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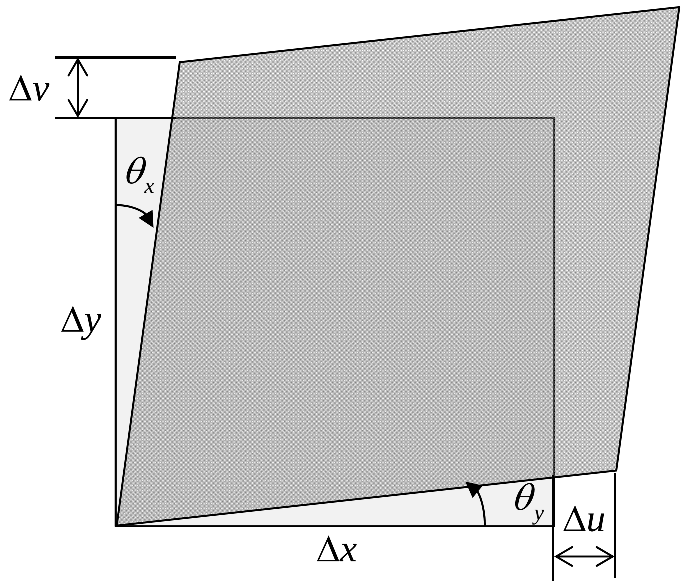
Displacement / Strain

displacement: $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

strain (infinitesimal):

$$\begin{cases} \text{elongation/contraction: } \varepsilon_x = \lim\limits_{\Delta x \to 0} \dfrac{(\Delta x + \Delta u) - \Delta x}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} = \dfrac{\partial u}{\partial x} \\ \text{angle change: } \gamma_{xy} = \theta_x + \theta_y = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\tan^{-1} \dfrac{\Delta u}{\Delta y} + \tan^{-1} \dfrac{\Delta v}{\Delta x} \right) = \lim\limits_{\Delta x \to 0} \left(\dfrac{\Delta u}{\Delta y} + \dfrac{\Delta v}{\Delta x} \right) = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{cases}$$



Strain / Stress

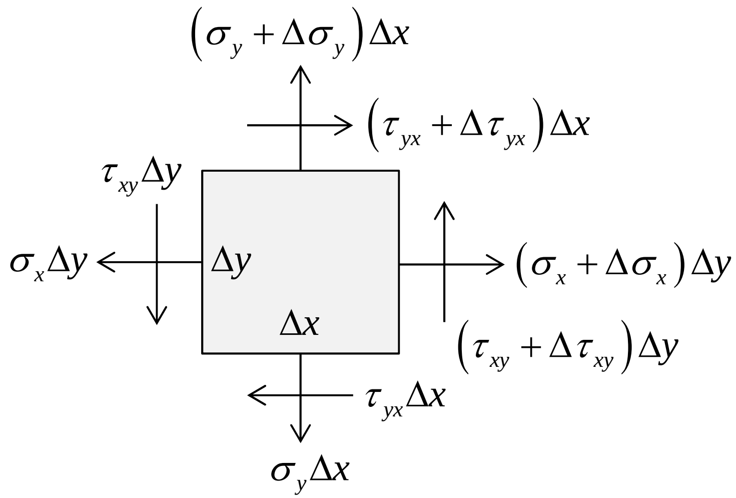
$$\boldsymbol{\varepsilon} = \left\{ \begin{matrix} \left.\begin{matrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{matrix}\right\} \text{normal strains} \\ \left.\begin{matrix} \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{matrix}\right\} \text{shear strains} \end{matrix} \right\} = \left\{ \begin{matrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{matrix} \right\} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \end{bmatrix} \left\{ \begin{matrix} u \\ v \\ w \end{matrix} \right\} \leftrightarrow \boldsymbol{\varepsilon} = \partial \mathbf{u}, \quad \boldsymbol{\sigma} = \left\{ \begin{matrix} \left.\begin{matrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{matrix}\right\} \text{normal stress} \\ \left.\begin{matrix} \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{matrix}\right\} \text{shear stress} \end{matrix} \right\}$$

Equilibrium Equations

$$\left(\sigma_x + \Delta\sigma_x\right)\Delta y - \sigma_x \Delta y + \left(\tau_{yx} + \Delta\tau_{yx}\right)\Delta x - \tau_{yx}\Delta x + \rho\left(b_x - \ddot{u}\right)\Delta x \Delta y = 0$$

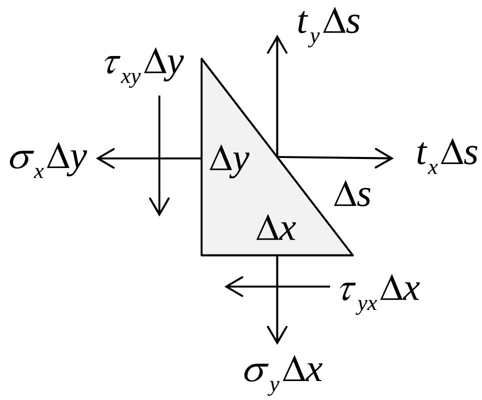
$$\xrightarrow{\div \Delta x \Delta y} \frac{\Delta\sigma_x}{\Delta x} + \frac{\Delta\tau_{yx}}{\Delta y} + \rho\left(b_x - \ddot{u}\right) = 0 \xrightarrow{\text{taking the limit}} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \rho\left(b_x - \ddot{u}\right) = 0$$

$$\xrightarrow{\text{3D}} \left\{ \begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} &= -\rho\left(b_x - \ddot{u}\right) \\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} &= -\rho\left(b_x - \ddot{v}\right) \\ \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} &= -\rho\left(b_x - \ddot{w}\right) \end{aligned} \right\}$$



$$\rightarrow \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} = \tau_{zy} \\ \tau_{zx} = \tau_{xz} \\ \tau_{xy} = \tau_{yx} \end{Bmatrix} = -\begin{Bmatrix} \rho\left(b_x - \ddot{u}\right) \\ \rho\left(b_y - \ddot{v}\right) \\ \rho\left(b_z - \ddot{w}\right) \end{Bmatrix} \leftrightarrow \partial^T \boldsymbol{\sigma} = -\rho\left(\mathbf{b} - \mathbf{a}\right)$$

Traction Vector

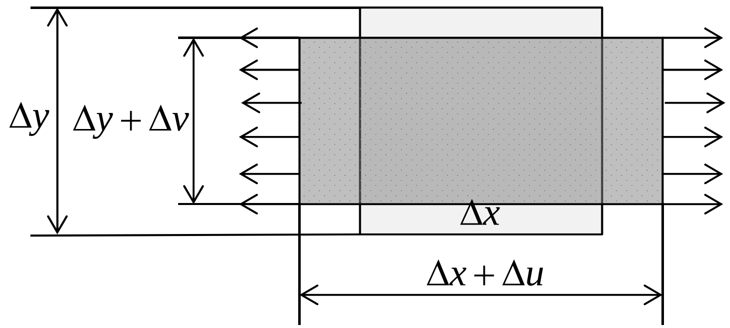


$$t_x \Delta s = \sigma_x \Delta y + \tau_{xy} \Delta x - \rho\left(b_x - \ddot{u}\right)\frac{1}{2}\Delta x \Delta y$$

$$\rightarrow \begin{cases} t_x = \lim\limits_{\Delta s \to 0}\left\{\sigma_x \dfrac{\Delta y}{\Delta s} + \tau_{xy}\dfrac{\Delta x}{\Delta s} - \rho\left(b_x - \ddot{u}\right)\dfrac{1}{2}\dfrac{\Delta x \Delta y}{\Delta s}\right\} = \sigma_x n_x + \tau_{xy} n_y \\ t_y = \lim\limits_{\Delta s \to 0}\left\{\tau_{yx} \dfrac{\Delta y}{\Delta s} + \sigma_y\dfrac{\Delta x}{\Delta s} - \rho\left(b_x - \ddot{v}\right)\dfrac{1}{2}\dfrac{\Delta x \Delta y}{\Delta s}\right\} = \tau_{yx} n_x + \sigma_y n_y \end{cases}$$

$$\xrightarrow{\text{3D}} \begin{Bmatrix} t_x = n_x \sigma_x + n_y \tau_{yx} + n_z \tau_{zx} \\ t_y = n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{zy} \\ t_z = n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z \end{Bmatrix} \rightarrow \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} n_x & 0 & 0 & 0 & n_z & n_y \\ 0 & n_y & 0 & n_z & 0 & n_x \\ 0 & 0 & n_z & n_y & n_x & 0 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} = \tau_{zy} \\ \tau_{zx} = \tau_{xz} \\ \tau_{xy} = \tau_{yx} \end{Bmatrix} \leftrightarrow \mathbf{t} = \mathbf{N}^T \boldsymbol{\sigma}$$

Strain-Stress Relation



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \\
\varepsilon_y &= -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \\
\varepsilon_z &= -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\gamma_{zx} &= \frac{\tau_{zx}}{G} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}\right\} \rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$$\leftrightarrow \boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma}$$

Stress-Strain Relation

$$\boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma} \xrightarrow{G=\frac{E}{2(1+\nu)}} \boldsymbol{\sigma} = \mathbf{C}^{-1}\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Special Case: Plane Stress

$\sigma_z = \tau_{zx} = \tau_{yz} = 0 \leftarrow$ if plane stress assumption in the (x, y) plane

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Special Case: Plane Strain

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \leftarrow \text{if plane strain assumption in the } (x, y) \text{ plane}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

temperature effect: $\boldsymbol{\varepsilon}_i = \begin{Bmatrix} \varepsilon_{i\_x} \\ \varepsilon_{i\_y} \\ \varepsilon_{i\_z} \\ \gamma_{i\_yz} \\ \gamma_{i\_zx} \\ \gamma_{i\_xy} \end{Bmatrix} = \alpha \Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_i)$$