Solid Mechanics (plane stress/strain)

Computational Design Laboratory
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- **Beam model: 2D**
  - ✓ **Textbook of "Solid Mechanics"**
- **Beam model: 3D**
- **2D plane stress/strain model**
  - ✓ **Kirsh's problem**
- **Assignment**

PLANE STRESS

PLANE STRESS

The plane stress variant of the 2D interface is useful for analyzing thin in-plane loaded plates. For a state of plane stress, the out-of-plane components of the stress tensor are zero.

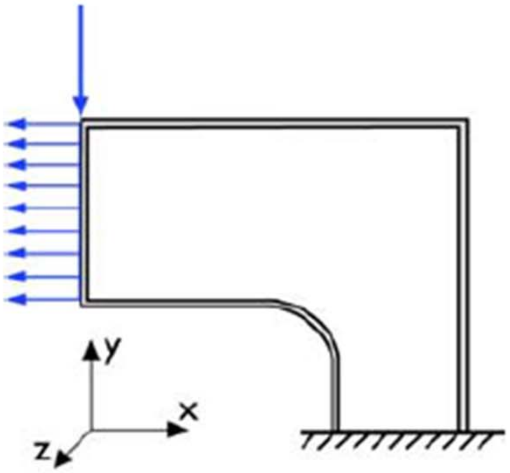


Figure 14-2: Plane stress models plates where the loads are only in the plane; it does not include any out-of-plane stress components.

The 2D interface for plane stress allows loads in the x and y directions, and it assumes that these are constant throughout the material's thickness, which can vary with x and y. The plane stress condition prevails in a thin flat plate in the xy-plane loaded only in its own plane and without any z direction restraint.

PLANE STRAIN

PLANE STRAIN

The plane strain variant of the 2D interface that assumes that all out-of-plane strain components of the total strain ε_z, ε_{yz}, and ε_{xz} are zero.

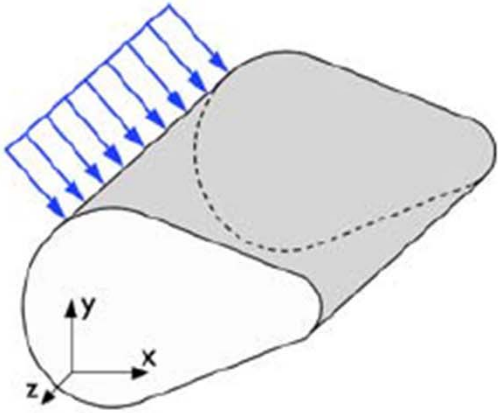


Figure 14-3: A geometry suitable for plane strain analysis.

Loads in the x and y directions are allowed. The loads are assumed to be constant throughout the thickness of the material, but the thickness can vary with x and y. The plane strain condition prevails in geometries, whose extent is large in the z direction compared to in the x and y directions, or when the z displacement is in some way restricted. One example is a long tunnel along the z-axis where it is sufficient to study a unit-depth slice in the xy-plane.

STIFFNESS MATRIX

Plane stress

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ 0\\ 0\\ 0\\ \sigma_{xy}\end{bmatrix}$$

Compliance matrix

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix}$$

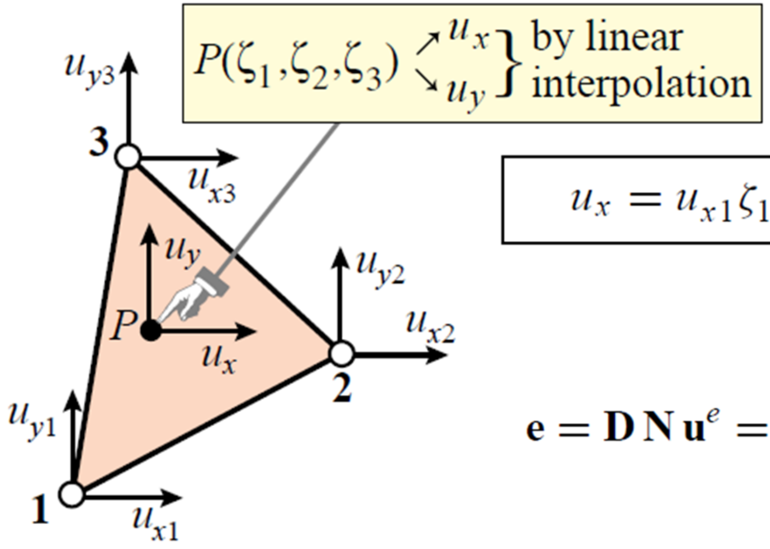
Stiffness matrix

Plane strain

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 0\\ 0\\ 0\\ \varepsilon_{xy}\end{bmatrix}$$

Stiffness matrix

TURNER TRIANGLE (1)



$$\boxed{u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3, \qquad u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3.}$$

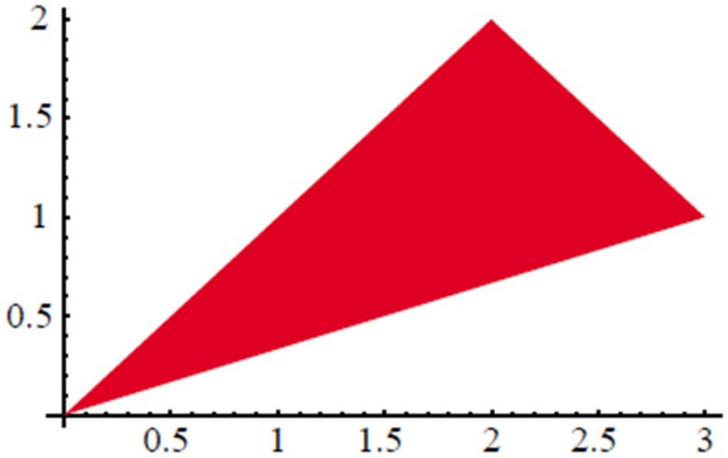
$$\mathbf{e} = \mathbf{D}\,\mathbf{N}\,\mathbf{u}^e = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B}\,\mathbf{u}^e$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E}\,\mathbf{e}$$

$$\boxed{\mathbf{K}^e = A\,h\,\mathbf{B}^T\,\mathbf{E}\,\mathbf{B} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}.}$$

- **Stiffness matrix of Turner triangle**
- **2D plane stress/strain model**
  - ✓ **Kirsh's problem**
- **Thick plate**
  - ✓ **Benchmark problem LE10**

TURNER TRIANGLE (2)

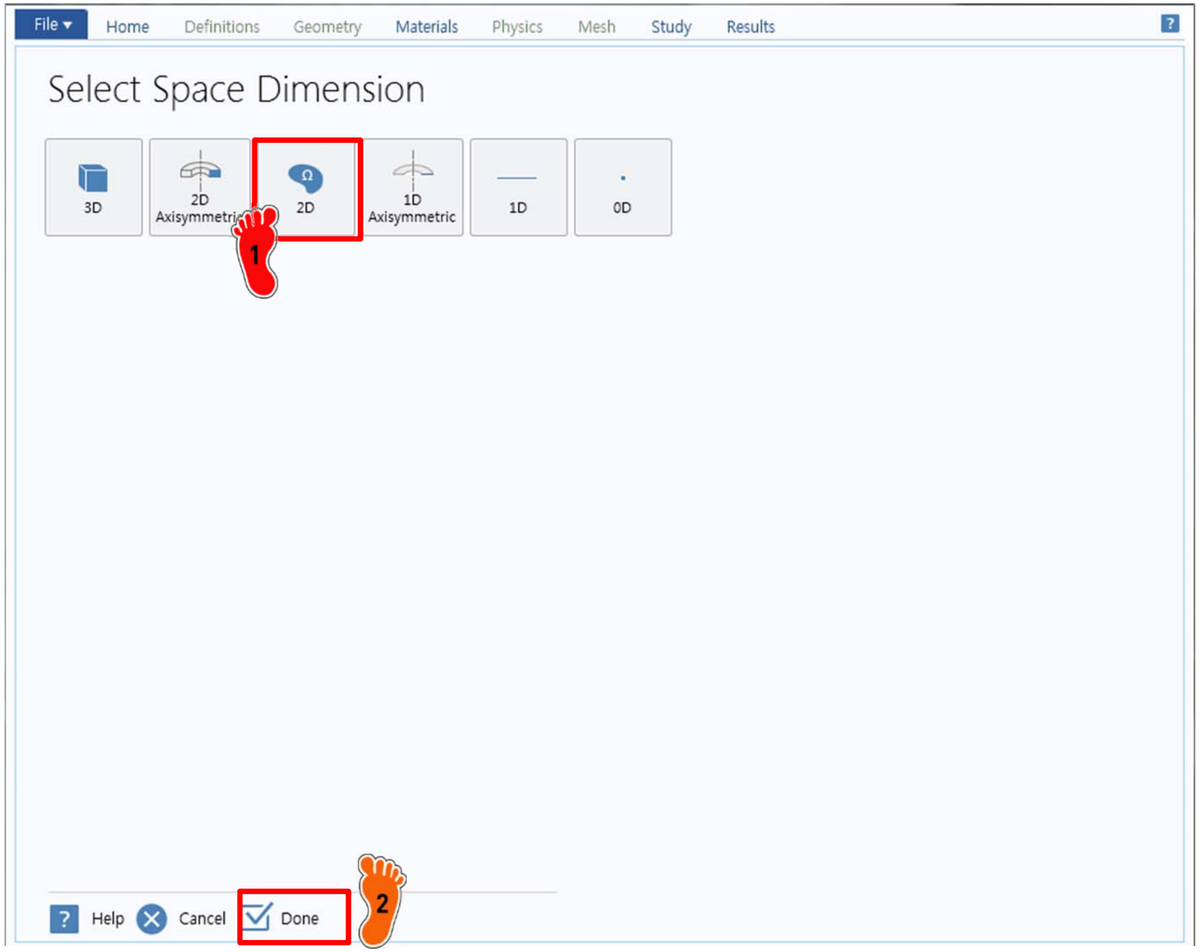


$$E_0 = 60 \ , \ \nu = 0.25 \ , \ h = 1$$

$$\mathbf{E}_{\text{plane stress}} = \begin{bmatrix} 64 & 16 & 0 \\ 16 & 64 & 0 \\ 0 & 0 & 24 \end{bmatrix} \qquad \mathbf{E}_{\text{plane strain}} = \begin{bmatrix} 72 & 24 & 0 \\ 24 & 72 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$\mathbf{Ke}_{\text{plane stress}} = \begin{bmatrix} 11 & 5 & -10 & -2 & -1 & -3 \\ 5 & 11 & 2 & 10 & -7 & -21 \\ -10 & 2 & 44 & -20 & -34 & 18 \\ -2 & 10 & -20 & 44 & 22 & -54 \\ -1 & -7 & -34 & 22 & 35 & 15 \\ 3 & -21 & 18 & -54 & -15 & 75 \end{bmatrix} \qquad \mathbf{Ke}_{\text{plane strain}} = \begin{bmatrix} 12 & 6 & -12 & 0 & 0 & -6 \\ 6 & 12 & 0 & 12 & -6 & -24 \\ -12 & 0 & 48 & -24 & -36 & 24 \\ 0 & 12 & -24 & 48 & 24 & -60 \\ 0 & -6 & -36 & 24 & 36 & -18 \\ -6 & -24 & 24 & -60 & -18 & 84 \end{bmatrix}$$

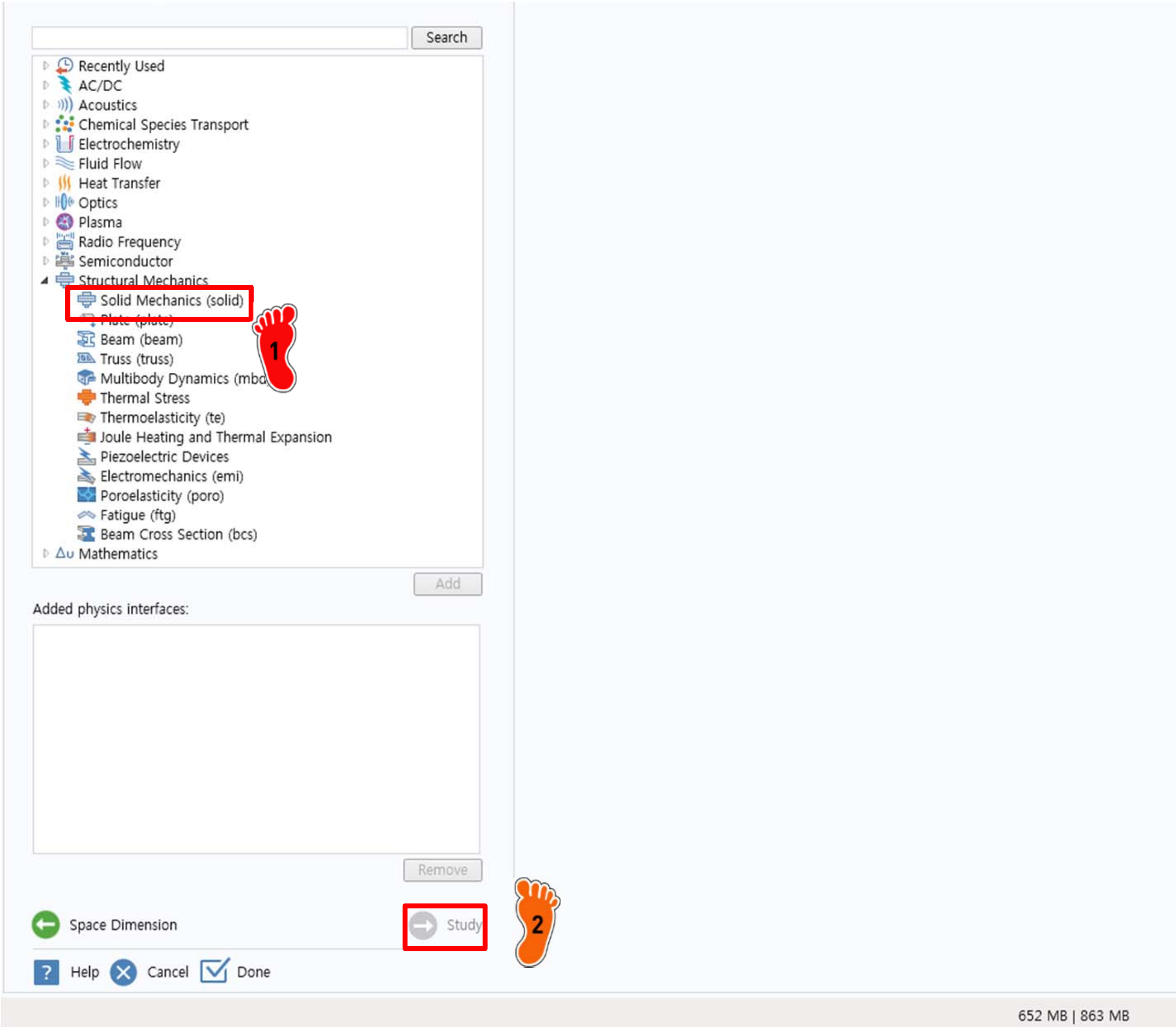
DIMENSION SELECTION



1 2D 선택

2 Done 클릭

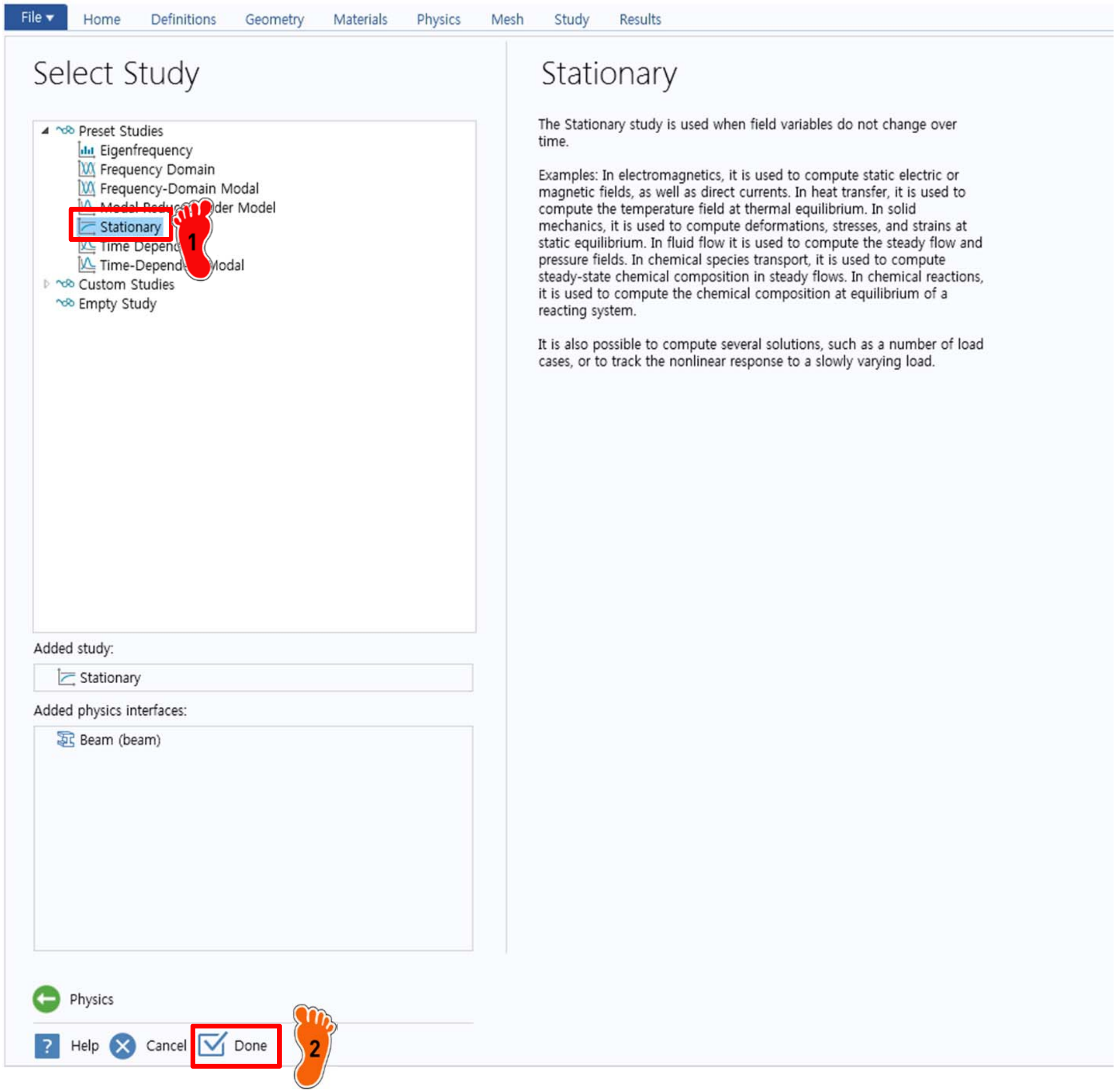
PHYSICS SELECTION



1 Structural Mechanics 의 Solid Mechanics 추가

2 Study 클릭

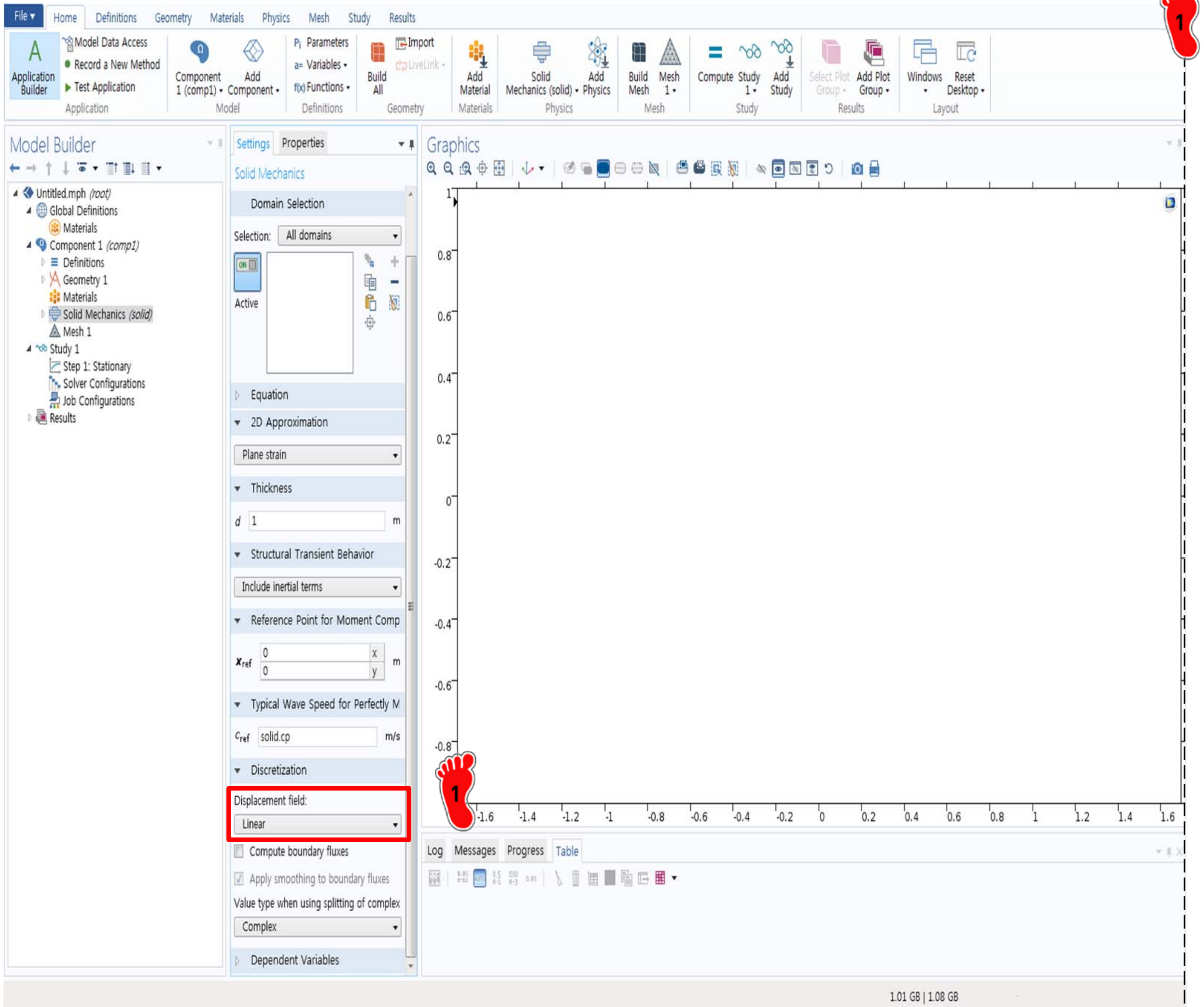
STUDY TYPE SELECTION



1 Stationary 선택

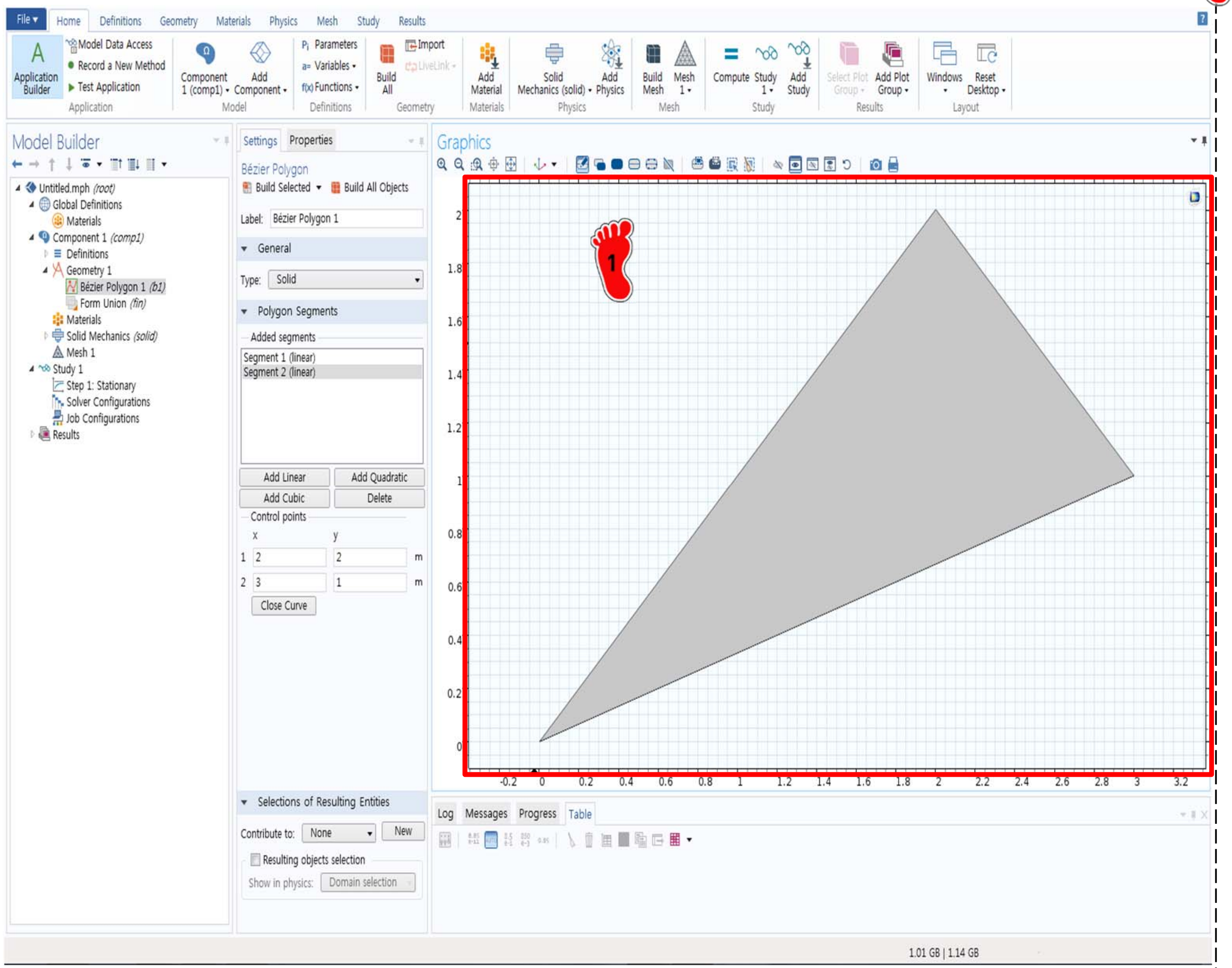
2 Done 클릭

DISCRETIZATION



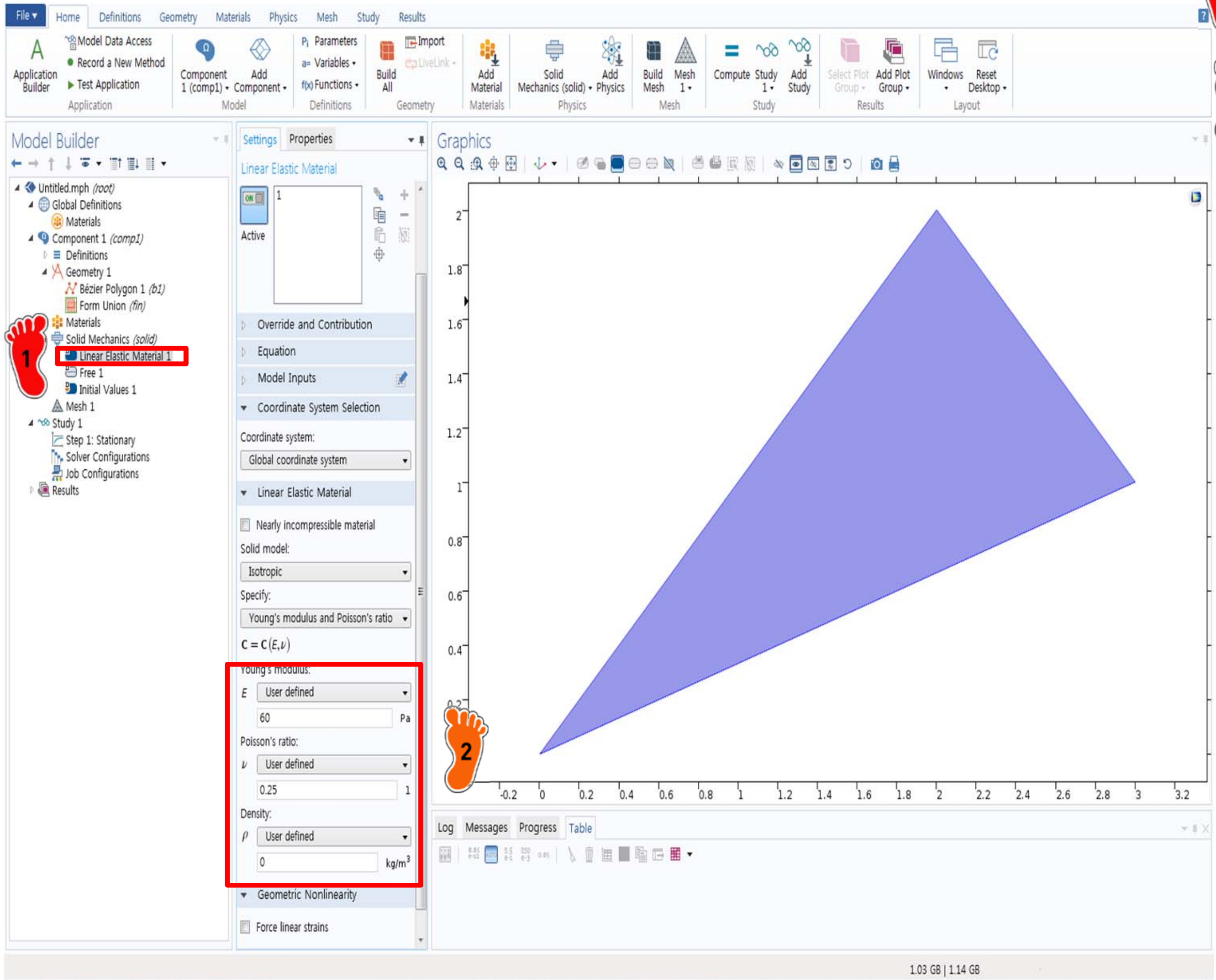
1. Discretization Displacement field 를 Linear 로 변경

GEOMETRY CREATION



1 Bezier Polygon을 이용하여
삼각형 기하형상 생성
(0,0) (2,2) (3,1)

MATERIAL PROPERTY

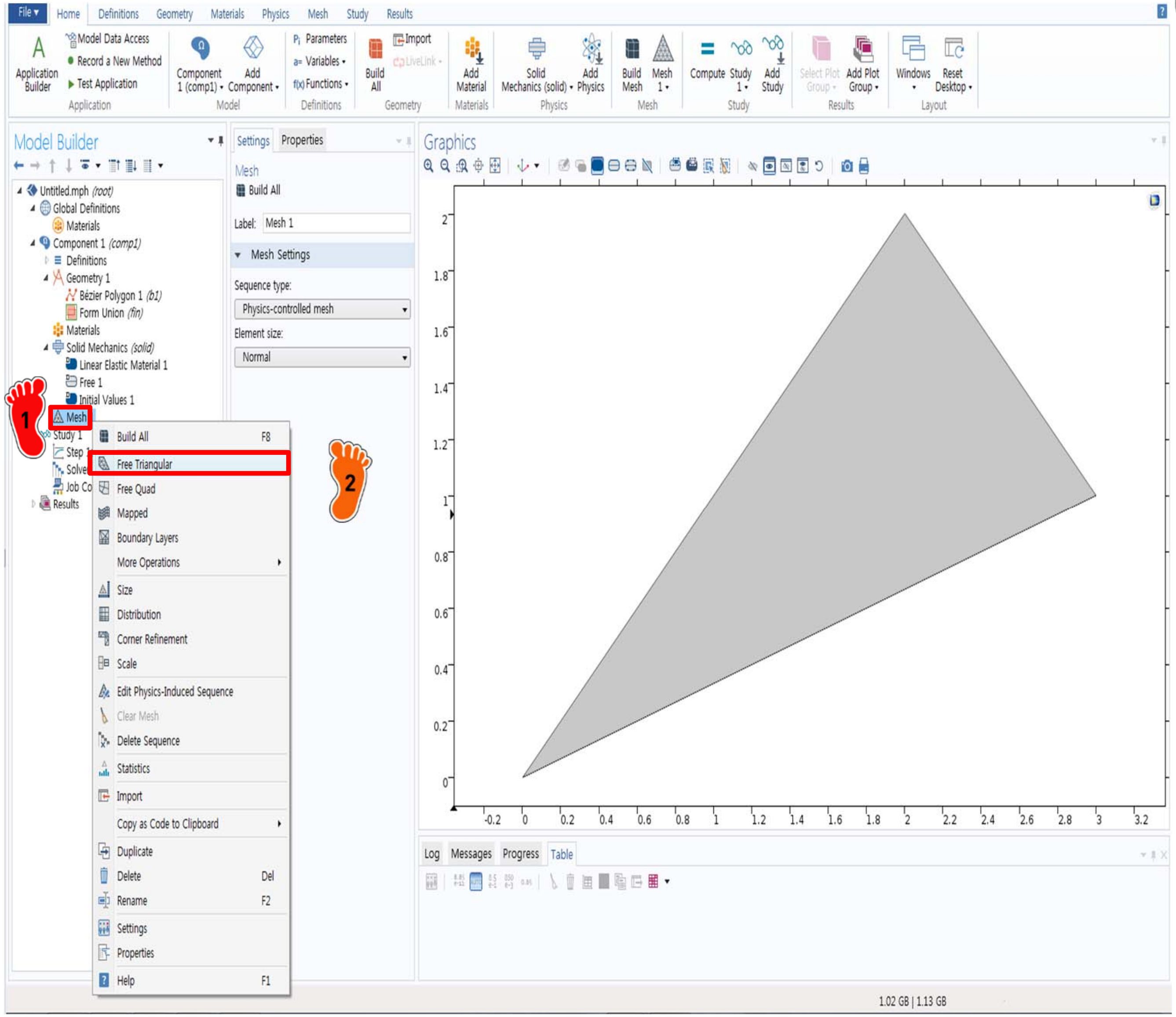


1 Linear Elastic Material 클릭

2 E: 60
mu: 0.25
rho : 0

입력

MESH



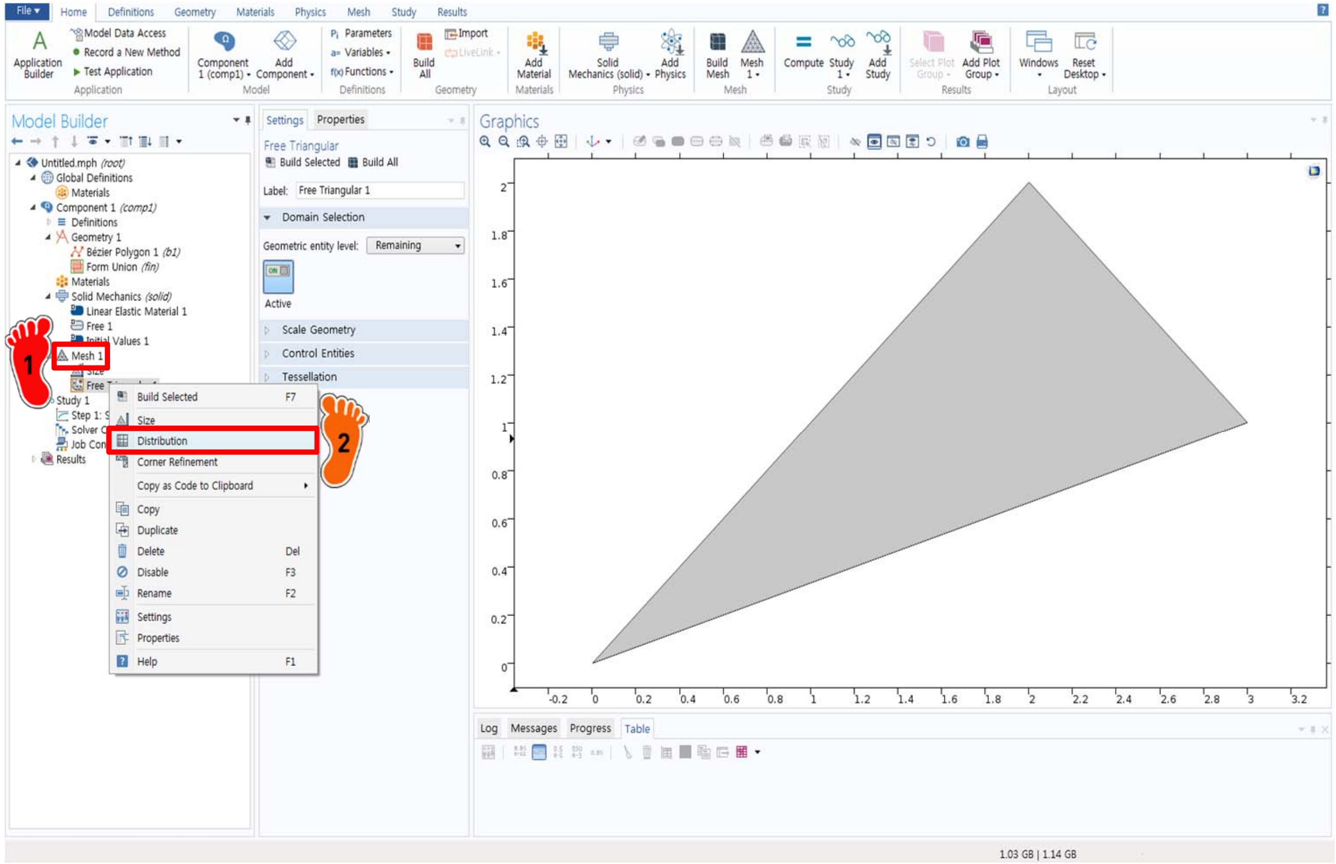
1 Mesh 마우스 우클릭

2 Free Triangular 클릭

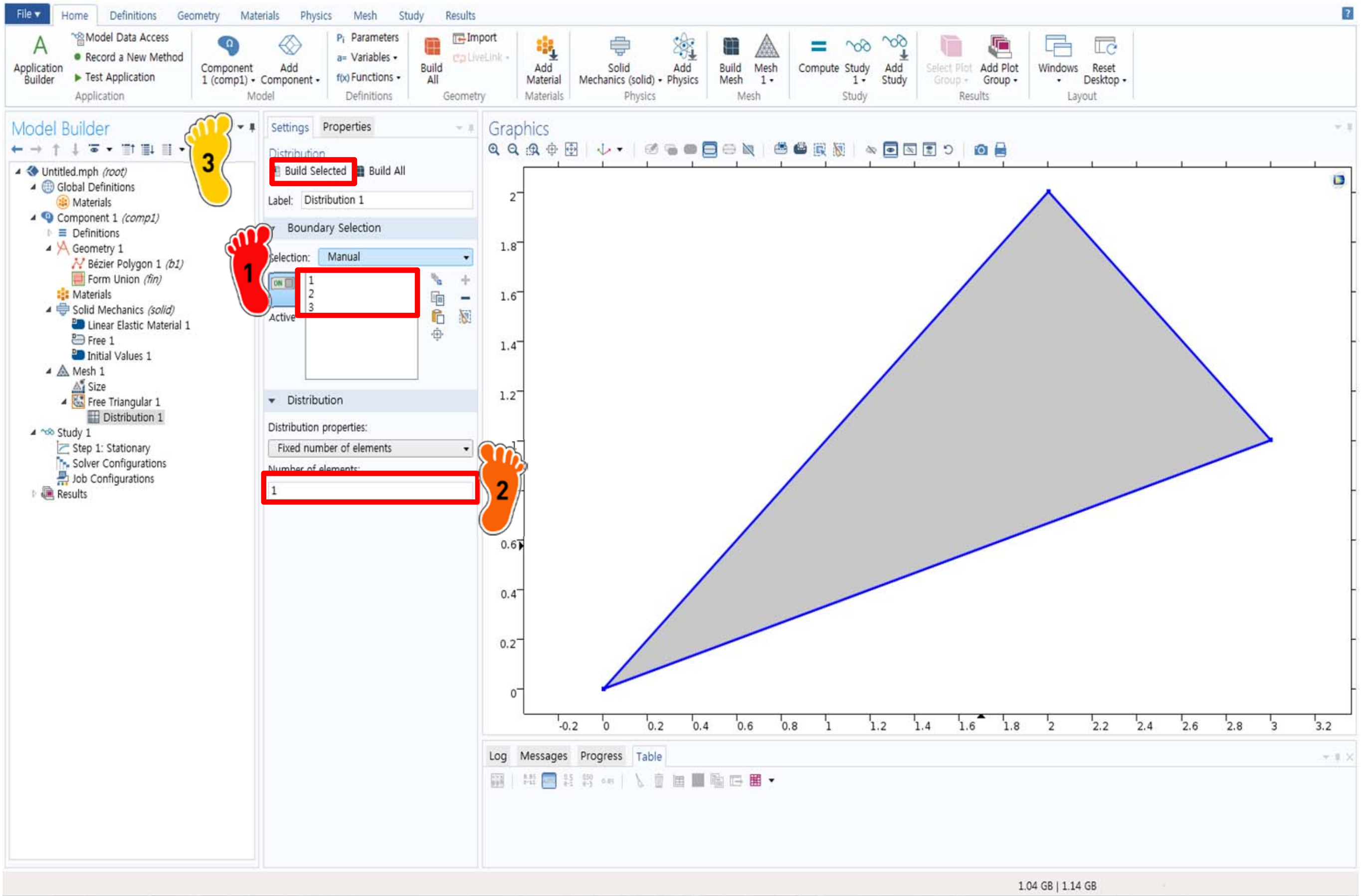
MESH

1 Free Triangular 마우스 우클릭

2 Distribution 클릭



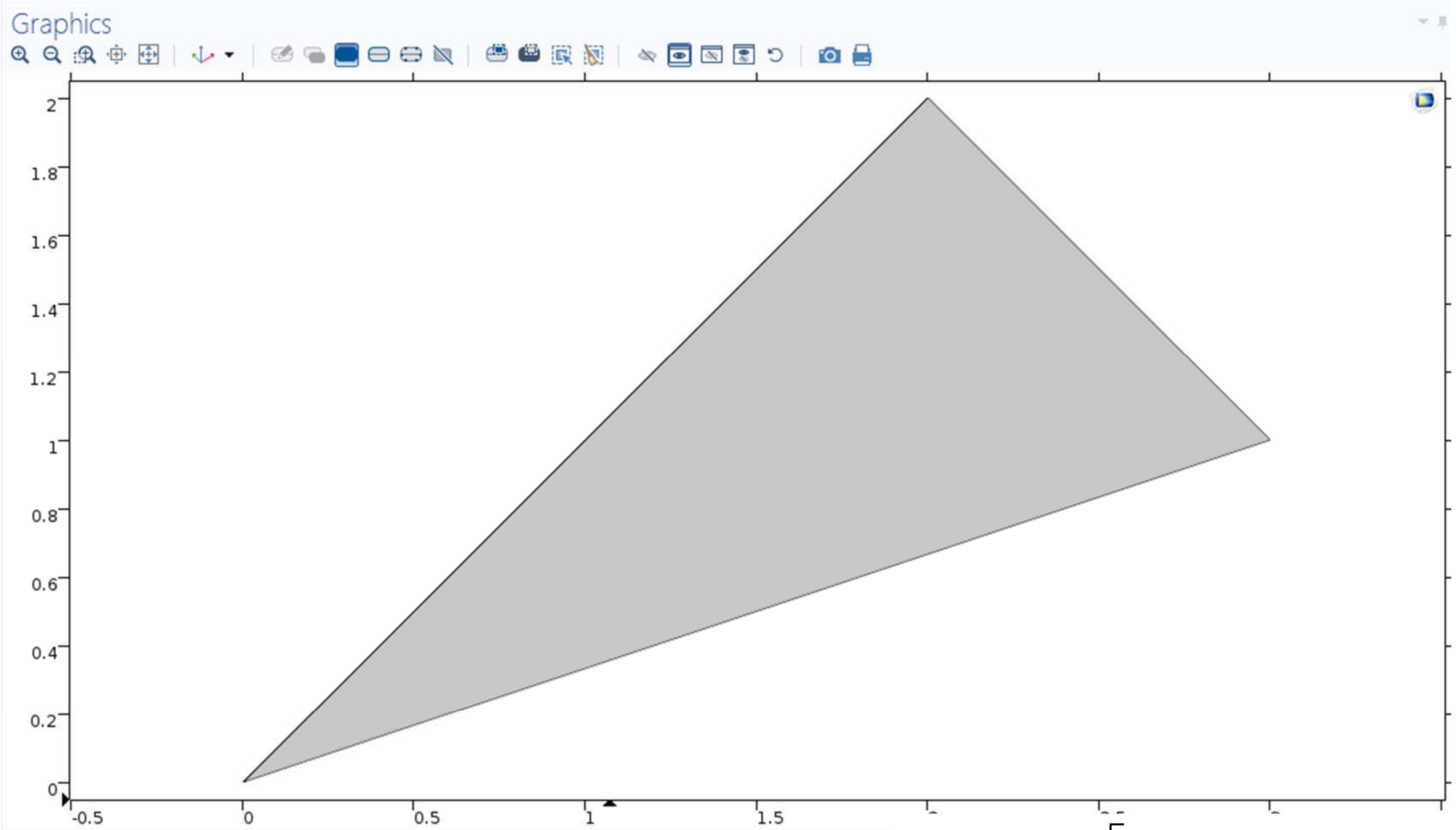
MESH

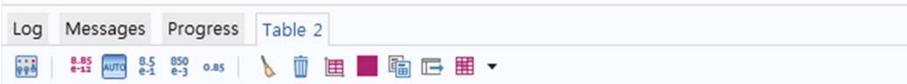


1. 3개의 boundary 선택
2. Number of elements를 1로 설정하여
3. 1개의 삼각형 요소 생성

해석 실행 후,
Assemble 탭에서 stiffness matrix를 선택하고 결과 확인

STIFFNESS MATRIX



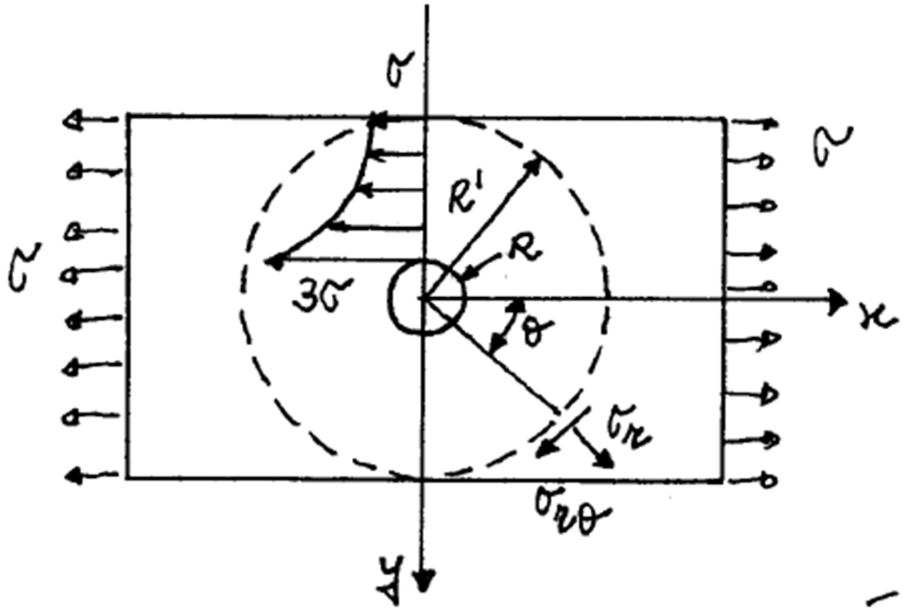


12.000	6.0000	-12.000	0.0000	0.0000	-6.0000
6.0000	12.000	0.0000	12.000	-6.0000	-24.000
-12.000	0.0000	48.000	-24.000	-36.000	24.000
0.0000	12.000	-24.000	48.000	24.000	-60.000
0.0000	-6.0000	-36.000	24.000	36.000	-18.000
-6.0000	-24.000	24.000	-60.000	-18.000	84.000

$$\mathbf{Ke}_{\text{plane strain}} = \begin{bmatrix} 12 & 6 & -12 & 0 & 0 & -6 \\ 6 & 12 & 0 & 12 & -6 & -24 \\ -12 & 0 & 48 & -24 & -36 & 24 \\ 0 & 12 & -24 & 48 & 24 & -60 \\ 0 & -6 & -36 & 24 & 36 & -18 \\ -6 & -24 & 24 & -60 & -18 & 84 \end{bmatrix}$$

KIRSCH'S PROBLEM: THEORY

Infinite plate containing a circular hole (Kirsh, G. (1898), V.D.I., 42, 797-807)



Consider portion of plate within concentric circle of radius $R' \gg R$ so that stress field is not perturbed by hole (Saint-Venant's Principle)

stress field at $r = R'$ (Mohr's cirle): $\begin{cases} \sigma_r = \dfrac{\sigma}{2}(1+\cos 2\theta) \\ \sigma_{r\theta} = -\dfrac{\sigma}{2}\sin 2\theta \end{cases}$

$$\text{solution: } \begin{cases} \sigma_r = \dfrac{\sigma}{2}\left(1-\dfrac{R^2}{r^2}\right)+\dfrac{\sigma}{2}\left(1+3\dfrac{R^4}{r^4}-4\dfrac{R^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta = \dfrac{\sigma}{2}\left(1+\dfrac{R^2}{r^2}\right)-\dfrac{\sigma}{2}\left(1+3\dfrac{R^4}{r^4}\right)\cos 2\theta \\ \sigma_{r\theta} = -\dfrac{\sigma}{2}\left(1-3\dfrac{R^4}{r^4}+2\dfrac{R^2}{r^2}\right)\sin 2\theta \end{cases}$$

for $r = R$:

$$\left.\begin{array}{l} \sigma_r = 0 \\ \sigma_\theta = \sigma(1-2\cos 2\theta) \\ \sigma_{r\theta} = 0 \end{array}\right\} \rightarrow \begin{cases} \text{max: } \sigma_\theta = 3\sigma \ @\ \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \\ \text{min: } \sigma_\theta = -\sigma \ @\ \theta = 0, \pi \end{cases}$$

stress concentration factor $= 3$ independent of R

solution applicable to finite plates with width $> 4R$

- **Stiffness matrix of Turner triangle**

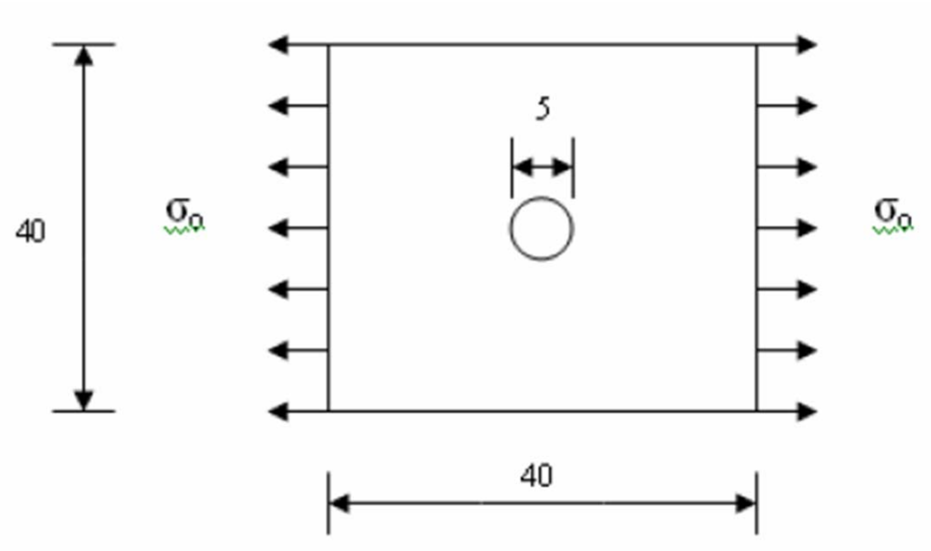
- **2D plane stress/strain model**

  - ✓ **Kirsh’s problem**

- **Thick plate**

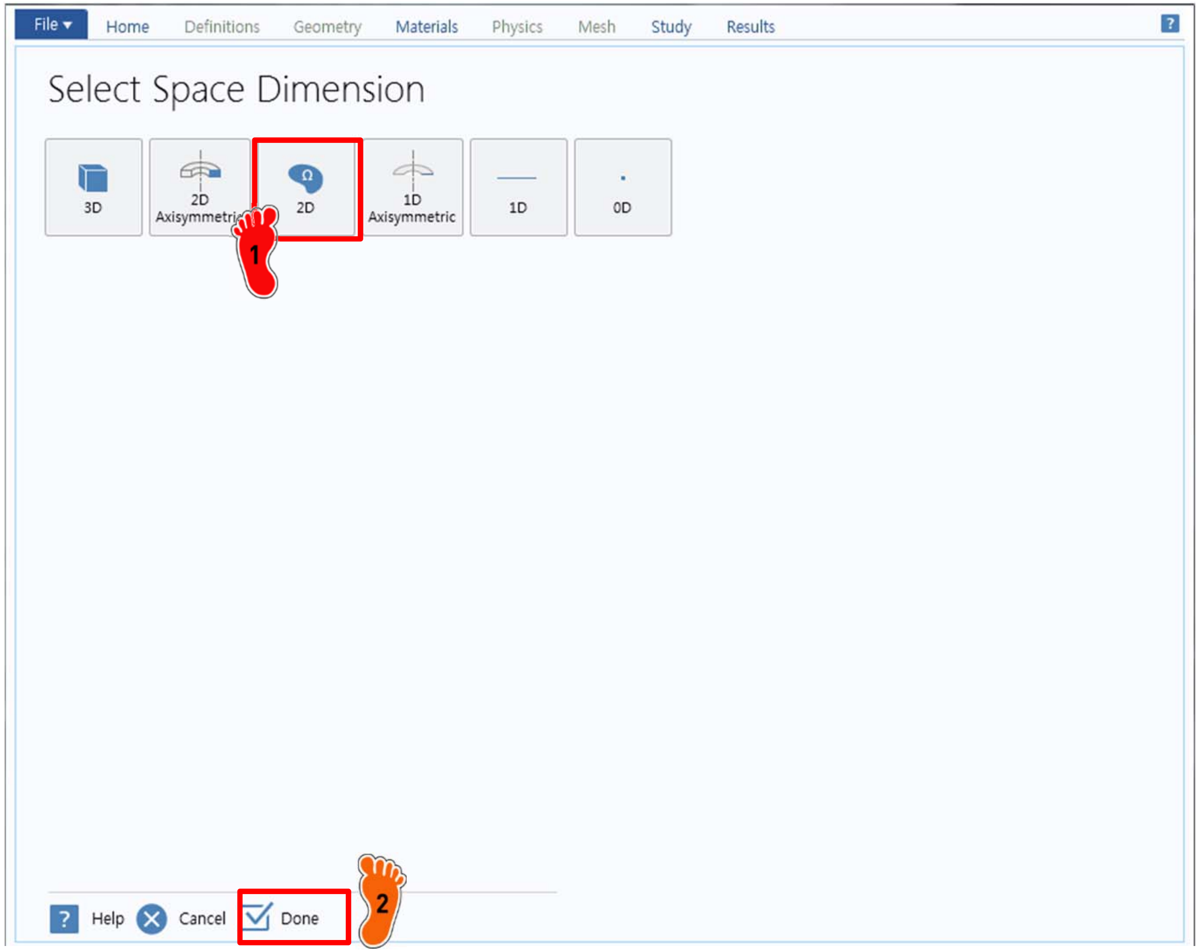
  - ✓ **Benchmark problem LE10**

KIRSCH'S PROBLEM: FEM



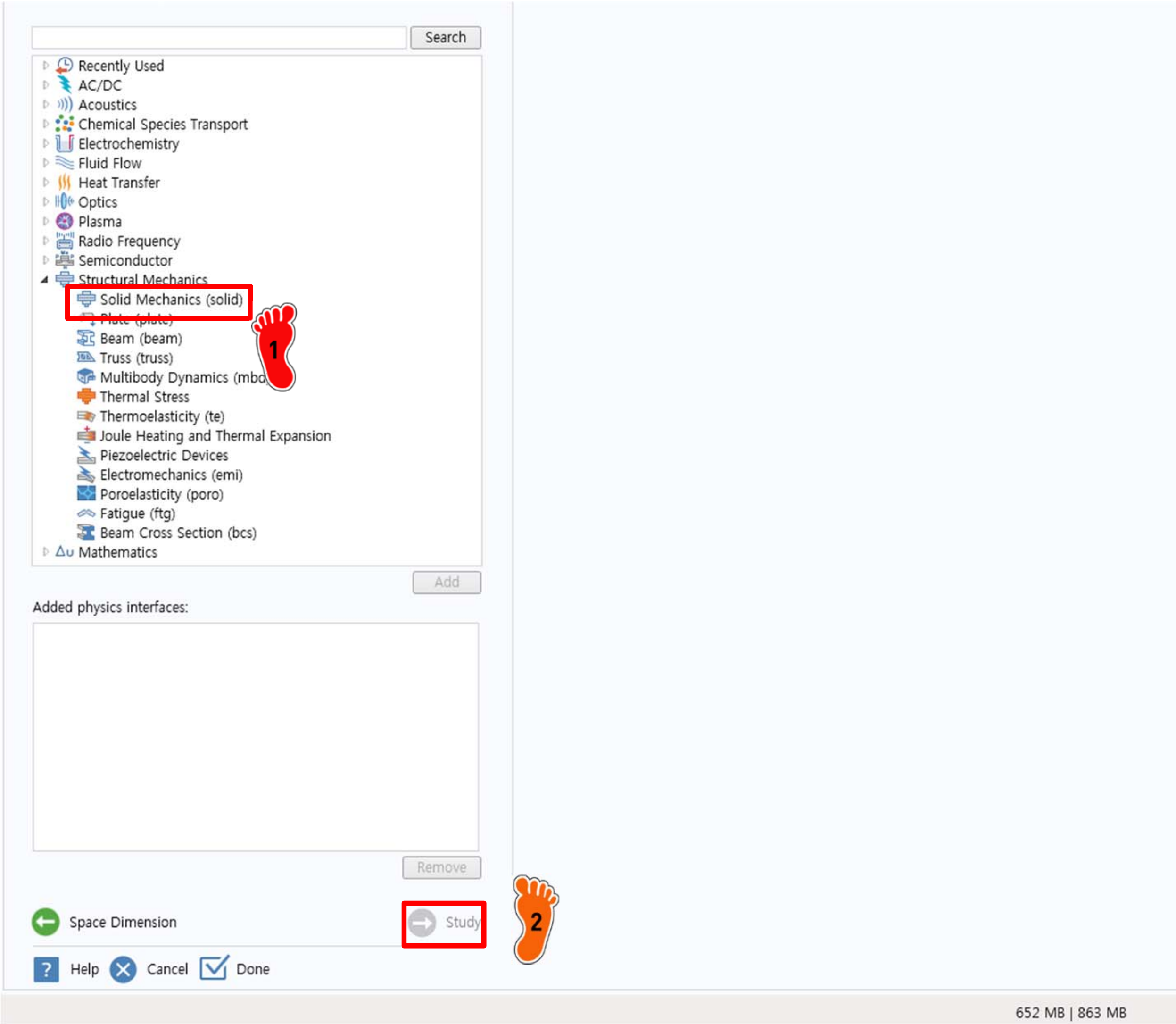
- 2D approximation
  - Plane stress
  - Plane stain
- Material Properties
  - E = 200 x 10^9
  - ν = 0.3
- Element Properties
  - Thickness = 1?
- Loads: $\sigma_0 = 1$
- BCs: none

DIMENSION SELECTION



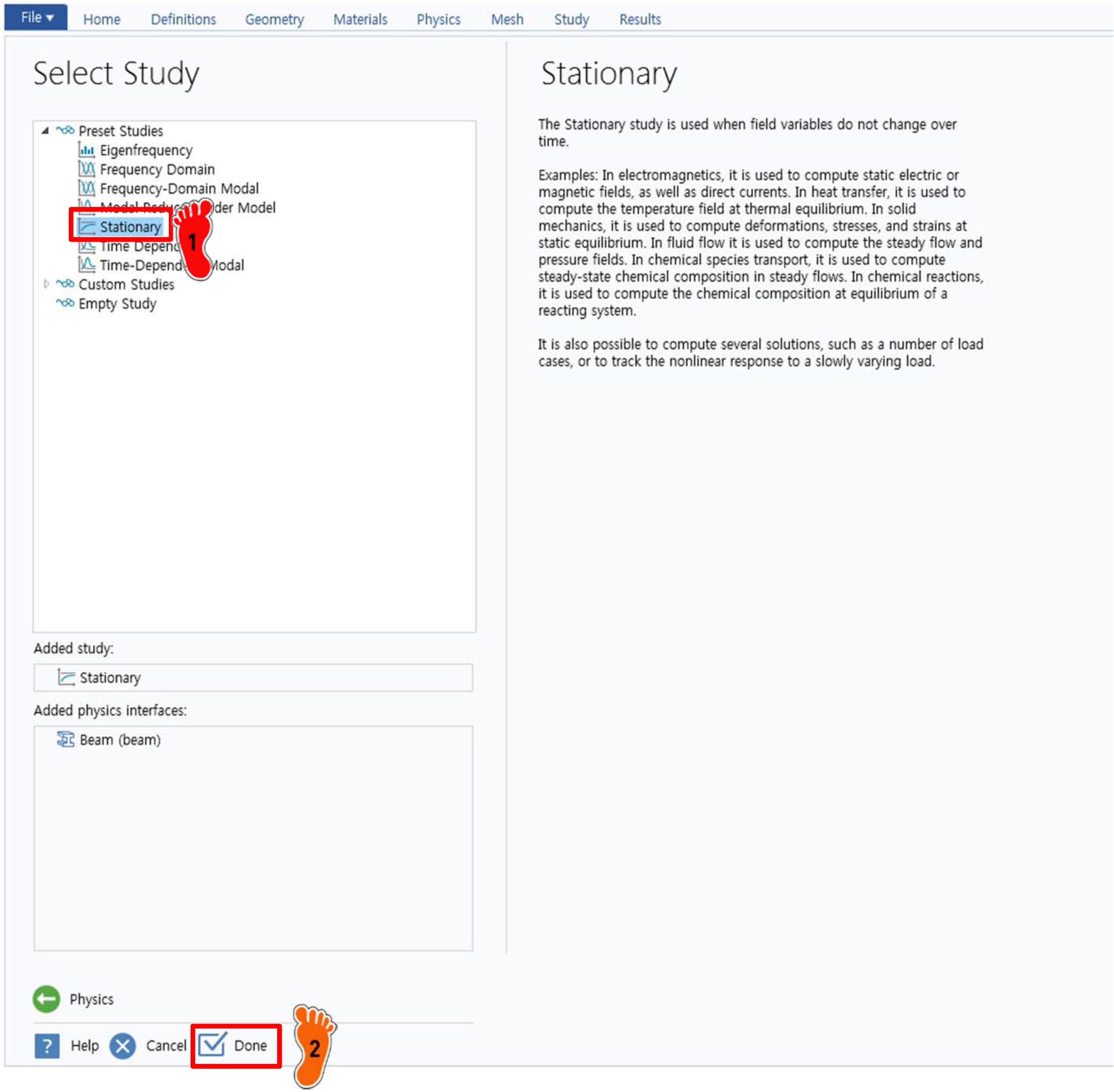
1. 2D 선택
2. Done 클릭

PHYSICS SELECTION



1. Structural Mechanics 의 Solid Mechanics 추가
2. Study 클릭

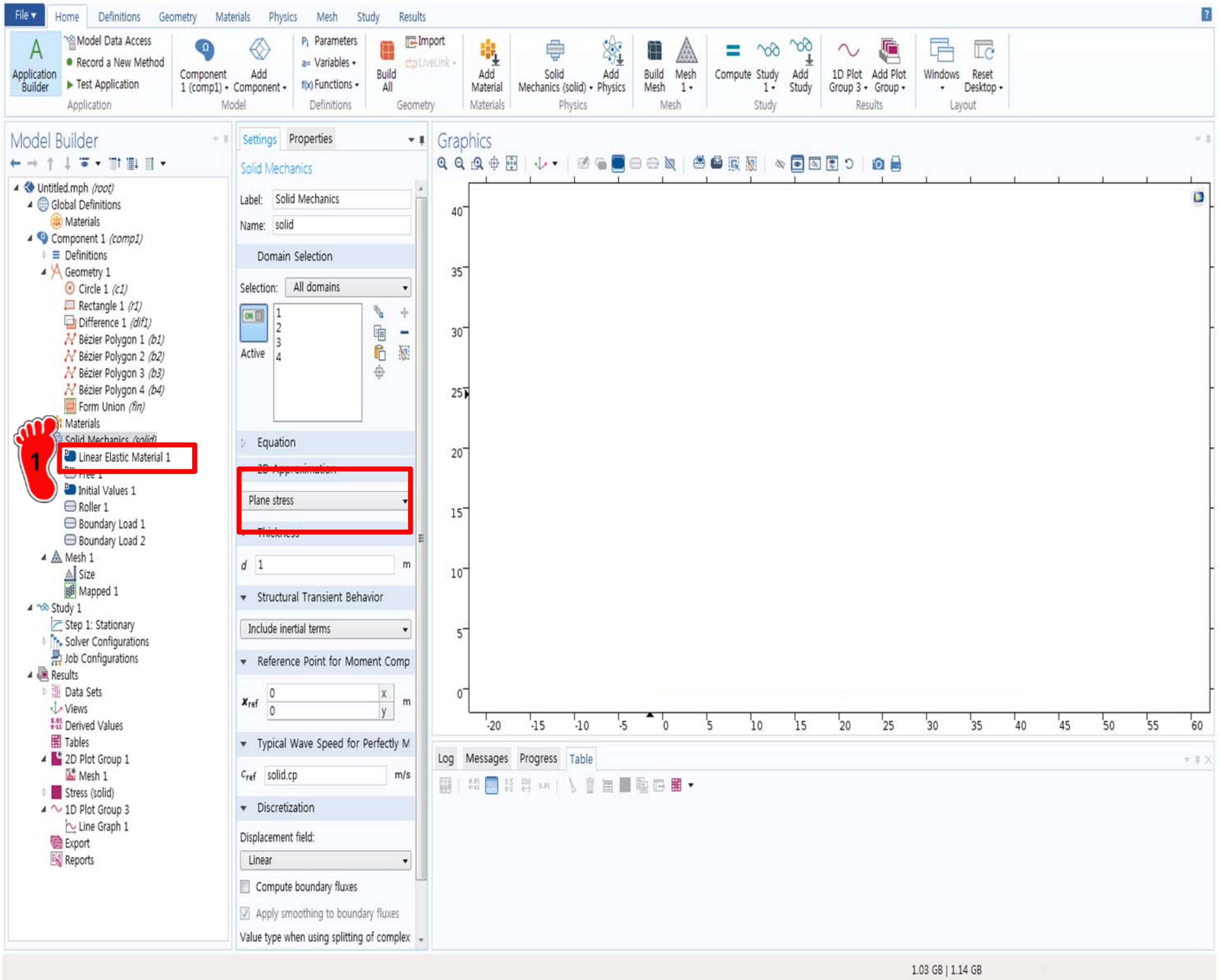
STUDY TYPE SELECTION



1 Stationary 선택

2 Done 클릭

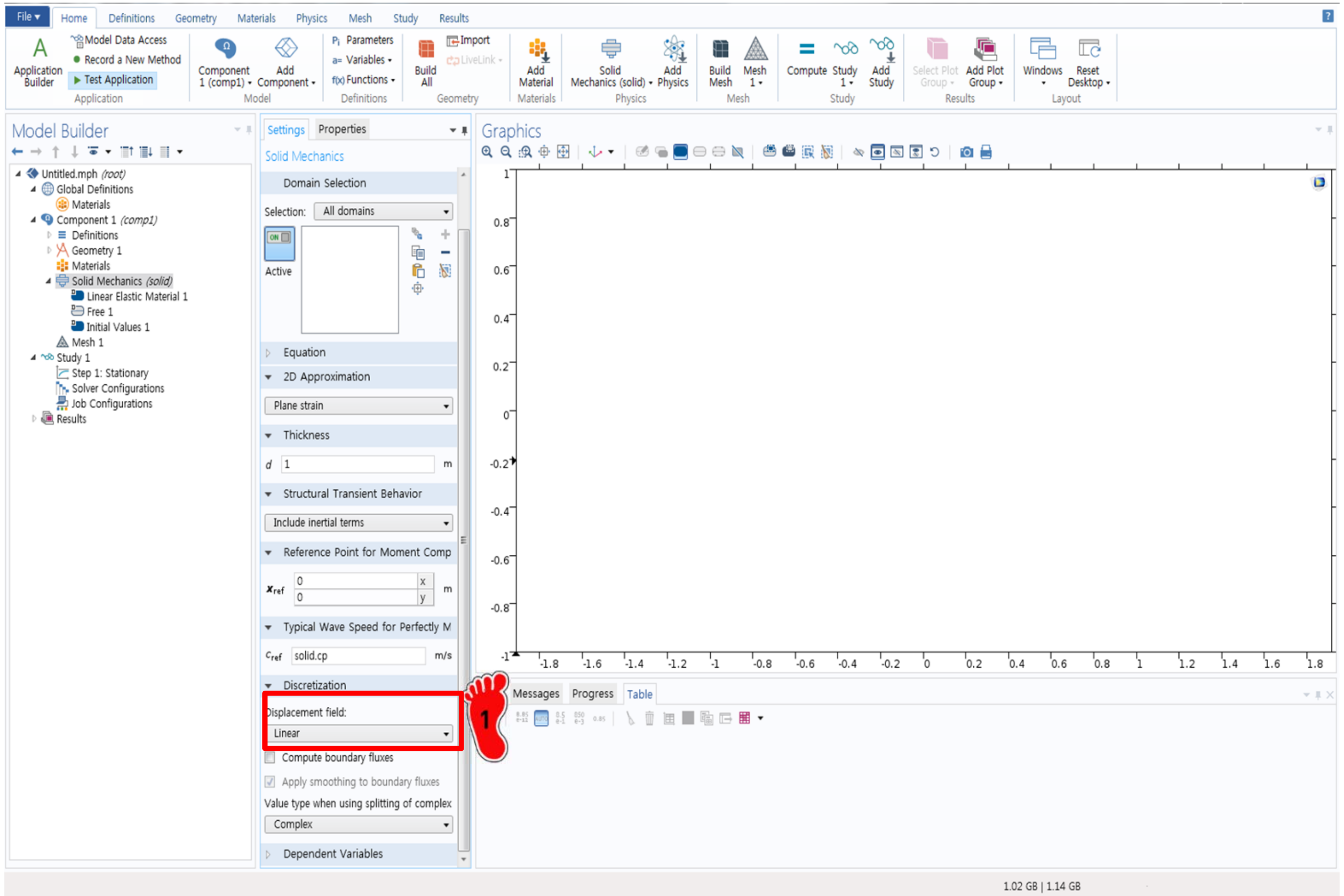
PLANE STRESS



1 Solid Mechanics 클릭

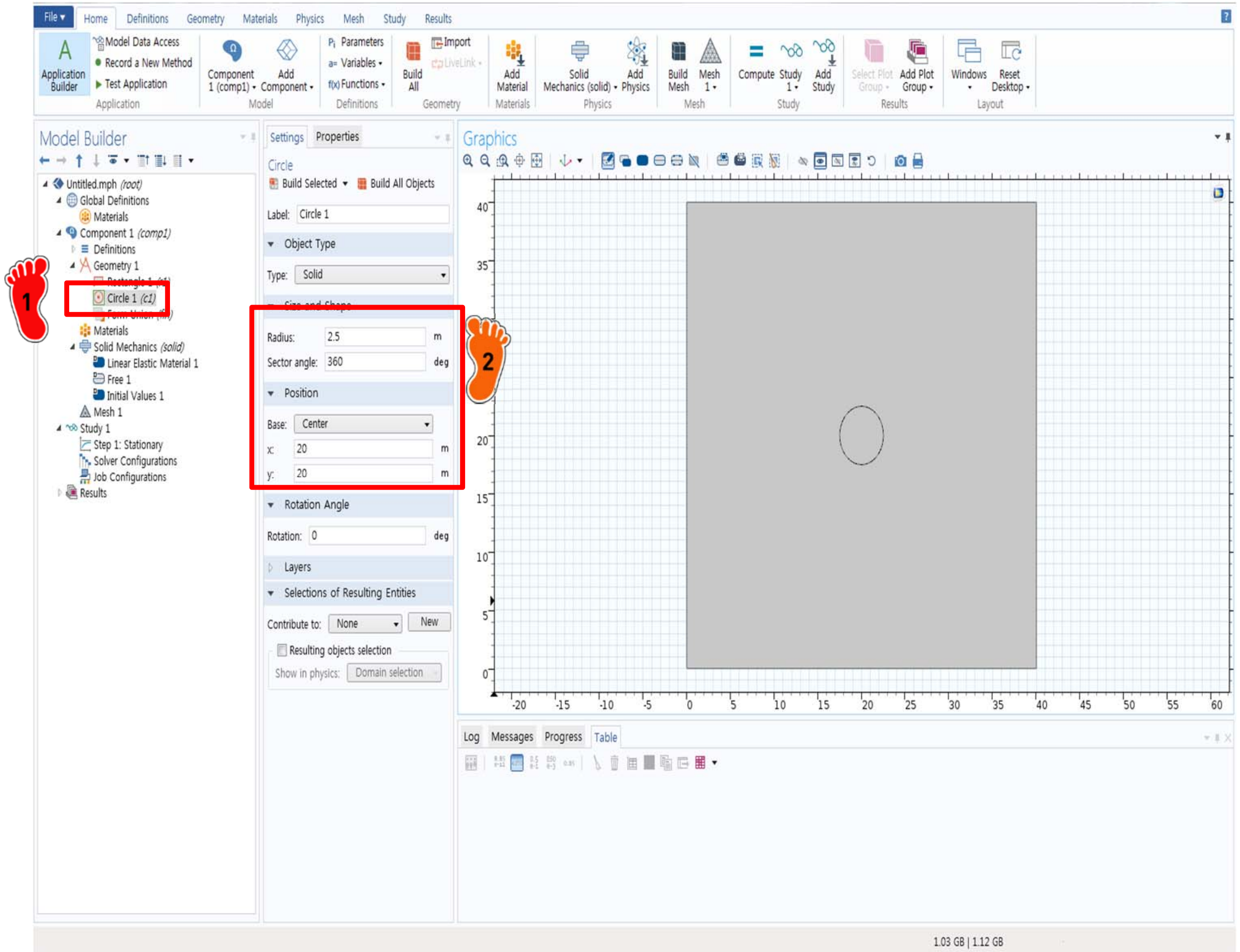
2 2D Approximation
Plane stress 로 변경

DISCRETIZATION



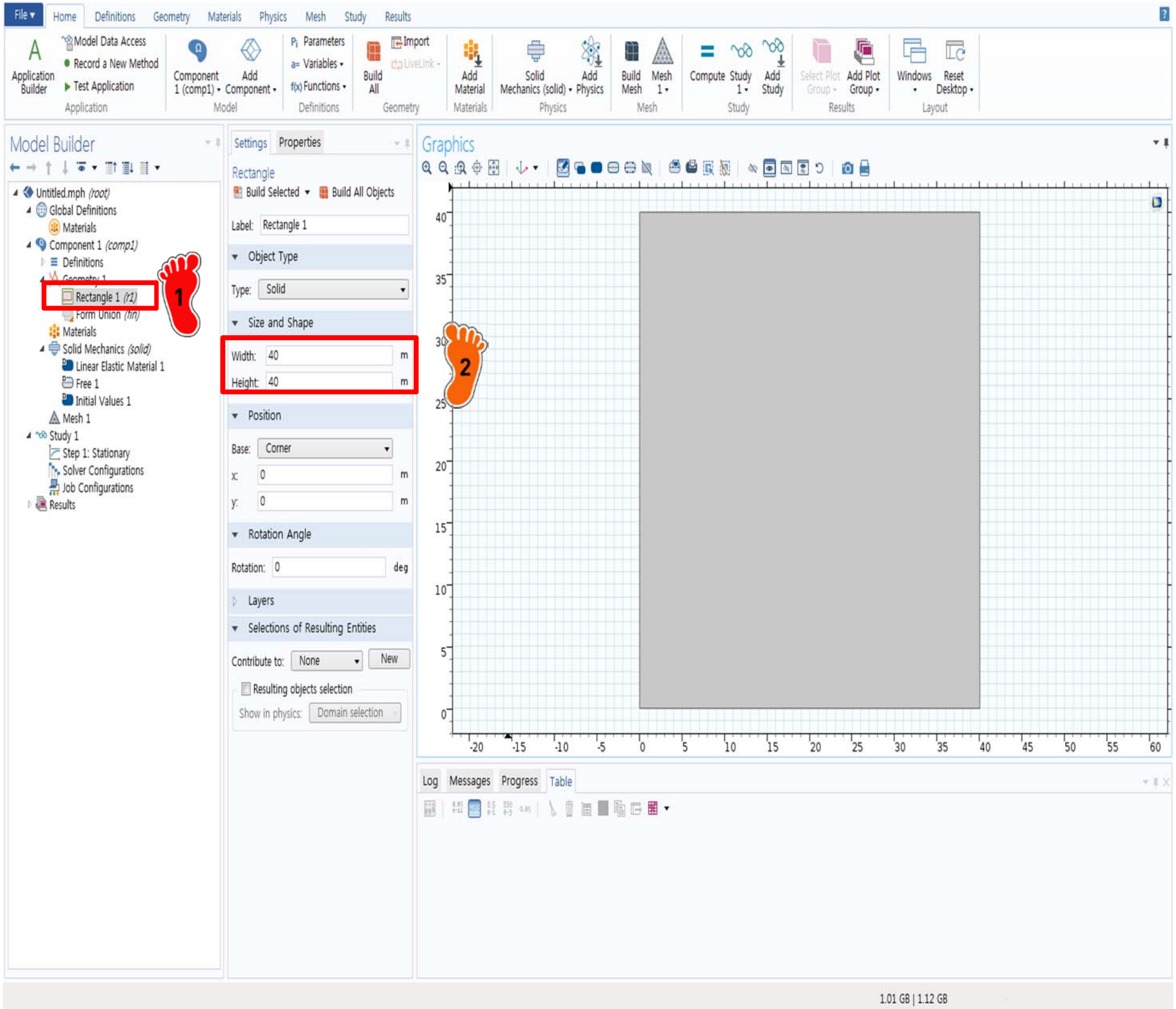
1 Discretization
Displacement field 를
Linear 로 변경

GEOMETRY CREATION



1. Circle 생성
2. 중심 (20,20)
   반지름 2.5
   인 원 생성

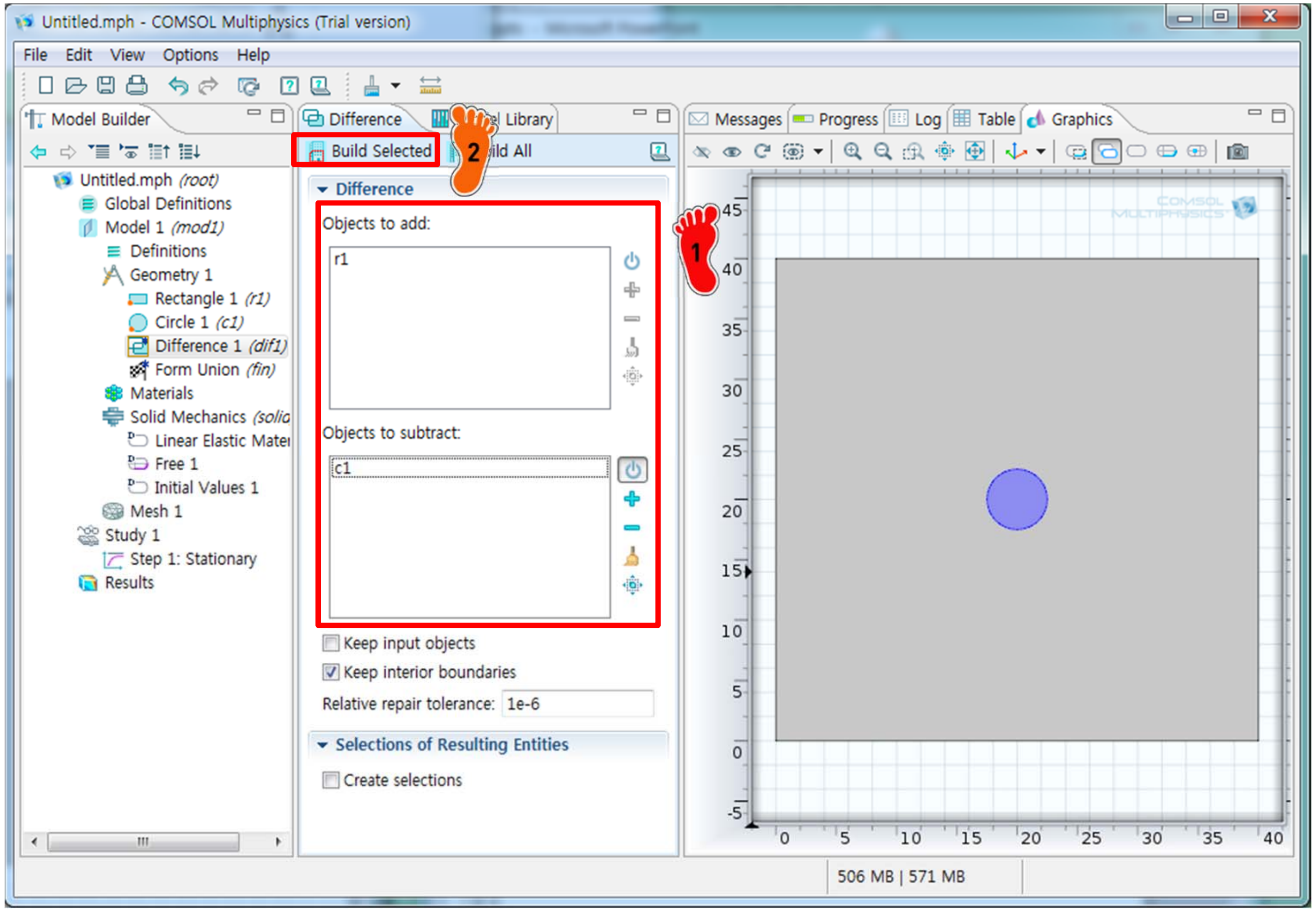
GEOMETRY CREATION



1 Rectangle 생성

2 가로 세로 40 길이의 사각형 생성

GEOMETRY CREATION



1 Objects to add
사각형 선택

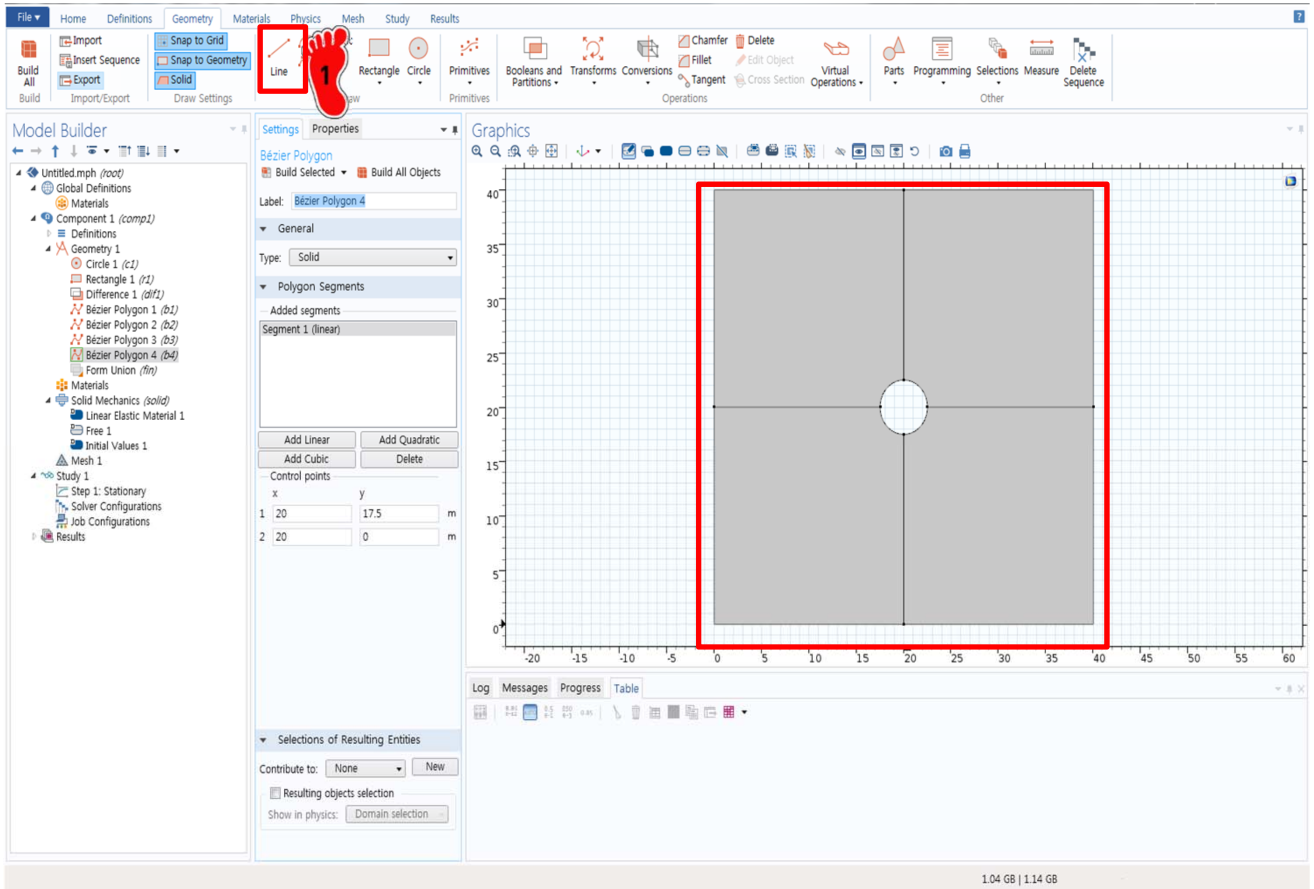
Objects to subtract
원 선택

2 Build Selected 클릭

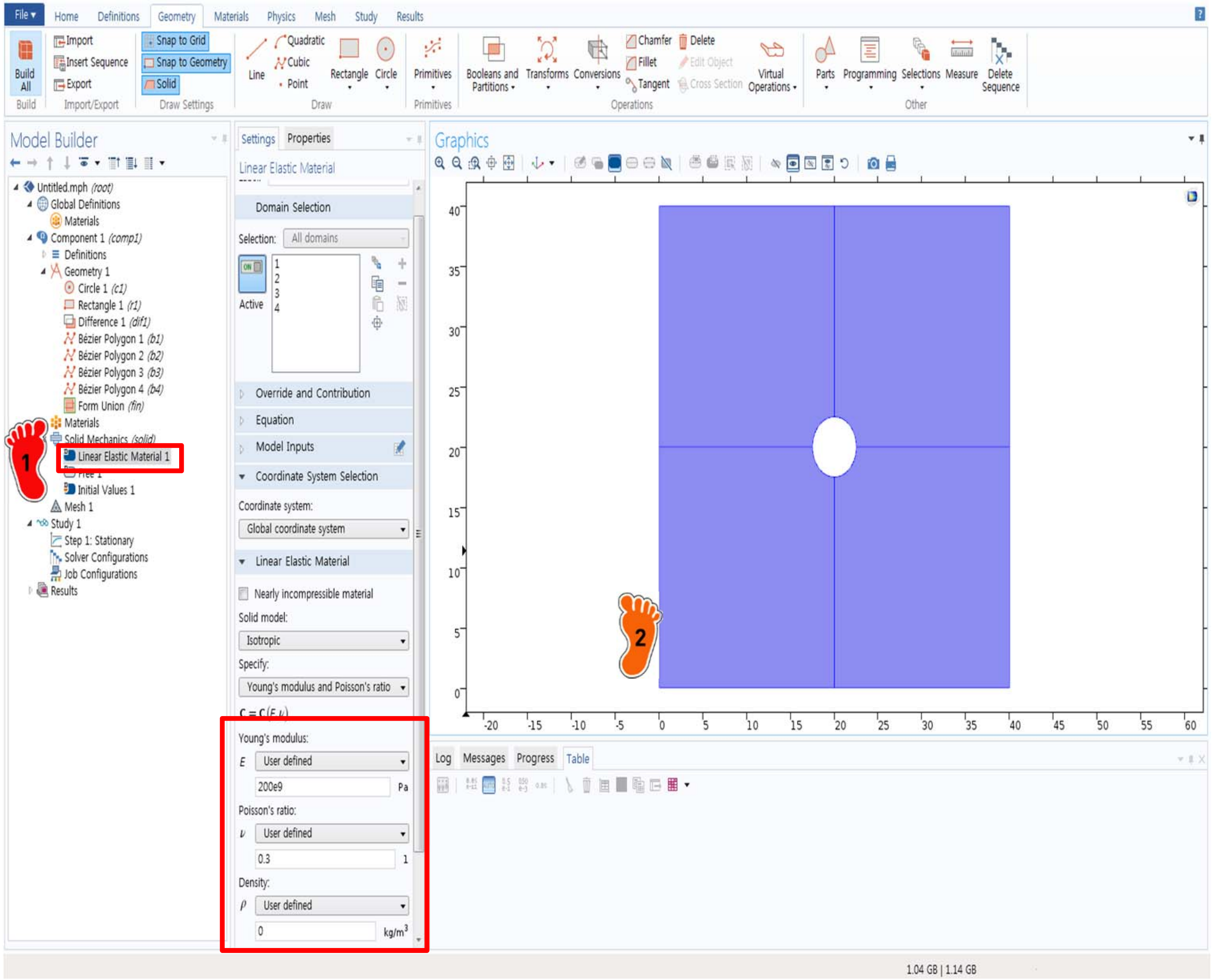
GEOMETRY CREATION

1 Draw line 메뉴를 이용

마우스 클릭으로 선분 4개 생성



MATERIAL PROPERTY

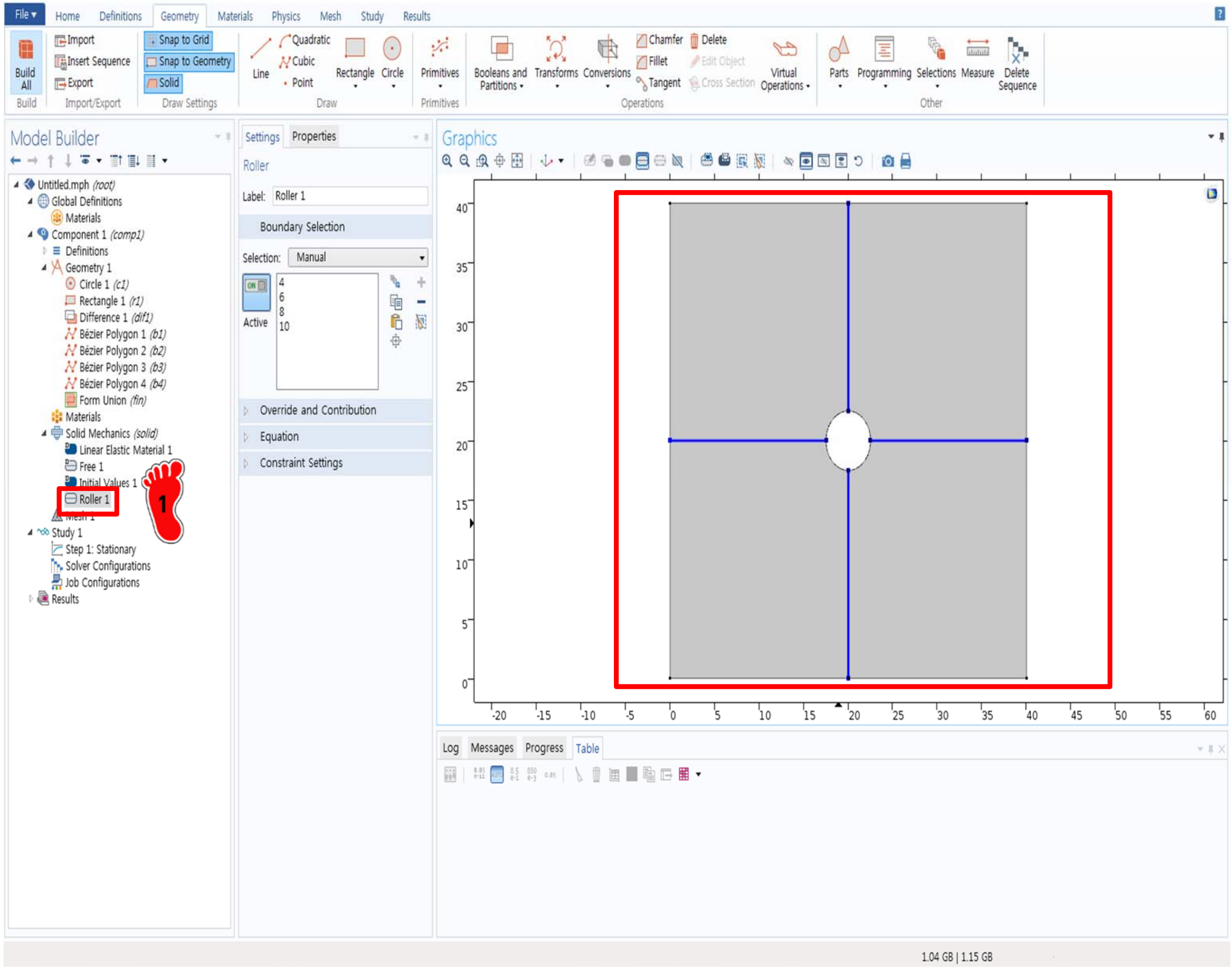


1 Linear Elastic Material 클릭

2 E: 200e9
mu: 0.3
rho : 0

입력

BOUNDARY CONDITION



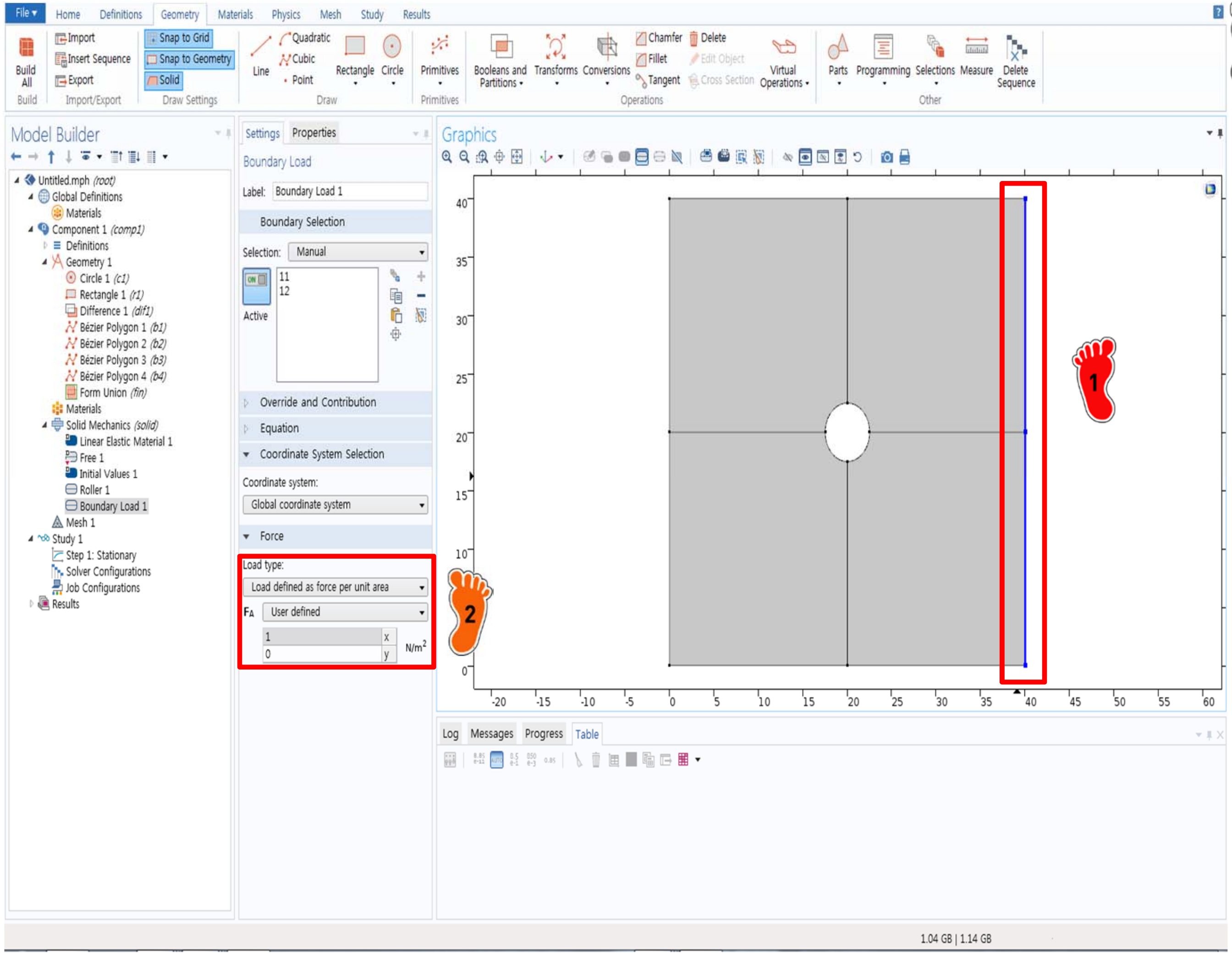
1 사각형 내부 4개 변을
Roller 조건으로 입력

LOADING CONDITION

1 제일 오른쪽 변 두 개 선택

2 x 방향 1 입력

단위는 N/m^2 을 유념

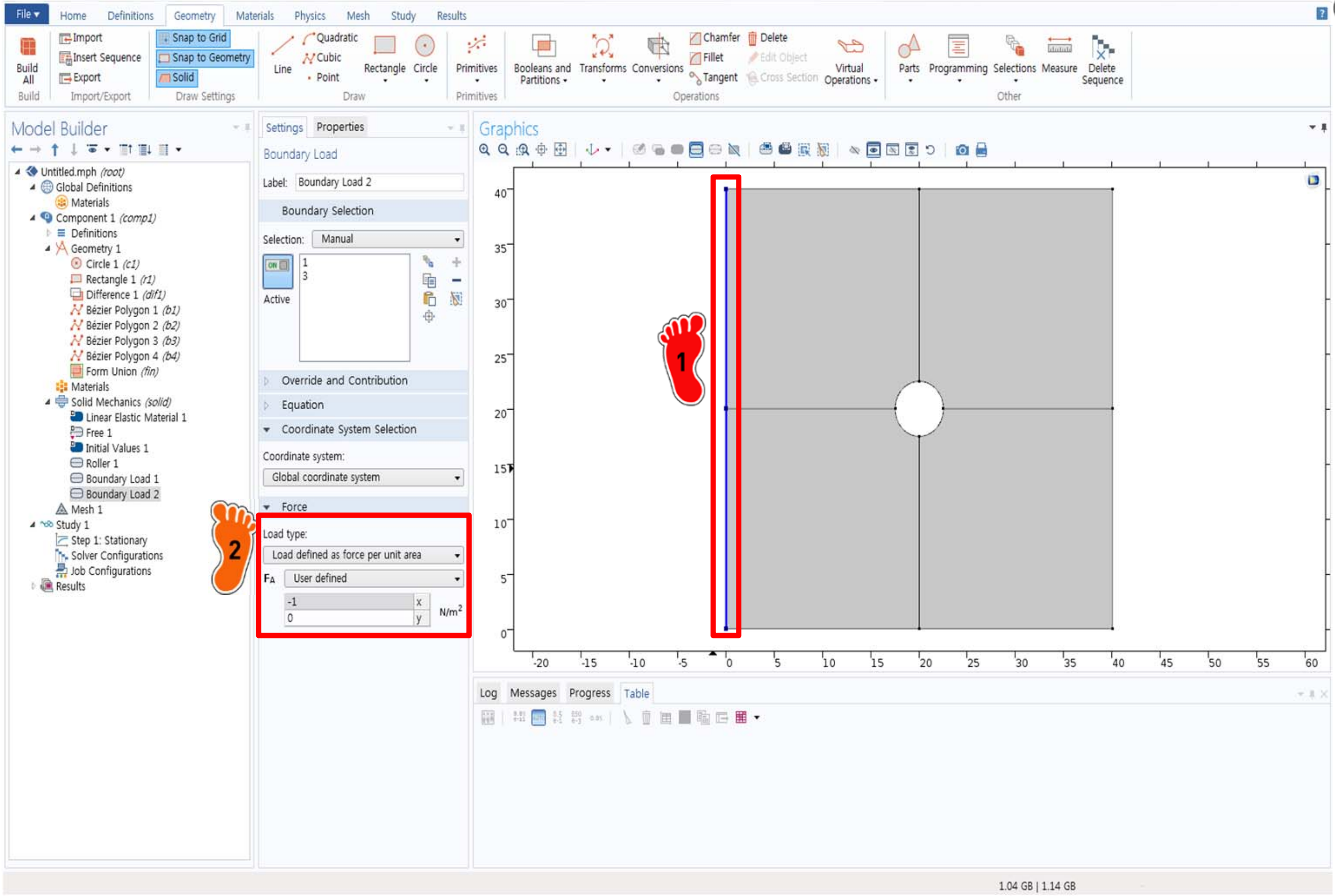


LOADING CONDITION

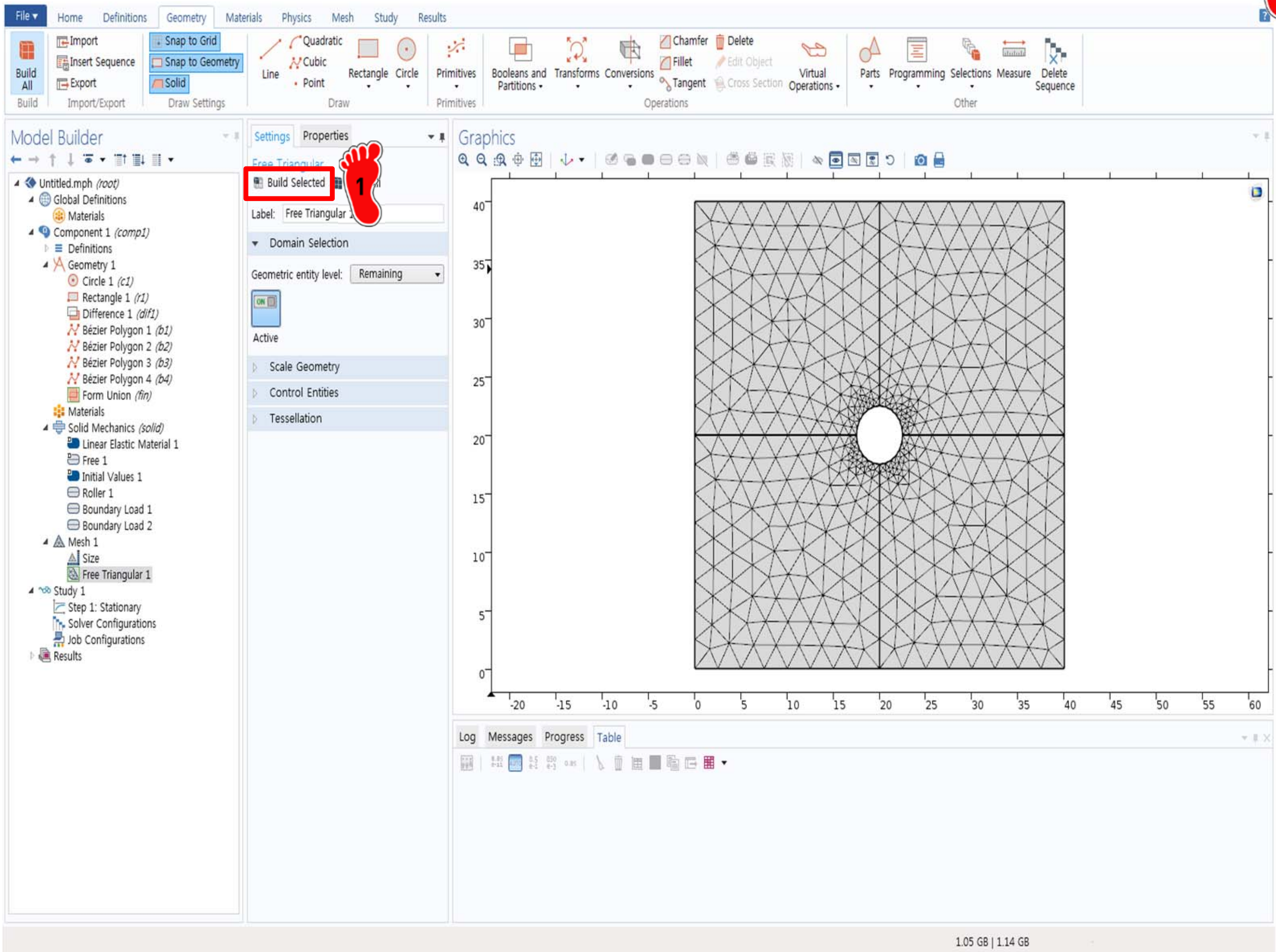
1 제일 왼쪽 변 두 개 선택

2 x 방향 -1 입력

단위는 N/m^2 을 유념



MESH



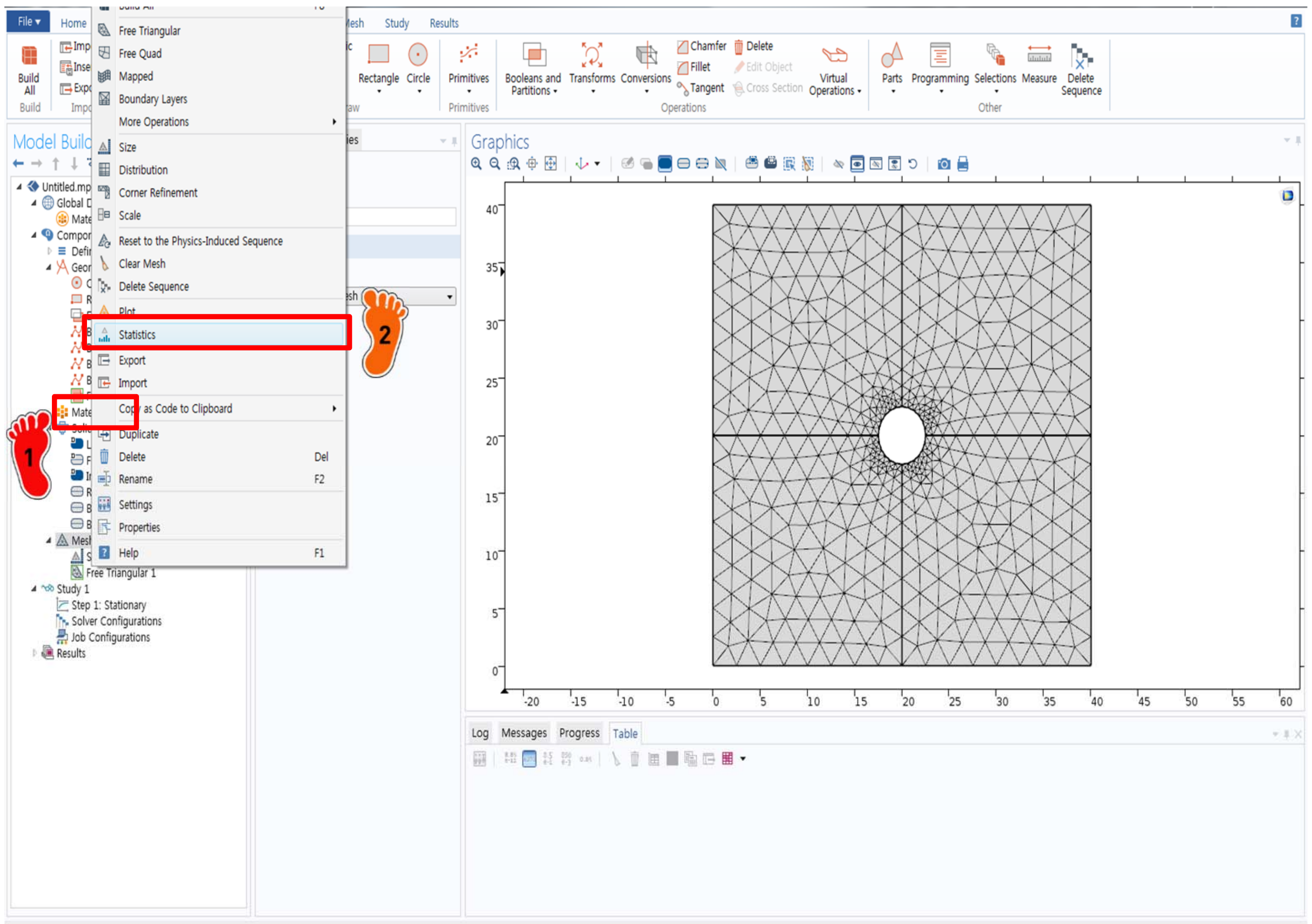
1 Build Selected 클릭

Mesh 생성

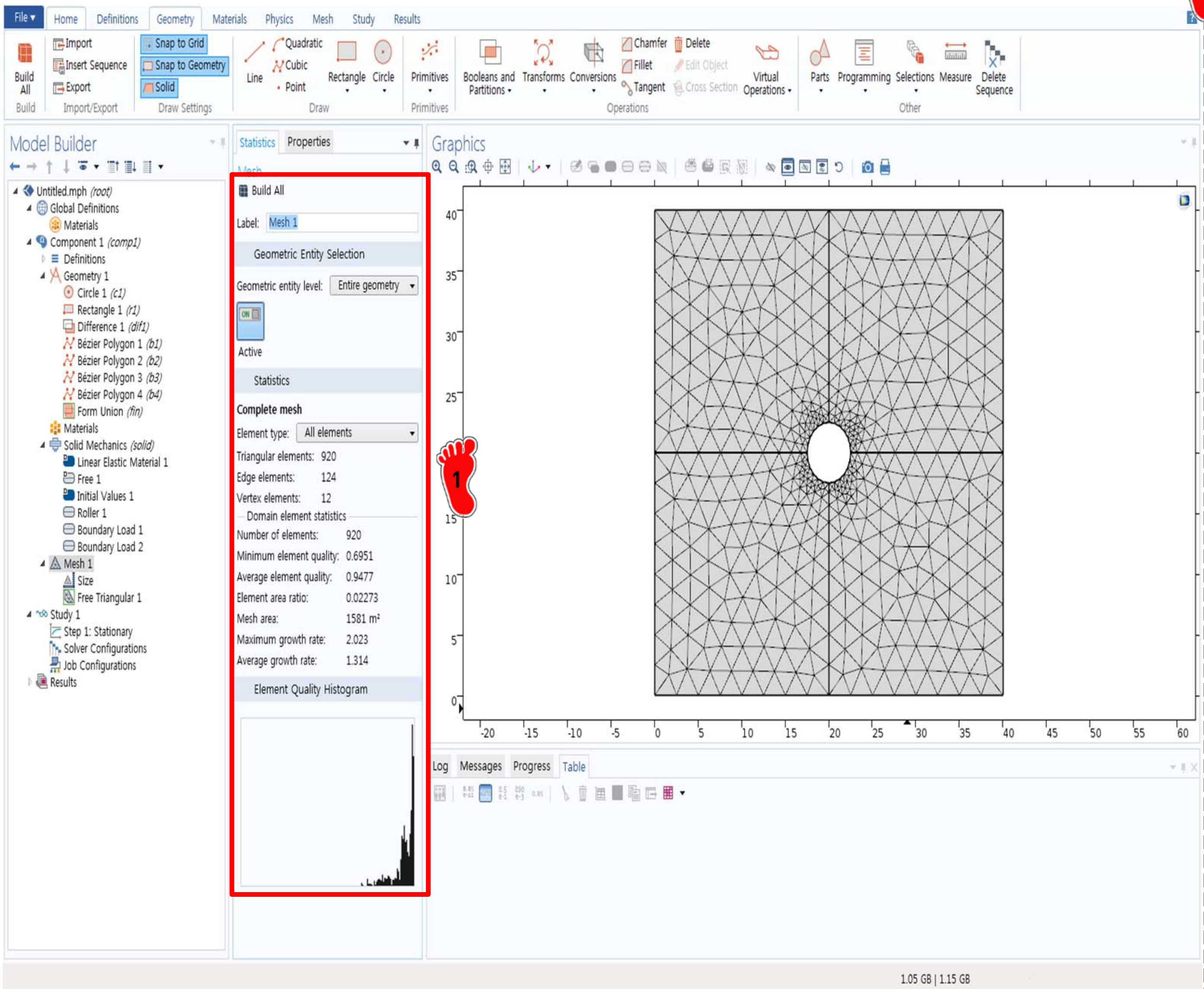
MESH

1 Mesh 마우스 우클릭

2 Statistics 클릭



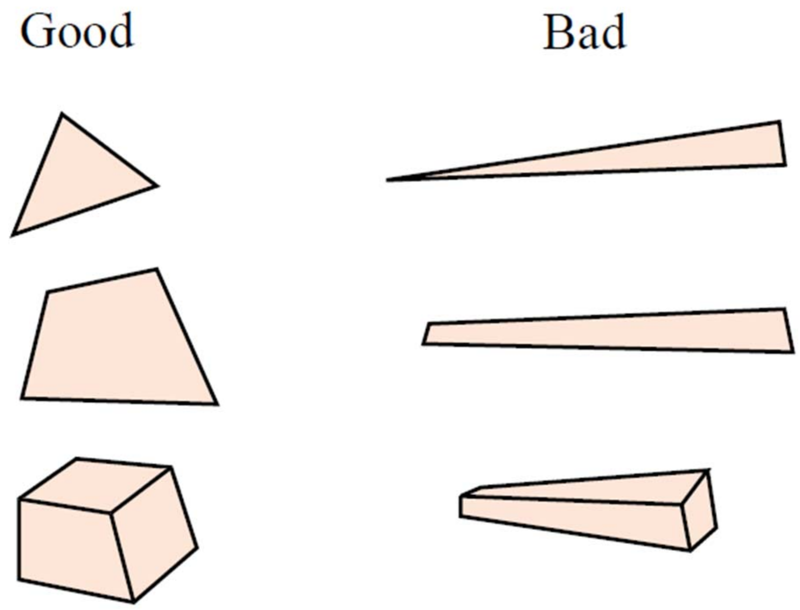
MESH



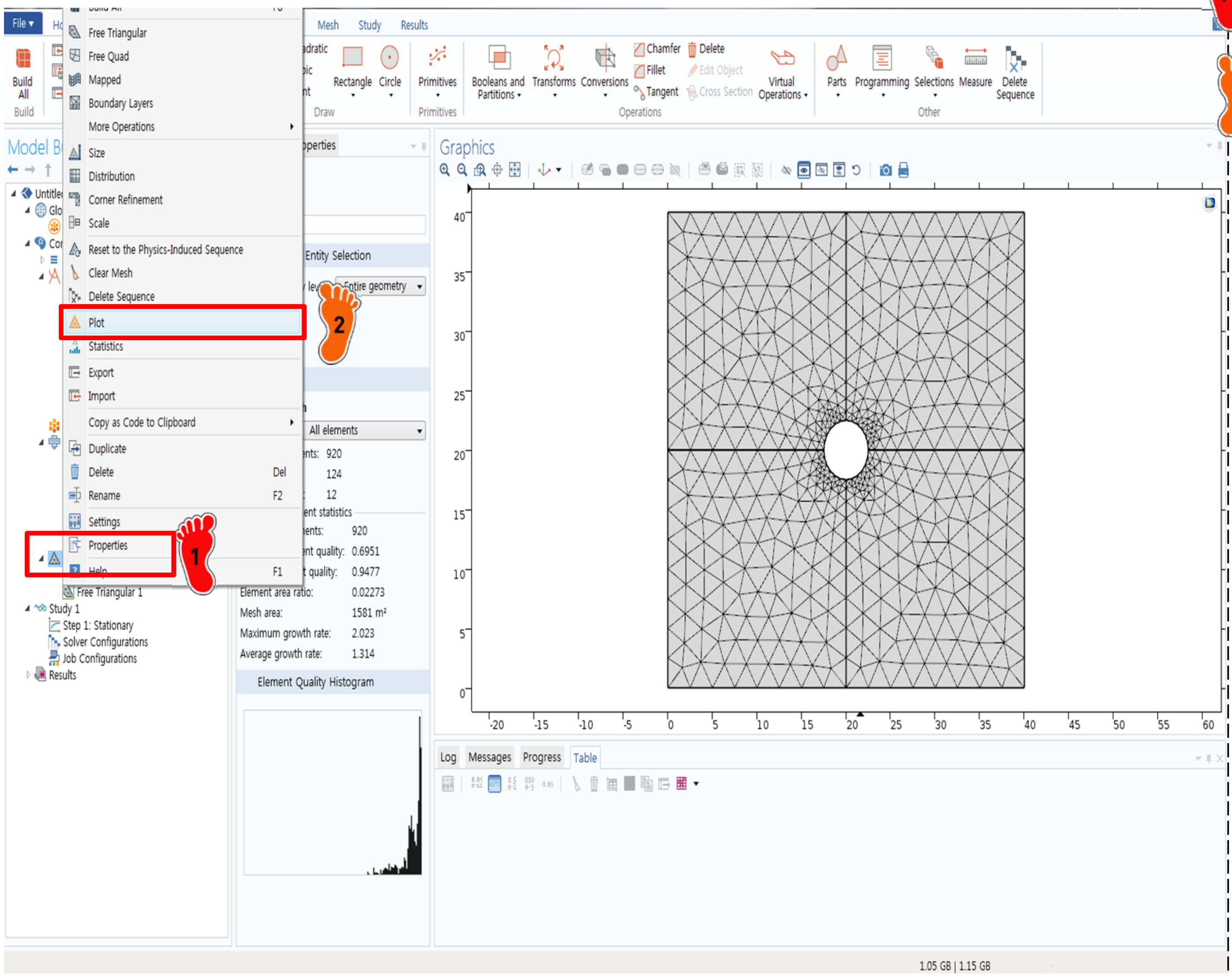
1 요소 개수, DOF, Quality 분포 정보 등을 자세히 나타내 줌

2D/3D BAD ASPECT RATIO ELEMENT

- “thin” structures modeled as continuous bodies
  - Elongated or “skinny” element
- Aspect ratio
  - Ratio between its largest and smallest dimension
  - > 3: caution
  - > 10: alarm



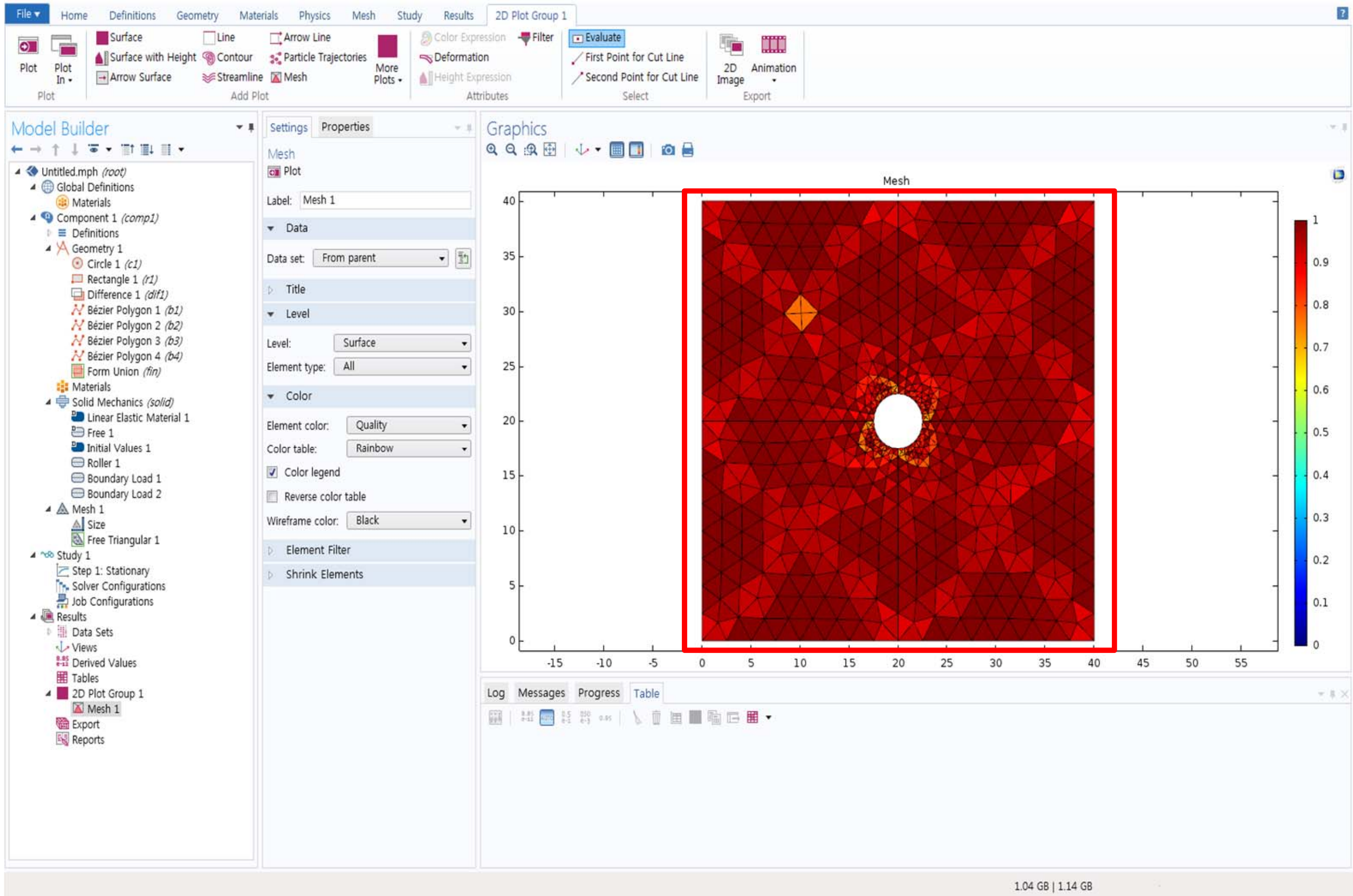
MESH

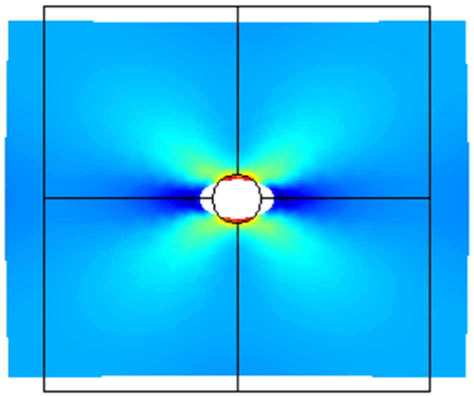


1 Mesh 마우스 우클릭

2 Plot 클릭

MESH



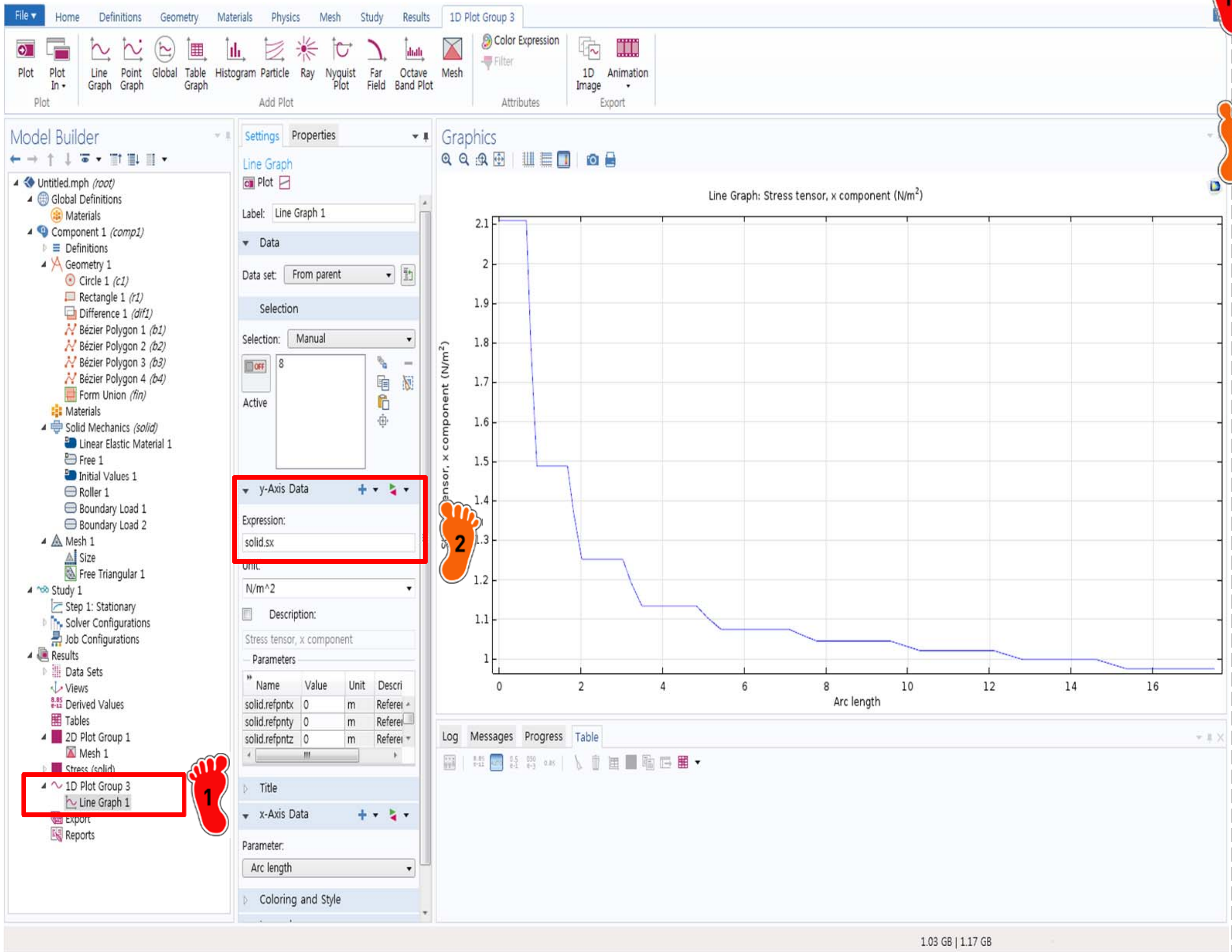


1 빨간색을 가지는 요소는 정삼각형에 가까운 요소

파란색에 가까운 요소일수록 aspect ratio 가 높은 요소

요소의 quality 를 확인 후 compute를 클릭하여 유한요소해석을 수행

POST-PROCESSING



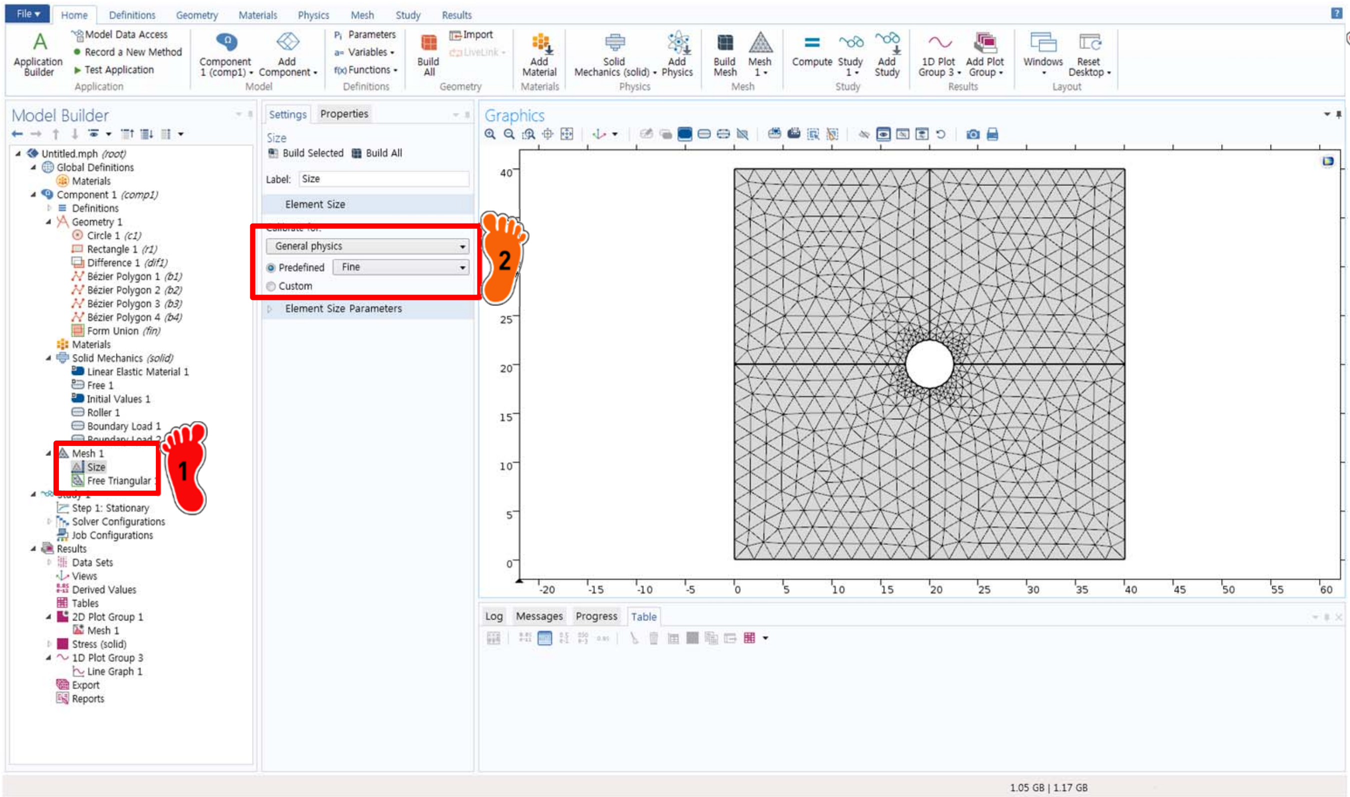
1D Plot Group 추가
Line Graph 추가

기하형상 중 8번 경계 선택

Expression
solid.sx
변경 후 plot

x 방향 응력은 2.5 Pa

MESH REFINEMENT

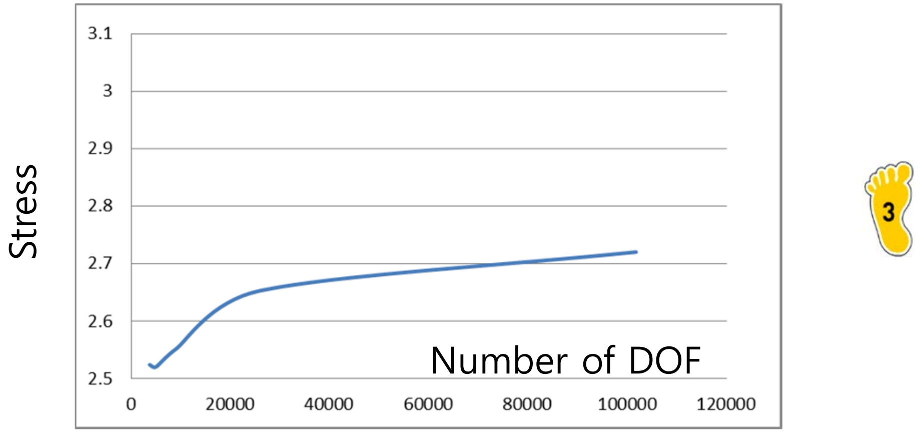


Free Triangular 의 Size 클릭

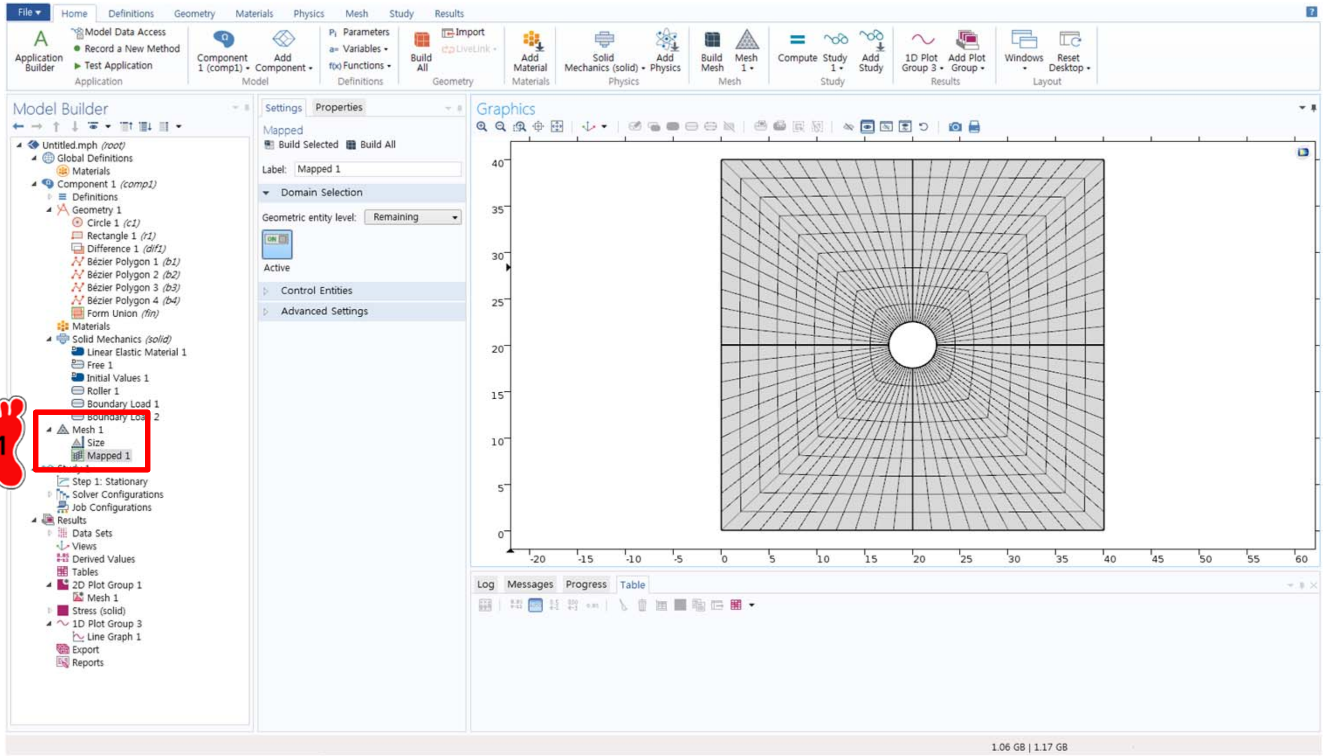
Fine 으로 변경 후 해석 수행

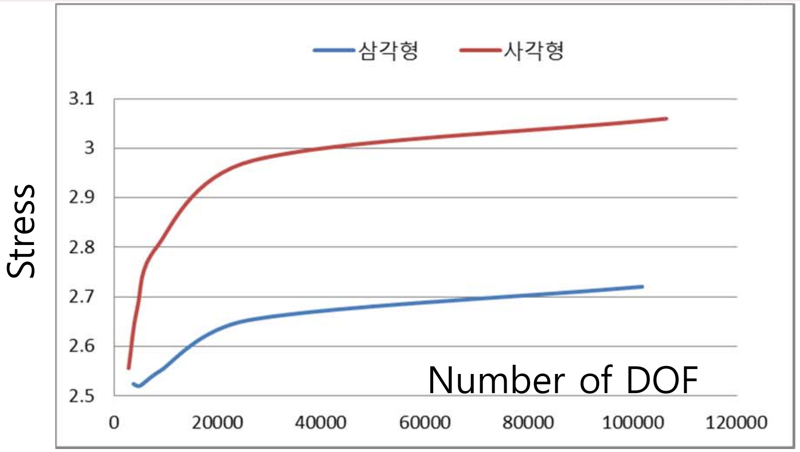
위와 같은 방식으로 요소 수를 증가시켜 응력 결과값을 출력한 그래프

Analytical solution 인 3*sigma 에 못 미치면서 수렴



QUADRILATERAL ELEMENT





Mapped 메뉴를 이용하여 사각 메시를 생성, Mapped 메시 외 Free quad 메뉴를 이용해도 사각 메시를 생성 가능

Fine 으로 변경 후 해석 수행

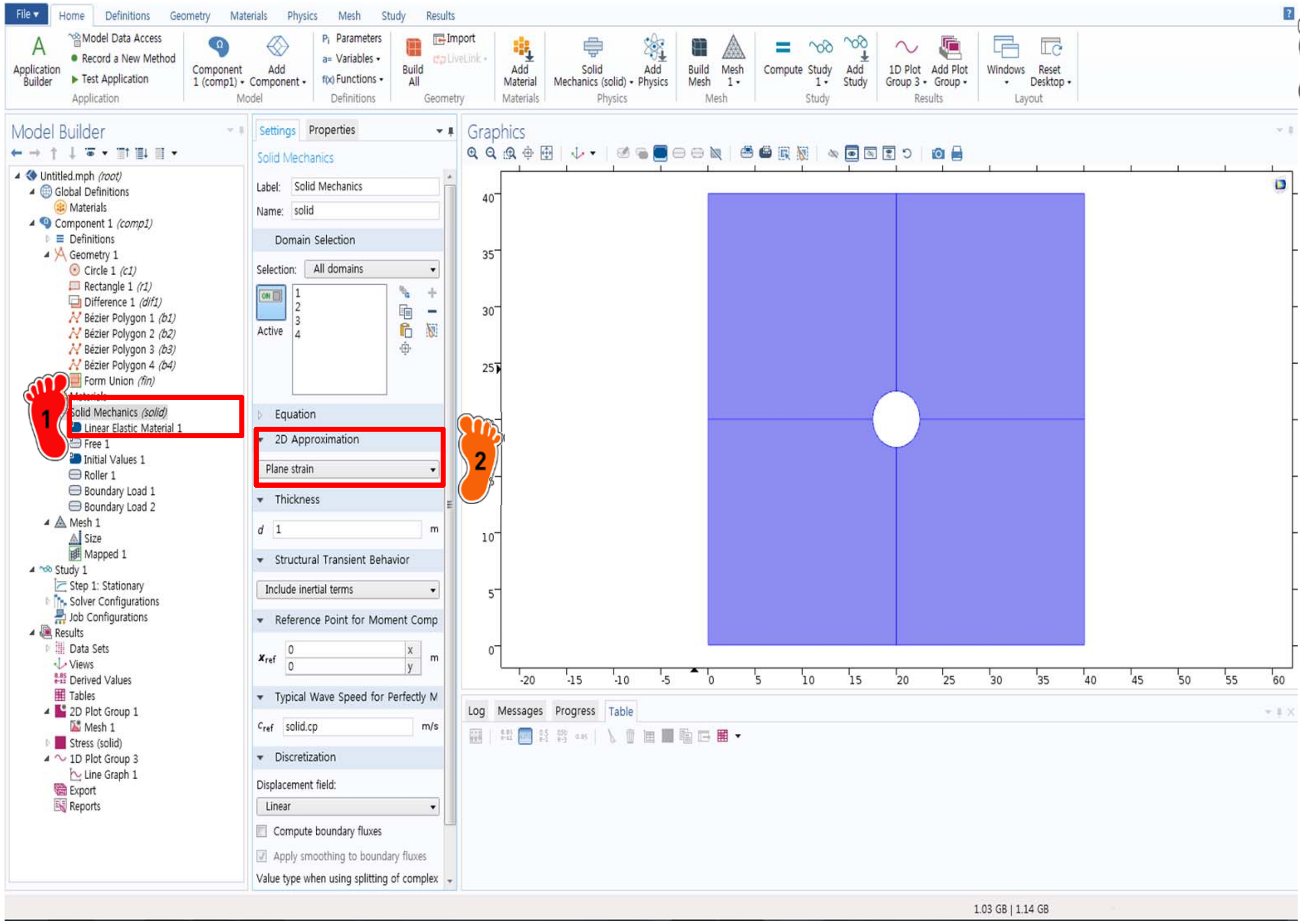
위와 같은 방식으로 요소 수를 증가시켜 응력 결과값을 출력한 그래프

삼각형보다 사각형 메시가 결과값에 빠르게 수렴하는 것을 확인

하지만 3*sigma 값을 넘어섬

현재 사각메시의 차수가 1 차이기 때문에 오차가 발생하는 것을 확인

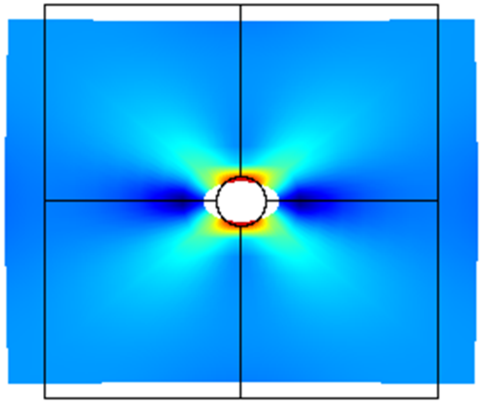
PLANE STRAIN



1 Solid Mechanics 클릭

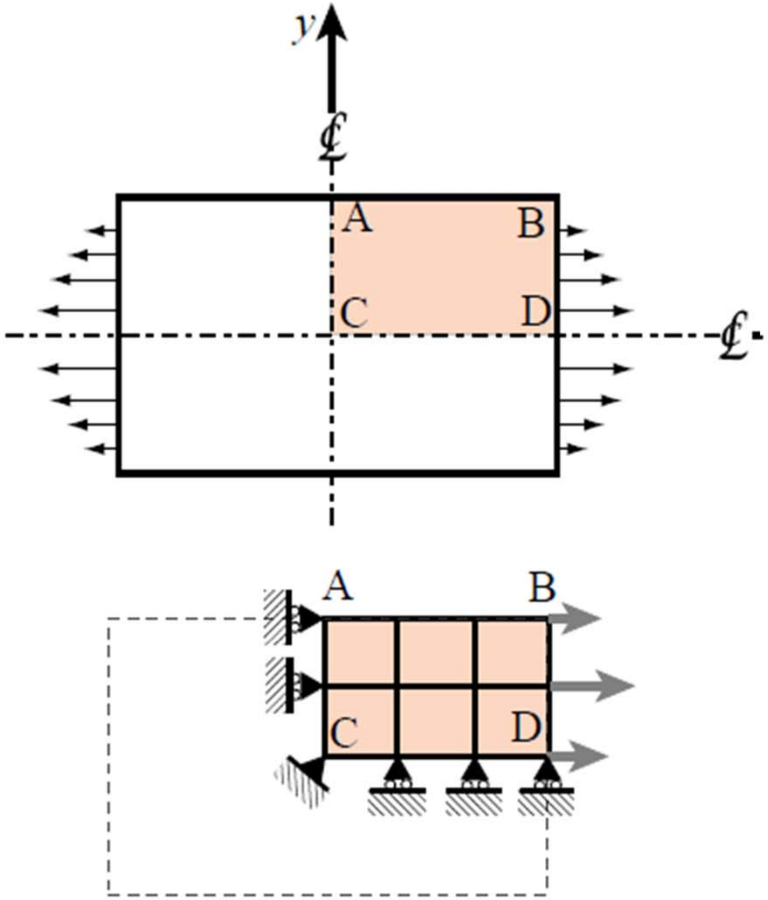
2 Plane strain 으로 변경 후 해석 수행

결과는 큰 차이가 없는 것을 확인

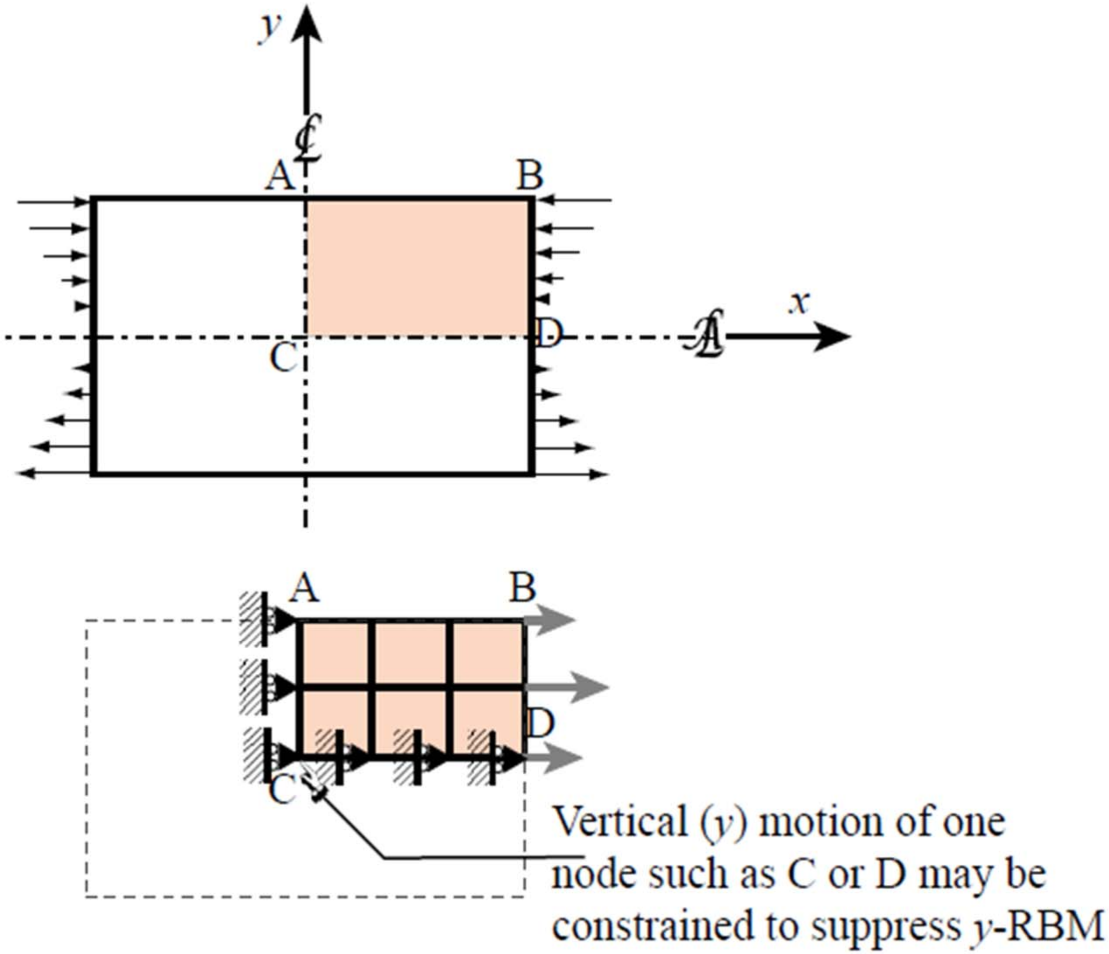


SYMMETRY CONDITION

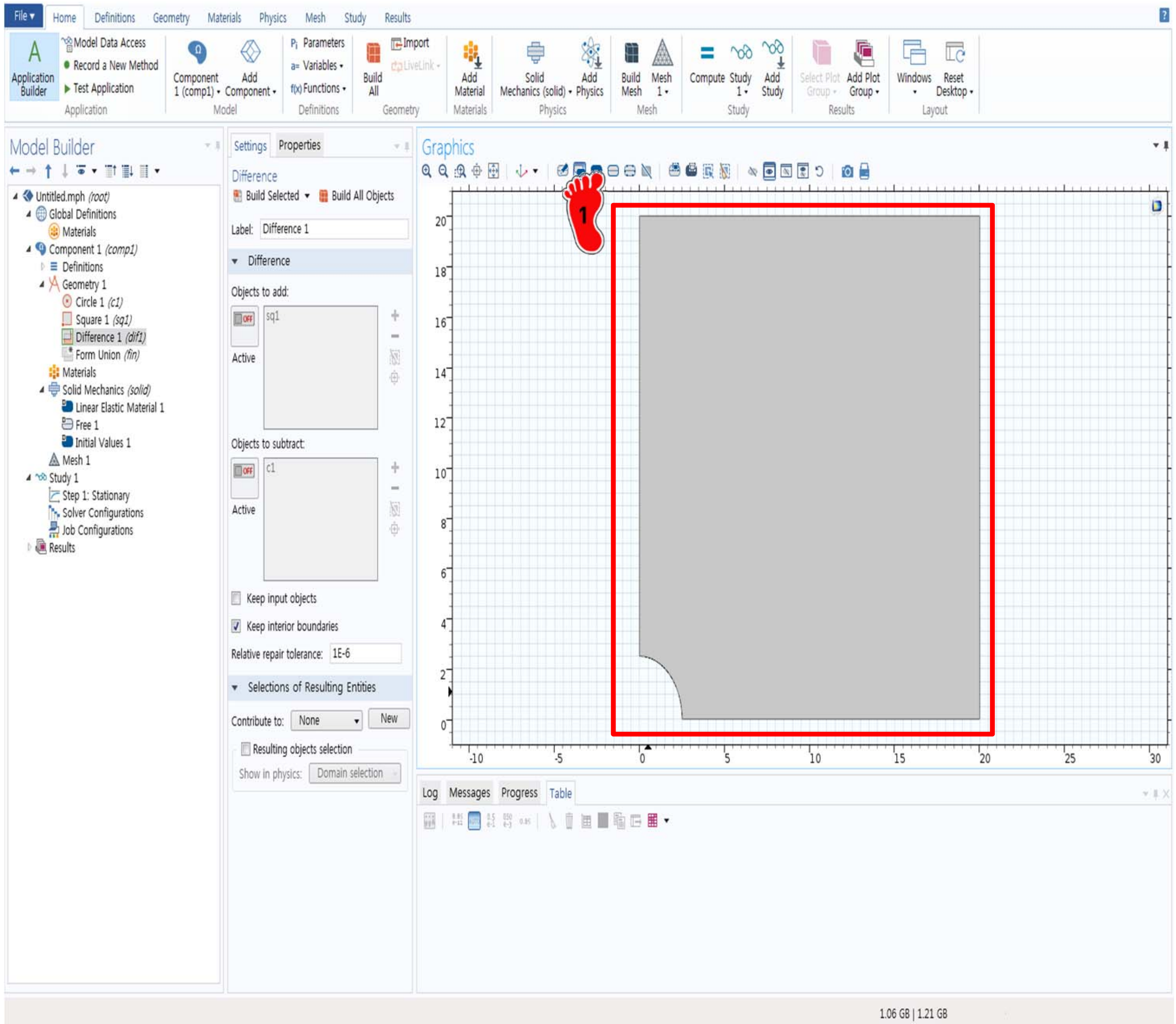
Symmetry condition



AntiSymmetry condition

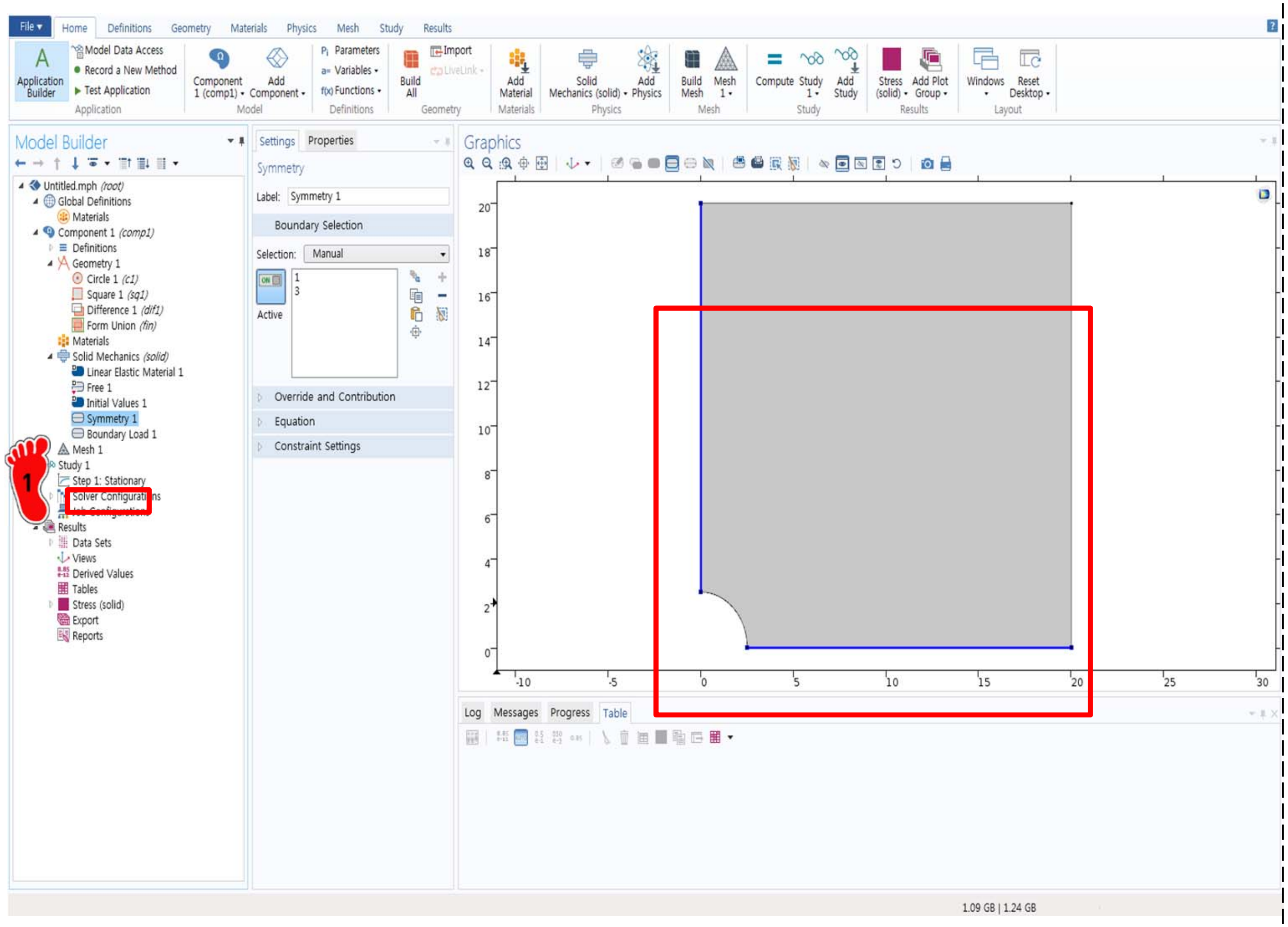


GEOMETRY CREATION



1 ¼ 모델 기하형상 생성

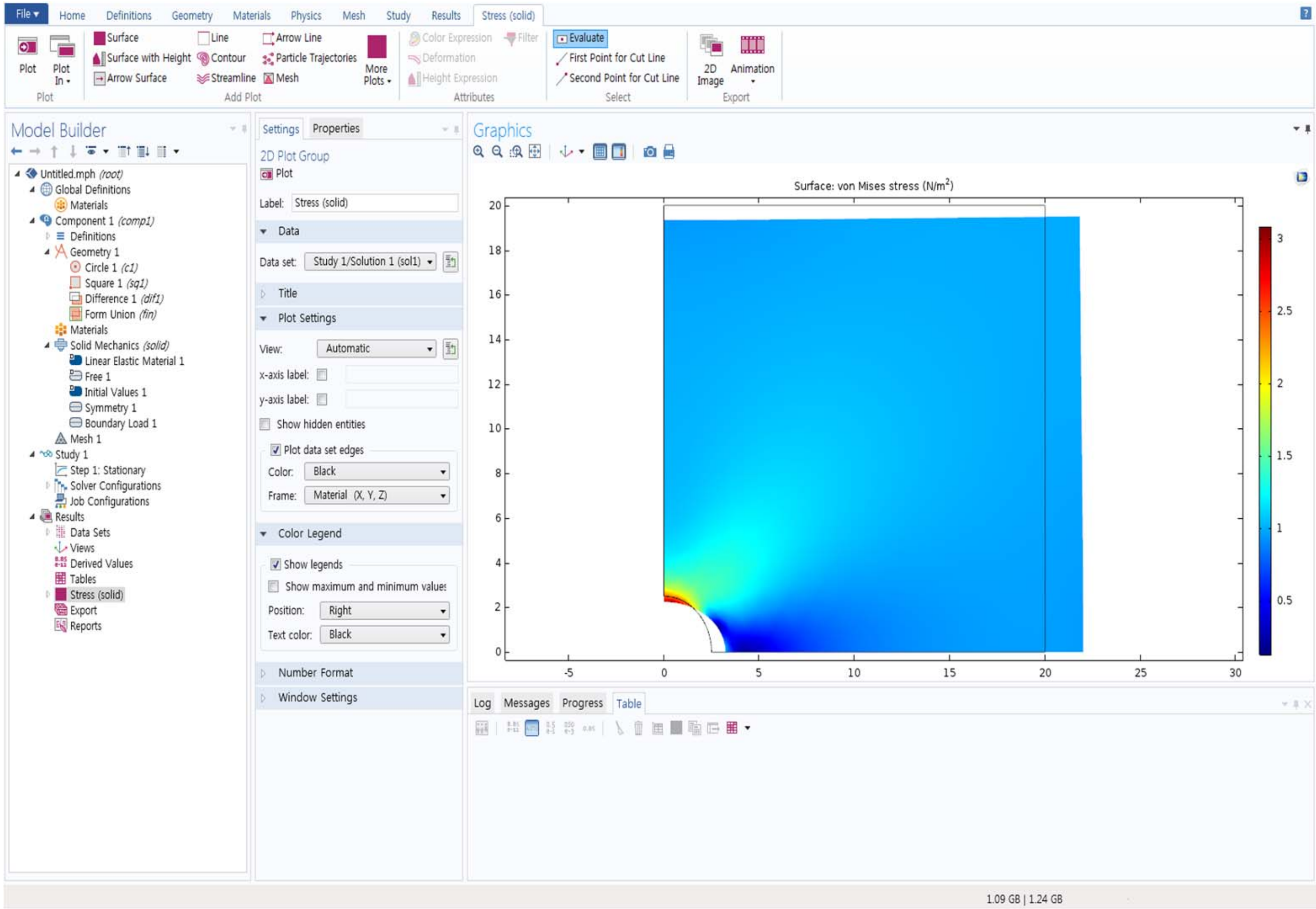
BC CONDITION



1 Symmetry 조건 생성 후 왼쪽과 아래쪽 경계 선택

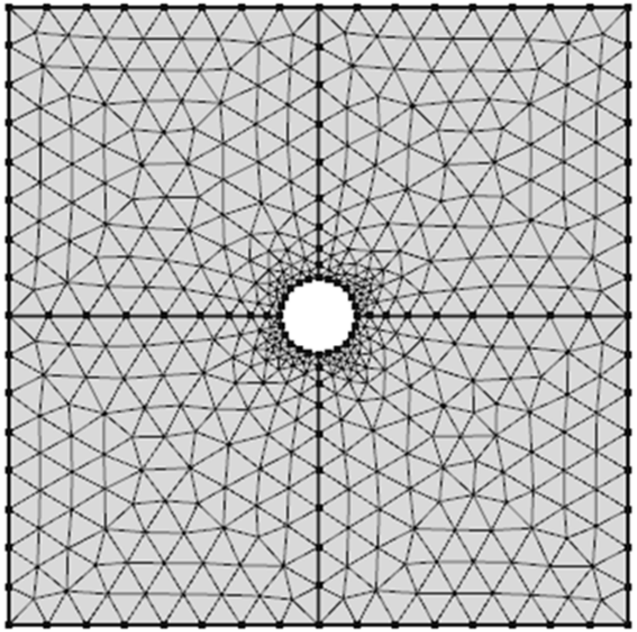
물성치, 하중 조건은 동일 삼각메시를 이용하여 해석 수행

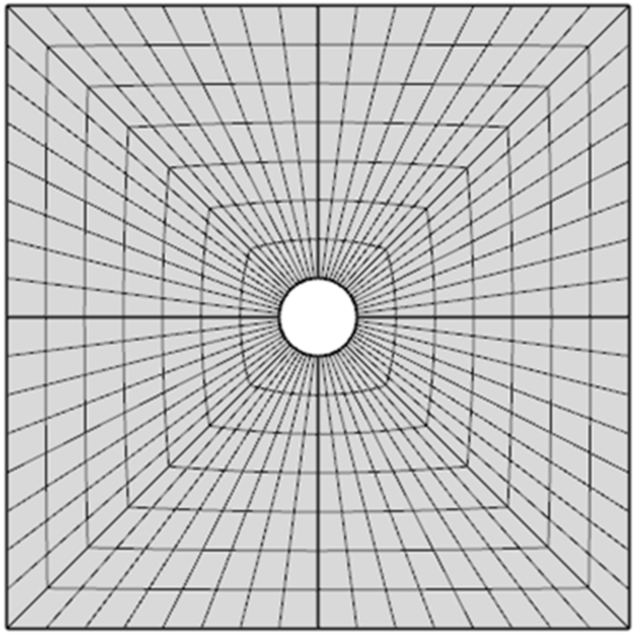
RESULT

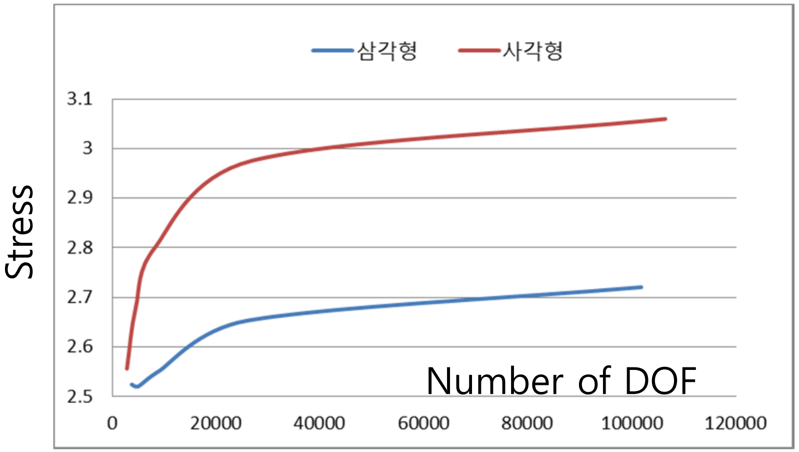


Full 모델과 비교하여 동일한 결과가 출력되는 것을 확인

ASSIGNMENT







삼각형요소, 사각형 요소에 따라 해석값이 달라짐

요소의 형태와 차수, 요소개수에 따른 해석값을 도출하고 경향 및 수렴 여부 파악

이 문제에서 가장 바람직한 요소망의 형태는?