y_2 - Y_c\right)}$

$$S_j : \begin{cases} +1 \text{ if defining a hole} \\ -1 \text{ if defining an exterior boundary} \end{cases}$$

$\left(X_1, Y_1\right)$ and $\left(X_2, Y_2\right)$ end points of the arc

$\left(X_c, Y_c\right)$ center of the arc

Solving for Geometry

$$F_i\left(\mathbf{x}, d\right) = 0 \rightarrow F_i\left(\mathbf{x}_0, d\right) + \frac{\partial F_i}{\partial x_j}\Delta\mathbf{x} = 0 \rightarrow \mathbf{J}\Delta\mathbf{x} = \mathbf{r}$$

$$\mathbf{J} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} \text{ where } f_{ij} = \frac{\partial F_i}{\partial x_j}$$

$$\Delta\mathbf{x} = \left\{\Delta x_1 \quad \Delta x_2 \quad \ldots \quad \Delta x_n\right\}^T$$

$$\mathbf{r} = \left\{-F_1 \quad -F_2 \quad \ldots \quad -F_m\right\}^T$$

m: number of constraint equations

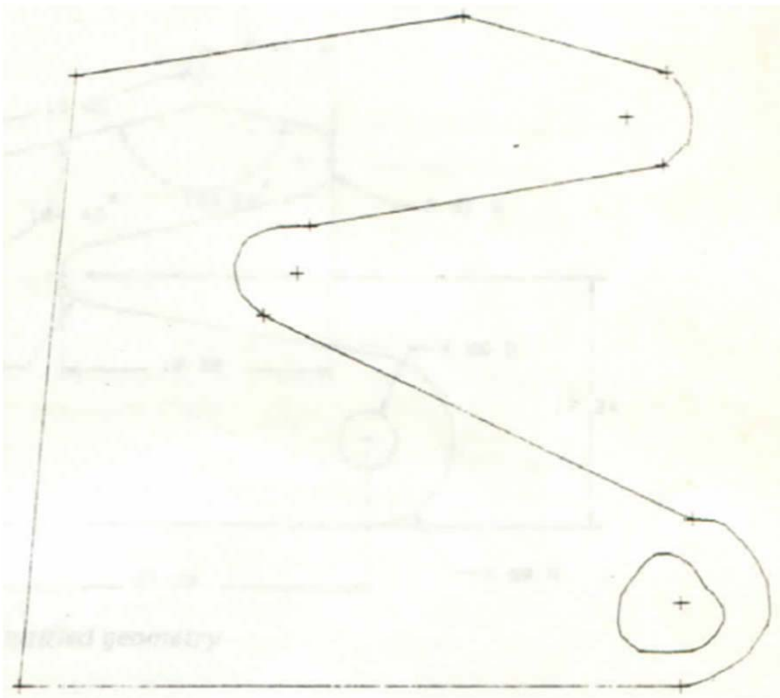
n: number of degrees of freedom (coordinates)

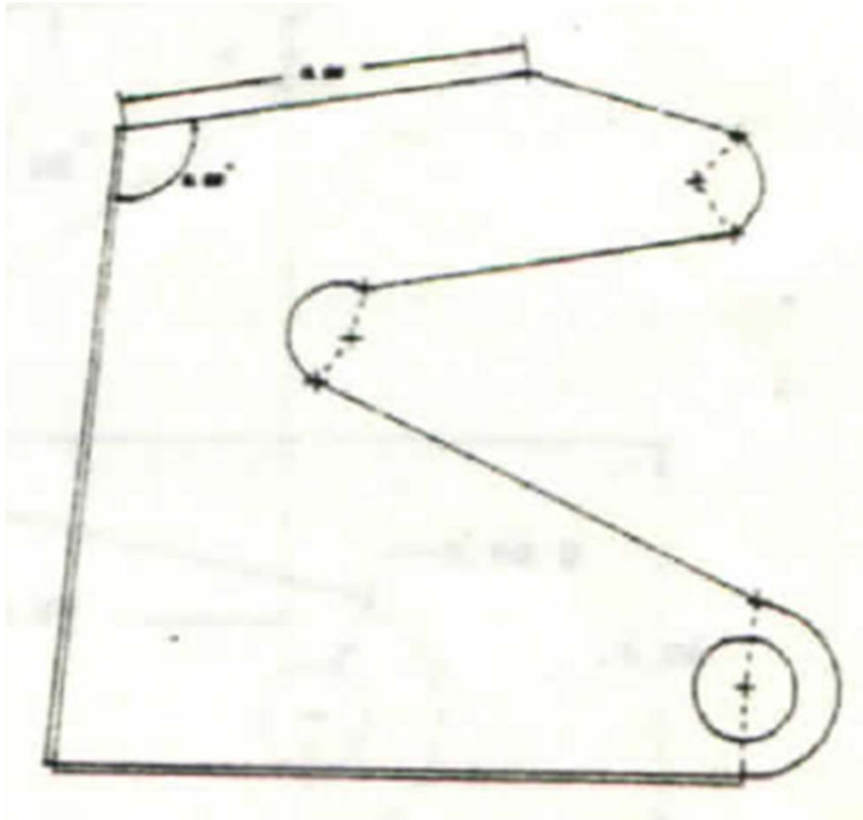
$$\begin{cases} m \neq n: \text{ Jacobian is structurally singular} \begin{cases} m > n: \text{ shape is overdimensioned (too many or redundant dimensions)} \\ m < n: \text{ shape is underdimensioned (too few dimensions)} \end{cases} \\ m = n: \text{ Jacobian can be singular satisfying a redundant dimensioning scheme} \end{cases}$$

$\rightarrow$ necessary and sufficient condition for a valid dimensiong scheme: non-singular Jacobian

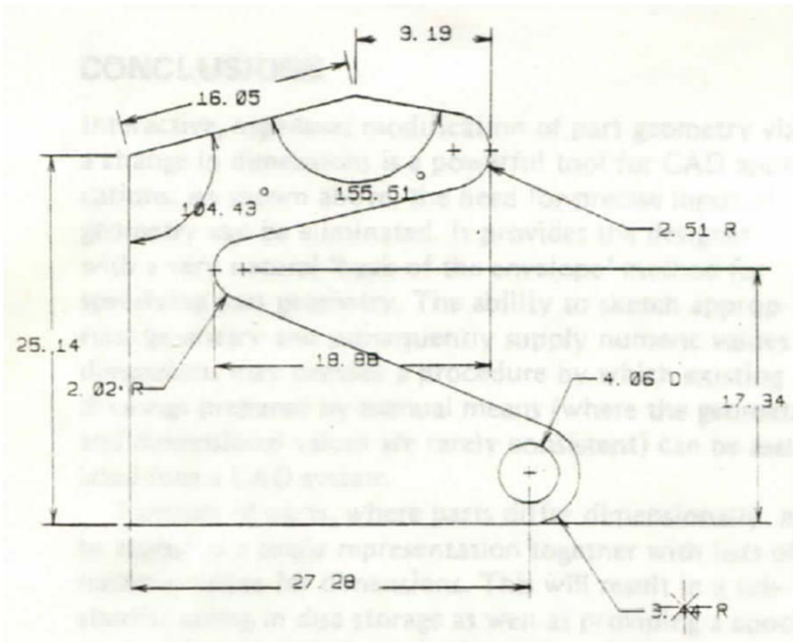
Implementation

- Define approximate geometry
- Assign dimensional constraints

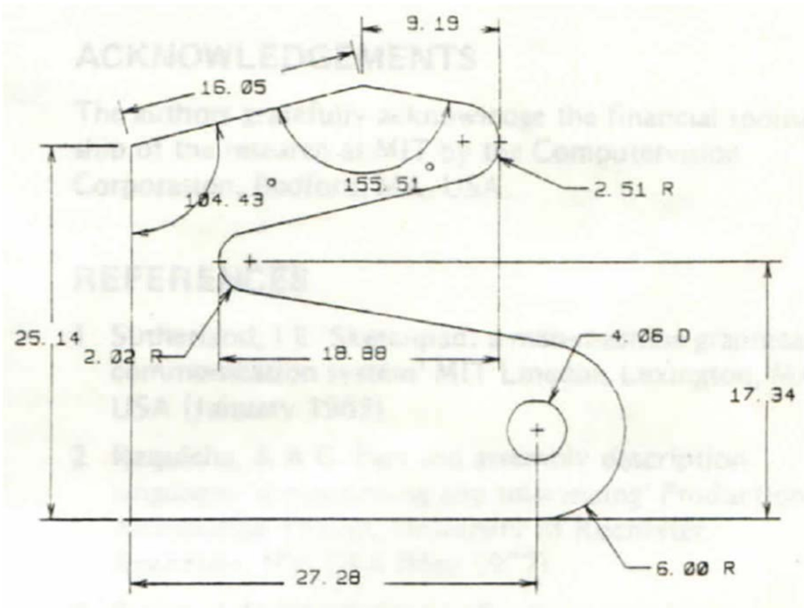




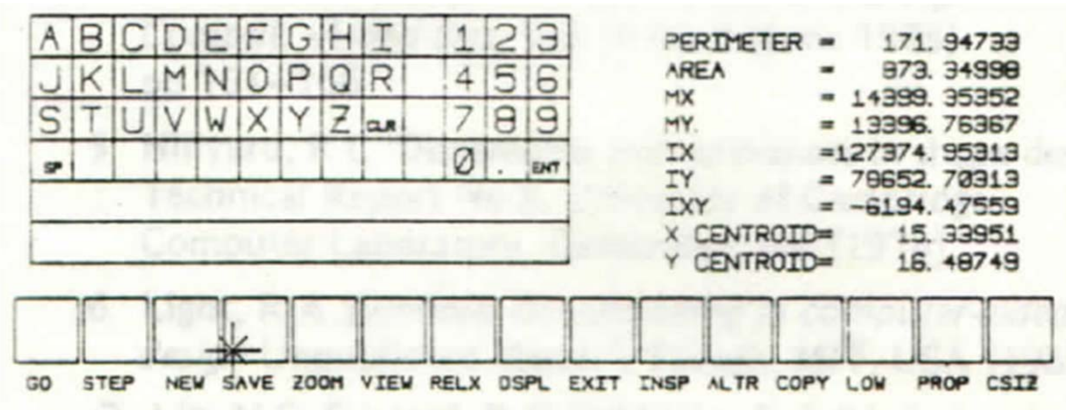
Example



Selection of a dimensional constraint



Modified geometry



Calculation of Geometric properties