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- Plane stress problem
- Linear elasticity equations
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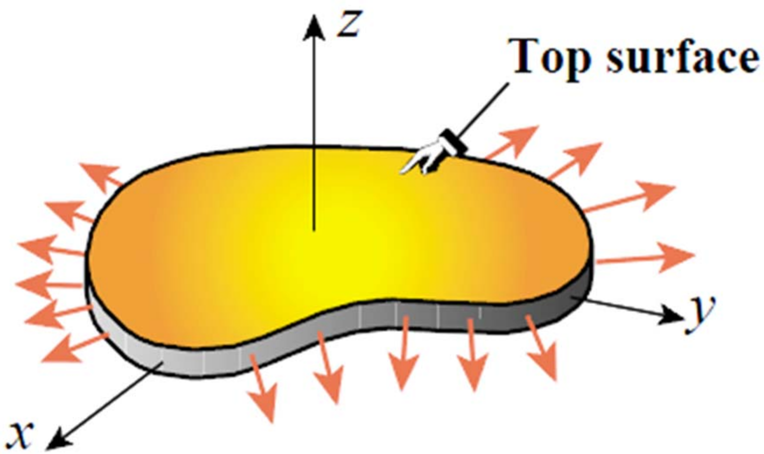
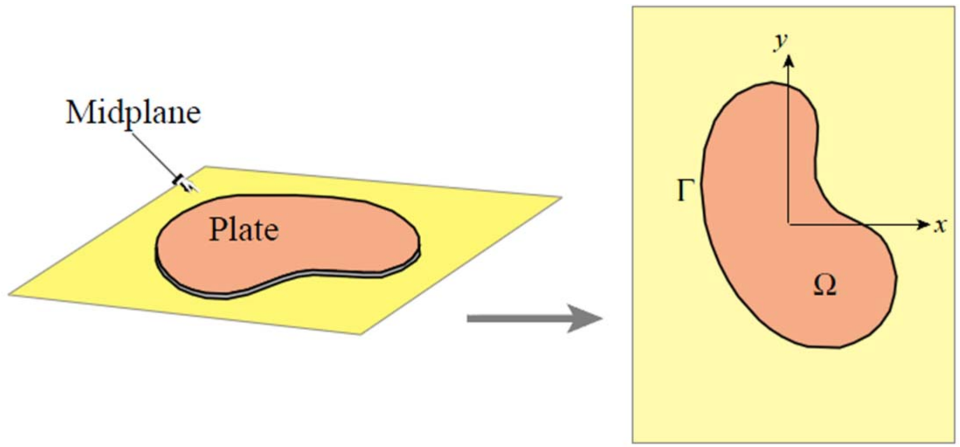
- Triangular coordinates
- Turner triangle (3-node plane stress triangle)

Plate in Plane Stress

- Plate: flat thin sheet of material
  - Thickness: distance between the plate faces
  - Midplane: lies halfway between the two faces (z=0)
  - Transverse (thickness) direction: normal to the midplane
    - +z (top surface) / -z (bottom surface)
  - In-plane direction: parallel to the midplane



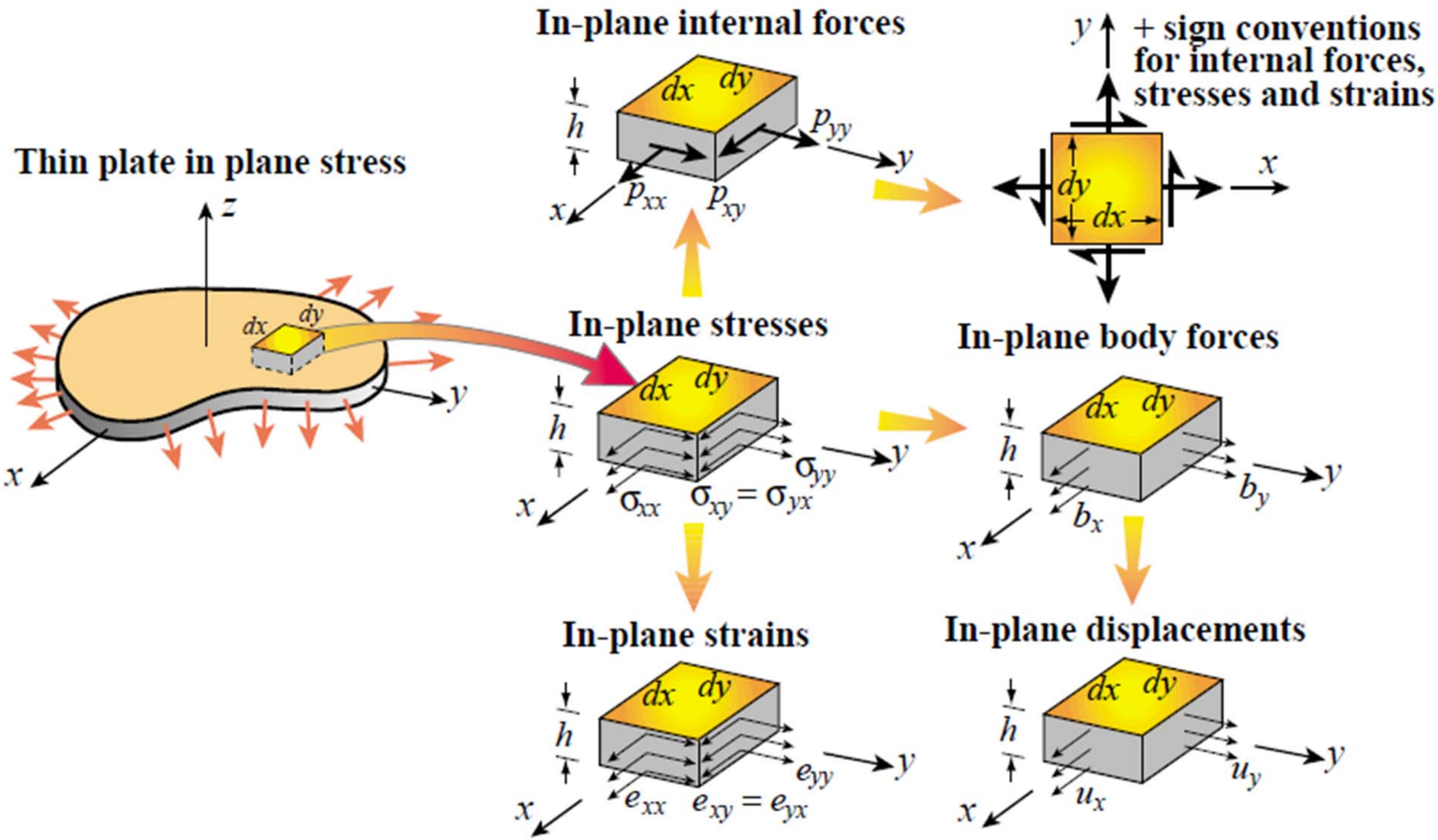


Plane Stress Physical Assumptions

- Plate is flat and has a symmetry plane (the midplane)
- All loads and support conditions are midplane symmetric
- Thickness dimension is much smaller than inplane dimensions (10% or less)
- In-plane displacements, strains and stresses uniform through thickness
- Transverse stresses σ_{zz}, σ_{xz} and σ_{yz} negligible, set to 0
- Plate fabricated of homogeneous material through thickness: transversely homogeneous

Notation for Stresses, Strains, Forces, Displacements

(also called membrane forces)



In-plane forces are obtained by stress integration through thickness.

Plane Stress Problem

- Given
  - Domain geometry
  - Thickness: constant, varying
  - Material data: linearly elastic but not necessarily isotropic
  - Specified interior forces: body (volume), face
  - Specified surface forces: surface traction
  - Displacement boundary conditions: fixed, allowed to move in one direction, or subject to multipoint constraints
- Find:
  - In-plane displacements
  - In-plane strains
  - In-plane stresses and/or internal forces

Governing Plane Stress Elasticity

$$\underbrace{\mathbf{u}(x,y)=\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}}_{\text{displacements}}, \quad \underbrace{\mathbf{e}(x,y)=\begin{bmatrix} e_{xx}(x,y) \\ e_{yy}(x,y) \\ 2e_{xy}(x,y) \end{bmatrix}}_{\substack{\text{strains} \\ \text{(factor of 2 in simplifies "energy dot products")}}}, \quad \underbrace{\boldsymbol{\sigma}(x,y)=\begin{bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{bmatrix}}_{\text{stresses}}$$

$$\underbrace{\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix}}_{\mathbf{e}=\mathbf{D}\mathbf{u}}, \quad \underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}=\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}}_{\boldsymbol{\sigma}=\mathbf{D}\mathbf{e}}, \quad \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}+\begin{bmatrix} b_x \\ b_y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{D}^T\boldsymbol{\sigma}+\mathbf{b}=\mathbf{0}}$$

Moduli for Isotropic Linear Elastic Material (1)

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \end{cases} \xleftrightarrow{G=\frac{E}{2(1+\nu)}} \begin{cases} \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{zx} = \dfrac{1}{G}\tau_{zx} \end{cases}$$

$$\text{plane stress}\left(\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0\right): \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\text{plane strain}\left(e_{zz} = e_{xz} = e_{yz} = 0\right): \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(2-\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

Moduli for Isotropic Linear Elastic Material (2)

near incompressible isopropic materials (as well as plasticity and viscoelasticity)

Lame constants (λ, μ) instead of (E, ν)

$$\left.\begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu = G &= \frac{E}{2(1+\nu)} \end{aligned}\right\} \leftrightarrow \left\{\begin{aligned} E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \\ \nu &= \frac{\lambda}{2(\lambda+\mu)} \end{aligned}\right.$$

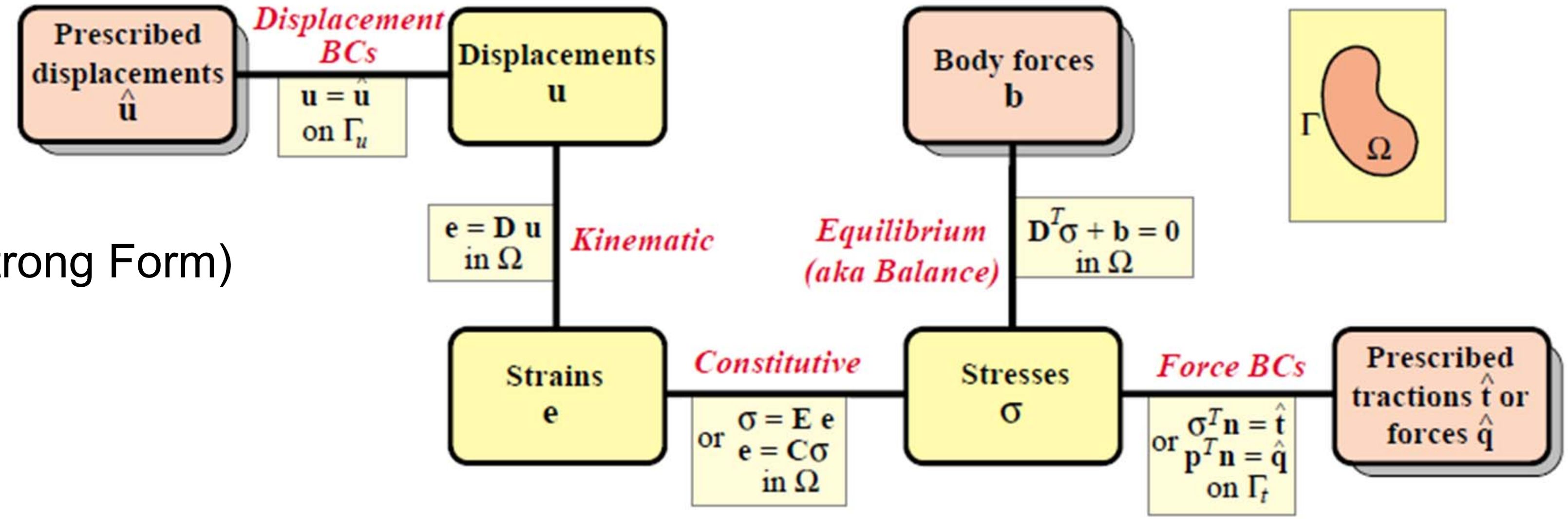
K : bulk modulus

M : P-wave modulus used in seismology

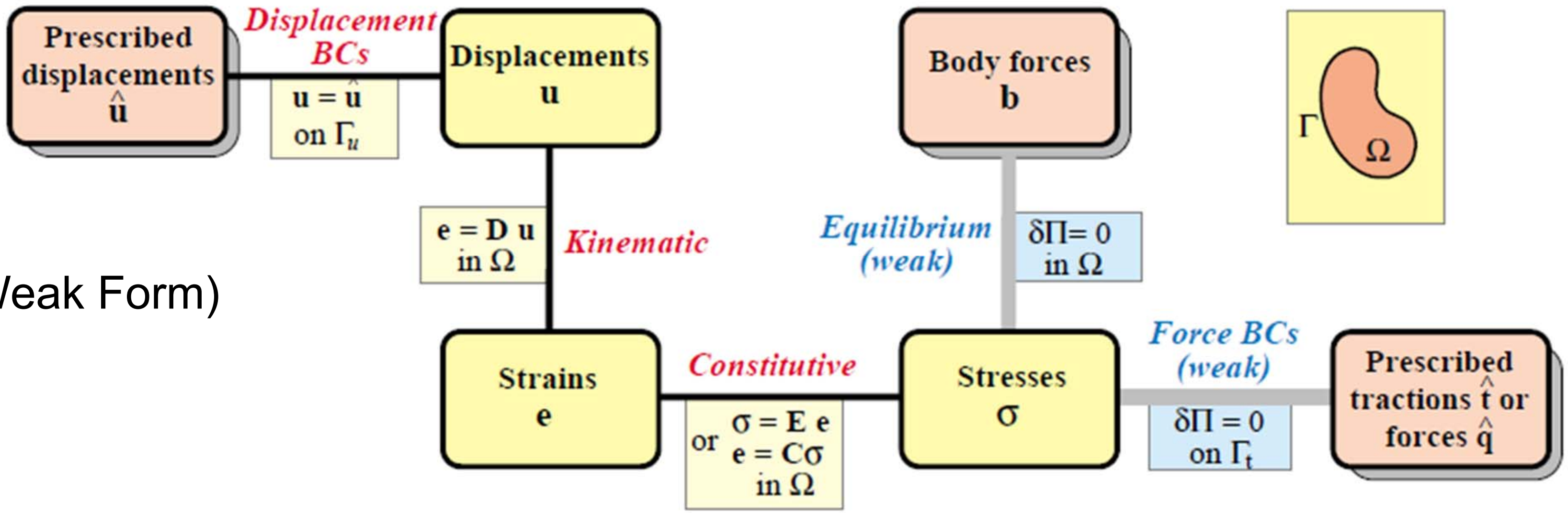
	(λ, μ)	(E, μ)	(K, λ)	(K, μ)	(λ, ν)	(μ, ν)	(E, ν)	(K, ν)	(K, E)
$K =$	$\lambda + \frac{2\mu}{3}$	$\frac{E\mu}{3(3\mu-E)}$			$\lambda\frac{1+\nu}{3\nu}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$		
$E =$	$\mu\frac{3\lambda+2\mu}{\lambda+\nu}$		$9K\frac{K-\lambda}{3K-\lambda}$	$\frac{9K\mu}{3K+\mu}$		$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2\mu(1+\nu)$		$3K(1-2\nu)$
$\lambda =$		$\mu\frac{E-2\mu}{3\mu-E}$		$K-\frac{2\mu}{3}$		$\frac{2\mu\nu}{1-2\nu}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{3K}{1+\nu}$	$\frac{3K(3K-E)}{9K-E}$
$\mu = G =$			$\mu\frac{K-\lambda}{2}$		$\lambda\frac{K-\lambda}{3K-\lambda}$	$\frac{9K\mu}{3K+\mu}$	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{E}{2(1+\nu)}$	$3K\frac{(1-2\nu)}{2(1+\nu)}$
$\nu =$		$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{E}{2\mu}-1$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2\mu}{2(3K+\mu)}$				$\frac{3K-E}{6K}$
$M =$	$\lambda+2\mu$	$\mu\frac{4\mu-E}{3\mu-E}$	$3K-2\lambda$	$K+\frac{4\mu}{3}$	$\lambda\frac{1-\nu}{\nu}$	$\mu\frac{2-2\nu}{1-2\nu}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$3K\frac{1-\nu}{1+\nu}$	$3K\frac{3K+E}{9K-E}$

Tonti Diagram of Governing Equations

(Strong Form)

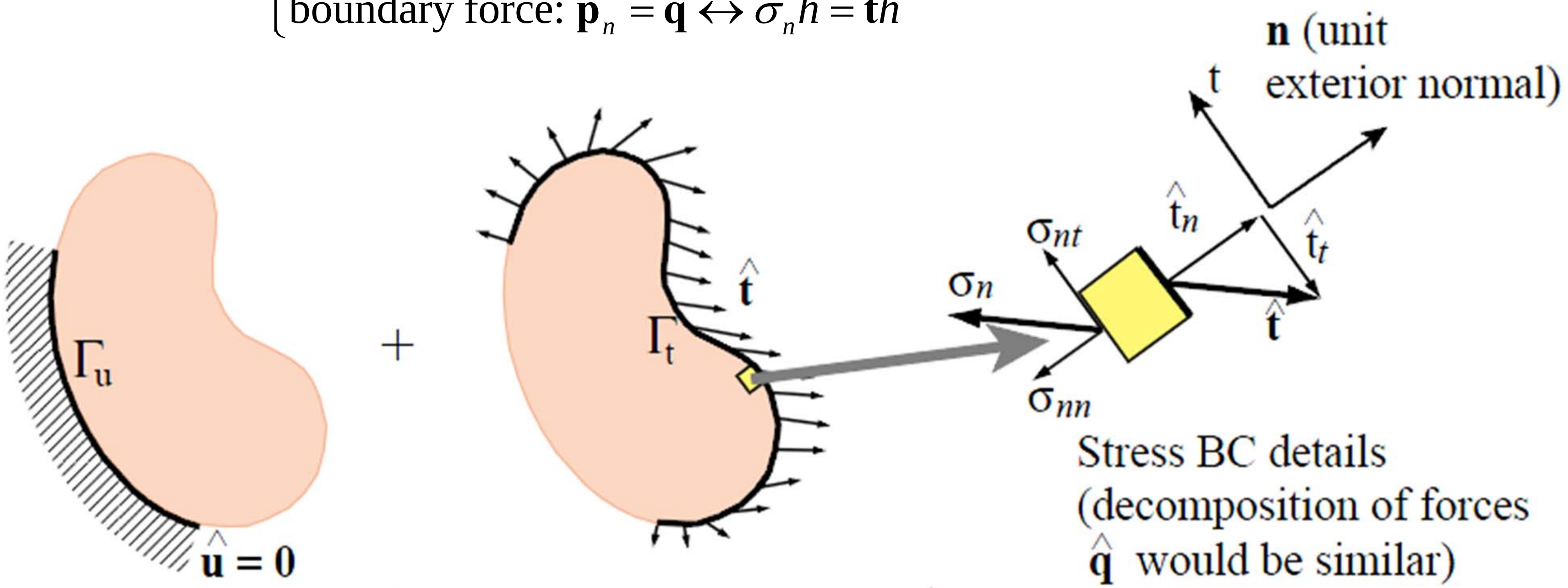


(Weak Form)



Plane Stress Boundary Conditions

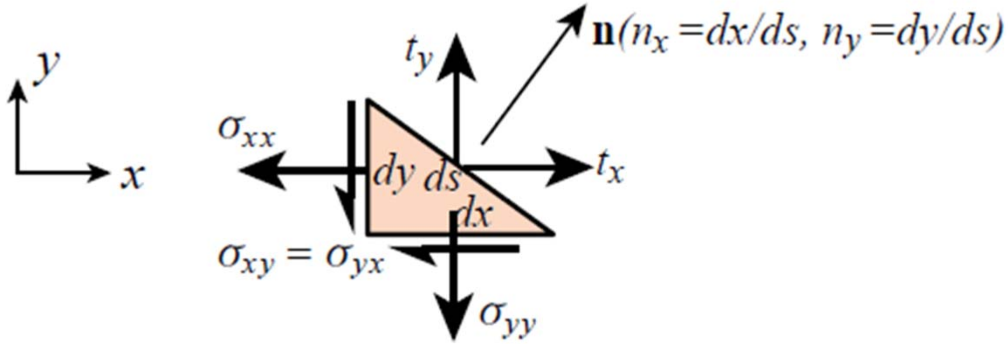
$$\begin{cases} \text{displacement BC on } \Gamma_u: \ \mathbf{u} = \hat{\mathbf{u}} \\ \text{force BC on } \Gamma_t: \ \begin{cases} \text{boundary traction: } \sigma_n = \hat{\mathbf{t}} \\ \text{boundary force: } \mathbf{p}_n = \hat{\mathbf{q}} \leftrightarrow \sigma_n h = \hat{\mathbf{t}} h \end{cases} \end{cases}$$



Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u (figure depicts fixity condition)

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on Γ_t

EXERCISE 14.4

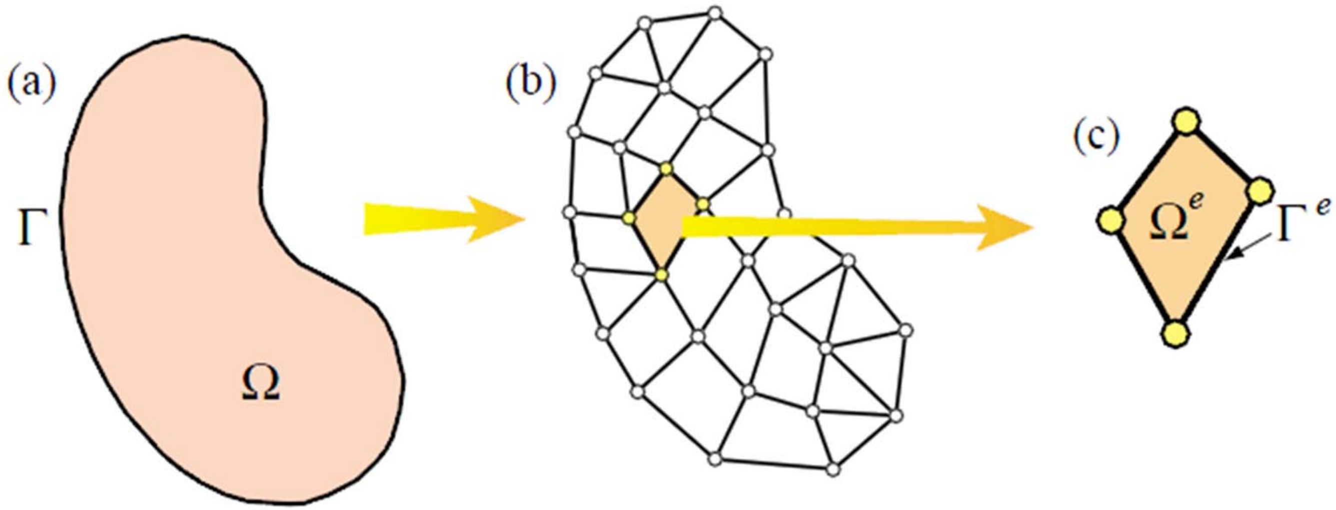


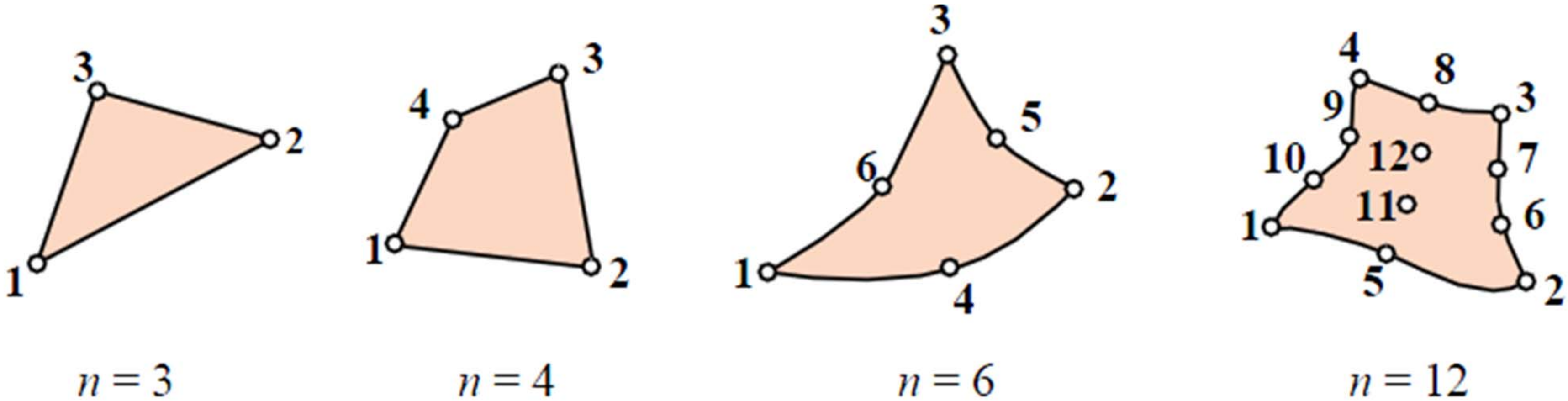
[Cauchy stress to traction equations]

$$t_x ds = \sigma_{xx} dy + \sigma_{yx} dx \rightarrow t_x = \sigma_{xx} \frac{dy}{ds} + \sigma_{yx} \frac{dx}{ds} = \sigma_{xx} n_y + \sigma_{yx} n_x$$

$$t_y ds = \sigma_{xy} dy + \sigma_{yy} dx \rightarrow t_y = \sigma_{xy} \frac{dy}{ds} + \sigma_{yy} \frac{dx}{ds} = \sigma_{xy} n_y + \sigma_{yy} n_x$$

Discretization into Finite Elements





Displacement Assumed Element

node displacement vector: $\mathbf{u}^e = \begin{bmatrix} u_{x1} & u_{y1} & \dots & u_{xn} & u_{yn} \end{bmatrix}$

displacement interpolation over element:

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \dots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e = \underbrace{\mathbf{N}}_{\substack{\text{shape function} \\ (2\times 2n)}} \mathbf{u}^e$$

strain-displacement relation:

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \mathbf{u}^e = \underbrace{\mathbf{B}}_{\substack{\text{strain-displacement} \\ \text{matrix} \\ =\text{D}\mathbf{N} \\ (3\times 2n)}} \mathbf{u}^e$$

stress-strain relation: $\boldsymbol{\sigma} = \mathbf{Ee} = \mathbf{EBu}^e$

Element Energy and Stiffness Equations

Internal energy: $U^e = \frac{1}{2}\int_V \sigma e dV^e = \frac{1}{2}\int_{\Omega^e} h\boldsymbol{\sigma}^T \mathbf{e} d\Omega^e = \frac{1}{2}\int_{\Omega} h\mathbf{e}^T \mathbf{E}\mathbf{e} d\Omega^e$

External work: $W^e = \int_{\Omega^e} h\mathbf{u}^T \mathbf{b} d\Omega^e + \int_{\Gamma_t^e} h\mathbf{u}^T \hat{\mathbf{t}} d\Gamma^e$ where $\begin{cases} \mathbf{b}: \text{ body forces} \\ \hat{\mathbf{t}}: \text{ boundary tractions} \end{cases}$

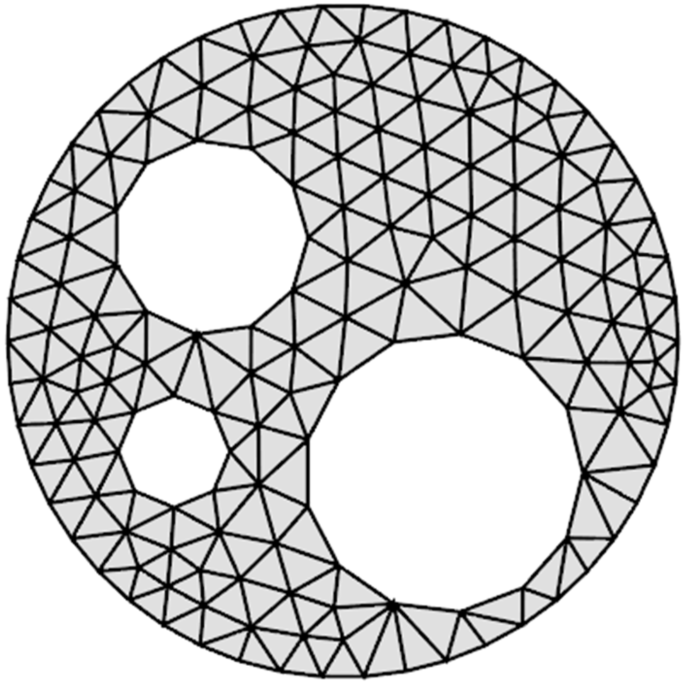
Total Potential Energy: $\Pi^e = U^e - W^e \leftarrow \begin{cases} \mathbf{u} = \mathbf{N}\mathbf{u}^e \\ \mathbf{e} = \mathbf{B}\mathbf{u}^e \\ \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \end{cases}$

$$= \frac{1}{2}\int_{\Omega^e} h\left(\mathbf{u}^e\right)^T \mathbf{B}^T\mathbf{E}\mathbf{B}\mathbf{u}^e d\Omega^e - \left[\int_{\Omega^e} h\left(\mathbf{u}^e\right)^T \mathbf{N}^T\mathbf{b} d\Omega^e + \int_{\Gamma_t^e} h\left(\mathbf{u}^e\right)^T \mathbf{N}^T\hat{\mathbf{t}} d\Gamma^e\right]$$

$$= \frac{1}{2}\left(\mathbf{u}^e\right)^T \int_{\Omega^e} h\left(\mathbf{u}^e\right)^T \mathbf{B}^T\mathbf{E}\mathbf{B} d\Omega^e \left(\mathbf{u}^e\right) - \left(\mathbf{u}^e\right)^T \left[\int_{\Omega^e} h\mathbf{N}^T\mathbf{b} d\Omega^e + \int_{\Gamma_t^e} h\mathbf{N}^T\hat{\mathbf{t}} d\Gamma^e\right]$$

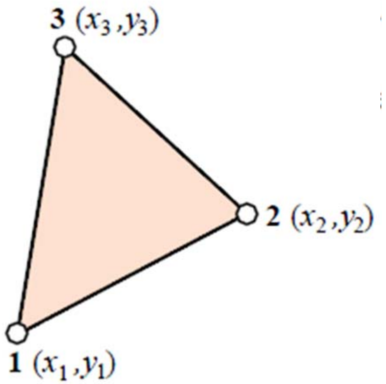
$$= \frac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e\mathbf{u}^e - \left(\mathbf{u}^e\right)^T \mathbf{f}^e \quad \text{where} \begin{cases} \mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B} d\Omega^e \\ \mathbf{f}^e = \int_{\Omega^e} h\mathbf{N}^T\mathbf{b} d\Omega^e + \int_{\Gamma_t^e} h\mathbf{N}^T\hat{\mathbf{t}} d\Gamma^e \end{cases}$$

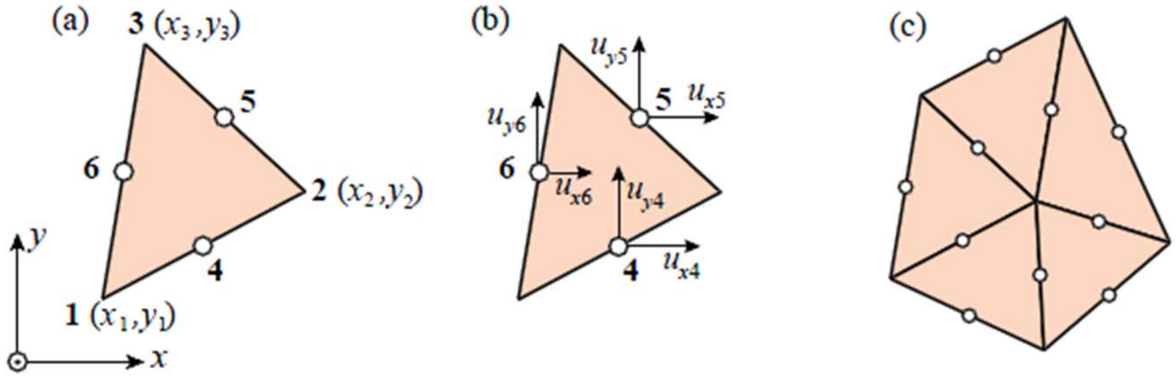
- Triangles are still popular because of geometric versatility and ease of automated mesh generation



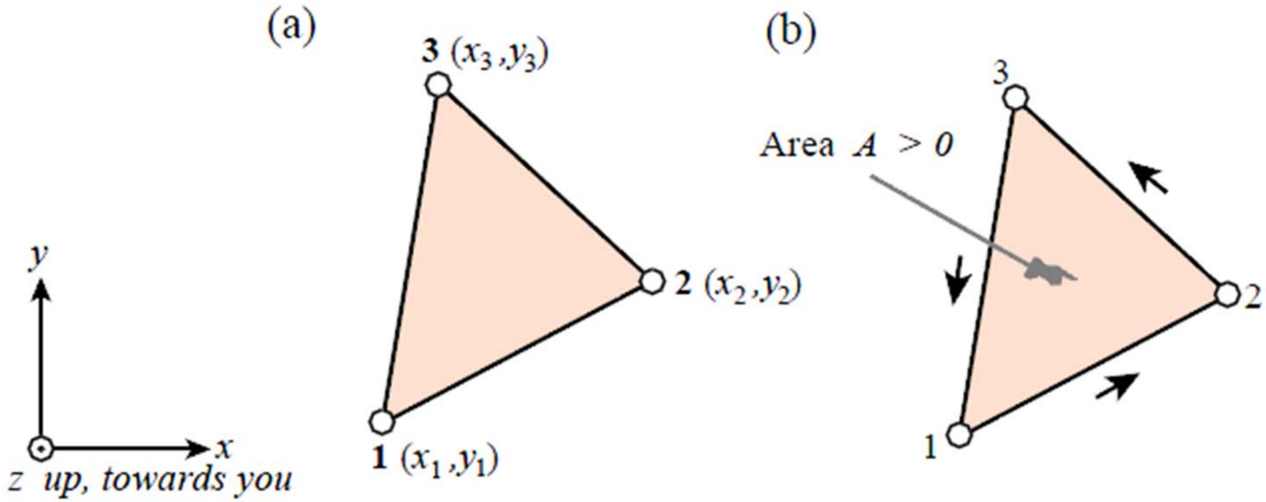
3-Node Plane Stress Triangles

- Turner triangle
  - both the isoparametric and subparametric element families
  - closed form derivations for the stiffness matrix and consistent force vector without need for numerical integration
  - cannot be improved by the addition of internal degrees of freedom
- Veubeke equilibrium triangle





Turner Triangle Geometry / Nodal Configuration



$$2A = \det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \left(x_2 y_3 - x_3 y_2\right) + \left(x_3 y_1 - x_1 y_3\right) + \left(x_1 y_2 - x_2 y_1\right)$$

First Berkeley and Engineering FEM paper

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Stiffness and Deflection Analysis of Complex Structures

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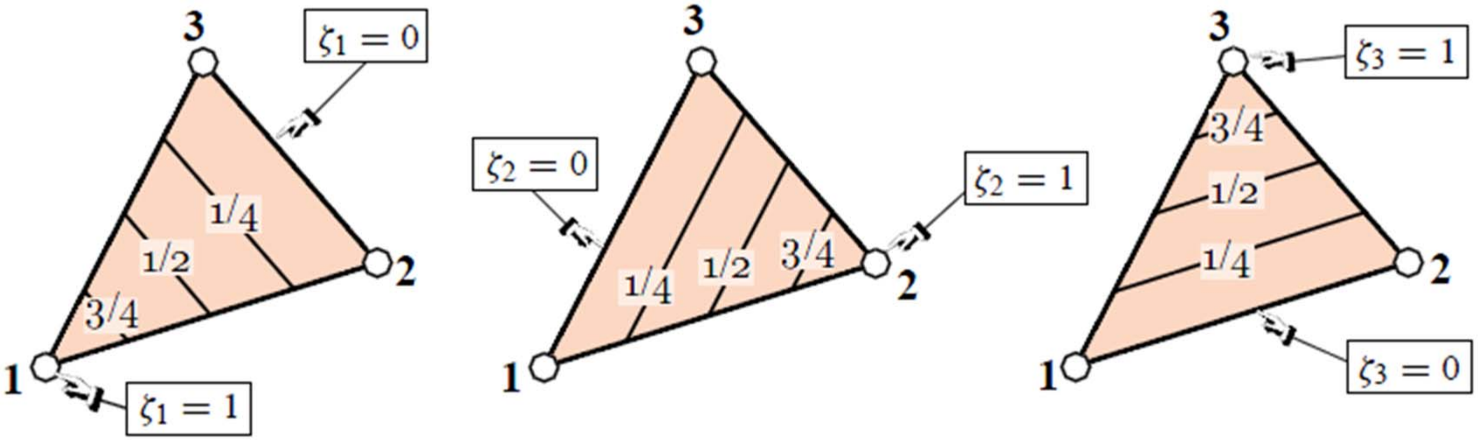
Triangular Coordinates (ζ_1, ζ_2, ζ_3)



Name	*Applicable to*
natural coordinates	all elements
isoparametric coordinates	isoparametric elements
shape function coordinates	isoparametric elements
barycentric coordinates	simplices (triangles, tetrahedra, ...)
Möbius coordinates	triangles
triangular coordinates	all triangles
area (also written "areal") coordinates	straight-sided triangles

Triangular coordinates normalized as per $\zeta_1 + \zeta_2 + \zeta_3 = 1$ are often qualified as "homogeneous" in the mathematical literature.

Triangular and Cartesian Coordinates

Consider a function $f\left(x, y\right)$ that varies linearly over the triangle domain.

Cartesian form: $f\left(x, y\right)=a_0+a_1 x+a_2 y$

nodal values taken by f at the corners: f_1, f_2, f_3

linear interpolant: $f\left(\zeta_1, \zeta_2, \zeta_3\right)=f_1\zeta_1+f_2\zeta_2+f_3\zeta_3=\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}=\begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

$$\underbrace{\text{triangular}}_{\text{Quantities that are linked with the element geometry}} \rightarrow \underbrace{\text{Cartesian}}_{\text{quantities such as displacements, strains and stresses}} : \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Cartesian $\rightarrow$ triangular: $\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}=\dfrac{1}{2A}\begin{bmatrix} x_2 y_3-x_3 y_2 & y_2-y_3 & x_3-x_2 \\ x_3 y_1-x_1 y_3 & y_3-y_1 & x_1-x_3 \\ x_1 y_2-x_2 y_1 & y_1-y_2 & x_2-x_1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}=\dfrac{1}{2A}\begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$

$x_{jk}=x_j-x_k,\ y_{jk}=y_j-y_k$

A_{jk} : area subtended by corners $j,\ k$ and the origin of the x - y system

If this origin is taken at the centroid of the triangle, $A_{23}=A_{31}=A_{12}=A/3$

Partial Derivatives

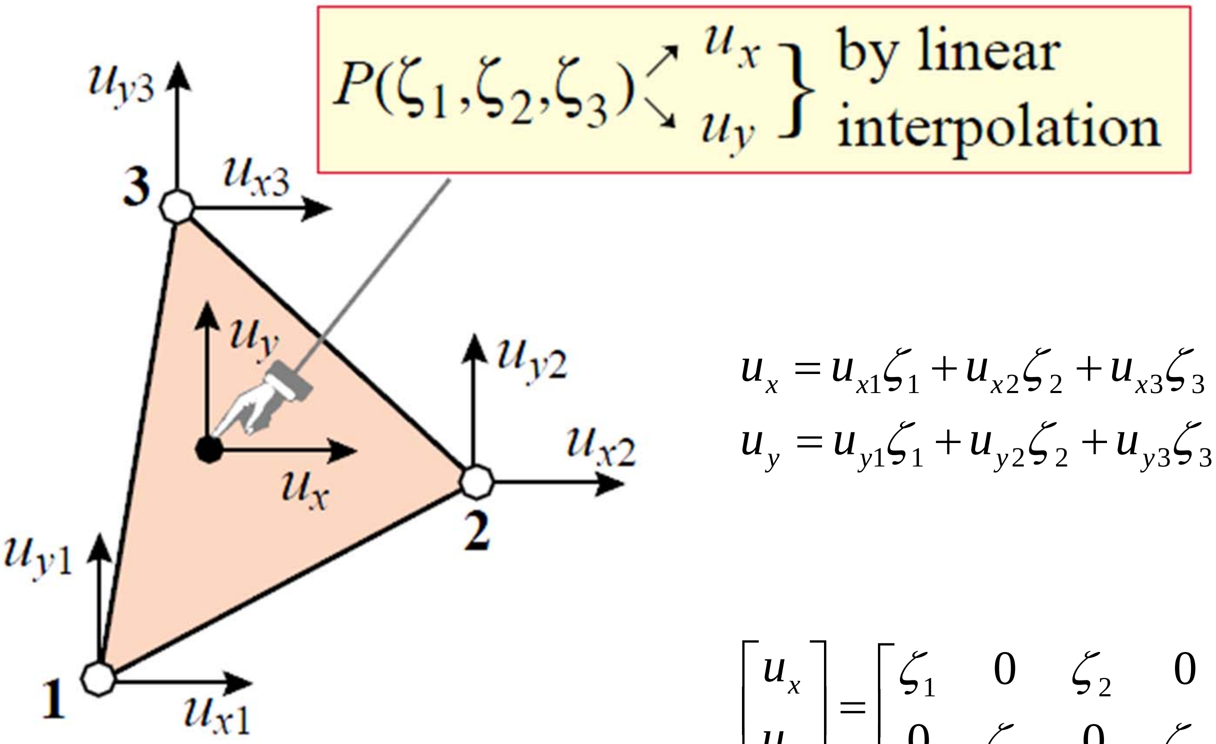
$$\frac{\partial x}{\partial \zeta_i} = x_i, \quad \frac{\partial y}{\partial \zeta_i} = y_i$$

$$2A\frac{\partial \zeta_i}{\partial x} = y_{jk}, \quad 2A\frac{\partial \zeta_i}{\partial y} = x_{kj} \begin{cases} i = 1,2,3 \\ j = 2,3,1 \\ k = 3,1,2 \end{cases}$$

$$f = f\left(\zeta_1, \zeta_2, \zeta_3\right)$$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = \dfrac{1}{2A}\left(\dfrac{\partial f}{\partial \zeta_1} y_{23} + \dfrac{\partial f}{\partial \zeta_2} y_{31} + \dfrac{\partial f}{\partial \zeta_3} y_{12}\right) \\ \dfrac{\partial f}{\partial y} = \dfrac{1}{2A}\left(\dfrac{\partial f}{\partial \zeta_1} x_{32} + \dfrac{\partial f}{\partial \zeta_2} x_{13} + \dfrac{\partial f}{\partial \zeta_3} x_{21}\right) \end{array}\right\} \rightarrow \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} \dfrac{\partial f}{\partial \zeta_1} \\ \dfrac{\partial f}{\partial \zeta_2} \\ \dfrac{\partial f}{\partial \zeta_3} \end{bmatrix}$$

Displacement Interpolation over Turner Triangle



$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \underbrace{\mathbf{N}}_{\substack{\text{shape functions} \\ N_i=\zeta_i,\ i=1,2,3}} \mathbf{u}^e$$

Displacement-Strain-Stress

strain-displacement relation:

$$\mathbf{e} = \mathbf{DNu}^{\mathrm{e}} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{Bu}^{\mathrm{e}}$$

* strains are constant over the element $\rightarrow$ constant strain triangle (CST)

stress-strain relation:

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{Ee}$$

Element Stiffness Matrix

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega^e \xrightarrow[\text{over the triangle area}]{\mathbf{B}\text{ and }\mathbf{E}\text{ are constant}} \mathbf{K}^e = \mathbf{B}^T\mathbf{E}\mathbf{B}\int_{\Omega^e} h\,d\Omega^e$$

$$= \frac{1}{4A^2}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\int_{\Omega^e} h\,d\Omega^e$$

$$\xrightarrow{\text{if } h \text{ is constant}} \mathbf{K}^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Consistent Node Force Vector for Body Forces

$$\mathbf{f}^e = \int_{\Omega^e} h\mathbf{N}^T \mathbf{b}\, d\Omega^e = \int_{\Omega^e} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} d\Omega^e \xrightarrow[\text{over the element}]{\text{if body forces and } h \text{ are constant}} \mathbf{f}^e = \frac{1}{3} A h \underbrace{\begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}}_{\text{same as "load limping"}}$$

$$\left\{ \begin{array}{l} \int_{\Omega^e} \zeta_1 d\Omega^e = \int_{\Omega^e} \zeta_2 d\Omega^e = \int_{\Omega^e} \zeta_3 d\Omega^e = \frac{1}{3} A \\ \text{instances of the general formula (intergrating triangular coordinate monomials):} \\ \frac{1}{2A} \int_{\Omega^e} \zeta_1^i \zeta_2^j \zeta_3^k d\Omega^e = \frac{i!\, j!\, k!}{(i+j+k+2)!} \quad (i \ge 0,\ j \ge 0,\ k \ge 0) \\ \text{valid for triangles with straight sides and constant metric} \end{array} \right.$$