Weighted Residual Method

- 1D heat conduction problem
  - Governing equation and analytic solution
  - Approximate solution and residual
  - Least square method
  - Weighted residual method
- Generalization of WRM → Weak Formulation
- Galerkin Finite Element Method
- 1D Elasticity
  - Governing equation
  - Weak formulation and discretization

1D Steady Heat Transfer Problem (1)

- BVP I: Dirichlet problem

$$kA\frac{d^2T}{dx^2} = \alpha s\left(T - T_a\right)$$

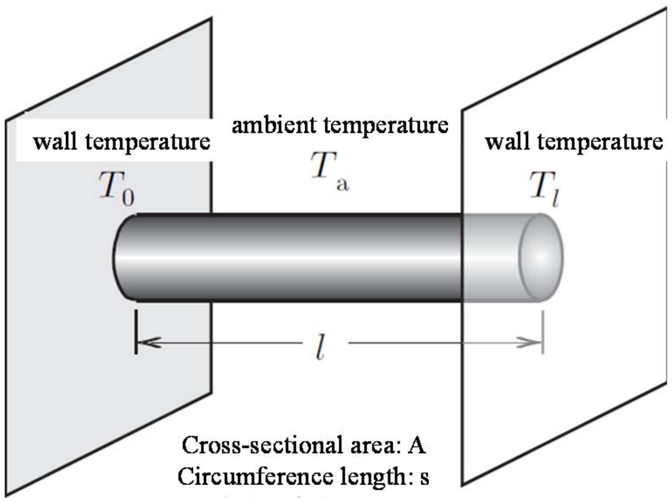
$$\begin{cases} T = T_0 & \text{on } x = 0 \\ T = T_l & \text{on } x = l \end{cases}$$

$$\xrightarrow{u = T - T_a,\ m^2 = \frac{\alpha s}{kA}}$$

$$\frac{d^2u}{dx^2} = m^2 u$$

$$\begin{cases} u = u_0 \left(= T_0 - T_a\right) & \text{on } x = 0 \\ u = u_l \left(= T_l - T_a\right) & \text{on } x = l \end{cases}$$

$$u\left(x\right) = \frac{u_0 \sinh\left[m\left(l - x\right)\right] + u_l \sinh\left(mx\right)}{\sinh\left(ml\right)}$$



1D Steady Heat Transfer Problem (2)

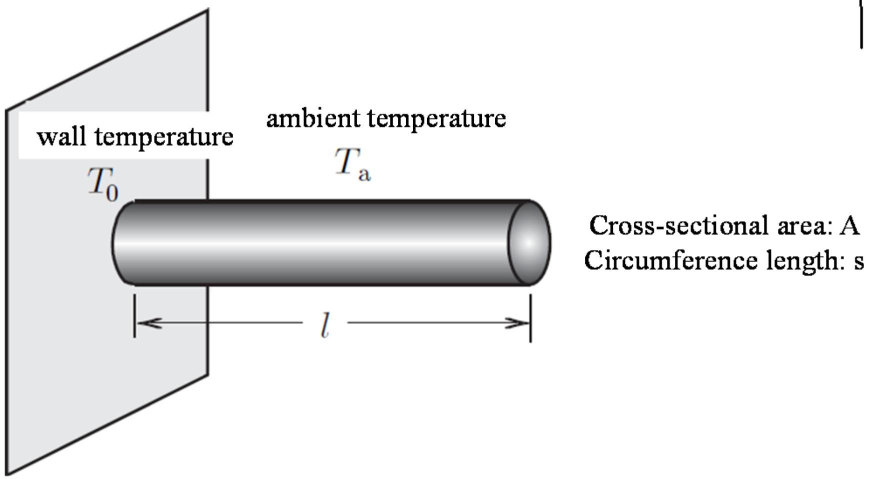
- BVP II: Mixed(Dirichlet + Neumann) problem

$$kA\frac{d^2T}{dx^2} = \alpha s\left(T - T_a\right)$$
$$T = T_0 \quad \text{on} \quad x = 0$$
$$\frac{dT}{dx} = Q_l \quad \text{on} \quad x = l$$

$$\xrightarrow{u = T - T_a,\ m^2 = \frac{\alpha s}{kA}}$$

$$\frac{d^2u}{dx^2} = m^2 u$$
$$u = u_0 \quad \text{on} \quad x = 0$$
$$\frac{du}{dx} = Q_l \quad \text{on} \quad x = l$$

$$\xrightarrow{Q_l = 0} u\left(x\right) = u_0 \frac{\cosh\left[m\left(l - x\right)\right]}{\cosh\left(ml\right)}$$



Approximate Solution

$$\underbrace{u(x)}_{\substack{\text{solution}\\ \text{of BVP}}} \approx \underbrace{U(x)}_{\substack{\text{approximate}\\ \text{solution}}} = \sum_{k=1}^{N} \underbrace{\psi_k(x)}_{\substack{\text{basis}\\ \text{function}}} \underbrace{c_k}_{\substack{\text{unknown}\\ \text{parameter}}}$$

Residual / Least Square Method

$$\underbrace{r\big(U(x)\big)}_{\text{residual}} = \frac{d^2U(x)}{dx^2} - m^2U(x)$$

$$= \frac{d^2}{dx^2}\left[g(x) + \sum_{k=1}^{M}\Phi_k(x)\alpha_k\right] - m^2\left[g(x) + \sum_{k=1}^{M}\Phi_k(x)\alpha_k\right]$$

$$= \sum_{k=1}^{M}\left(\frac{d^2\Phi_k(x)}{dx^2} - m^2\Phi_k(x)\right)\alpha_k - m^2g(x)$$

$$\underbrace{R(U)}_{\substack{\text{square}\\\text{error}}} = \frac{1}{2}\int_0^l \left[r(U)\right]^2 dx \xrightarrow{\text{Least Square Method}} \frac{\partial R(U)}{\partial \alpha_i} = \int_0^l \frac{\partial r(U)}{\partial \alpha_i} r(U)\,dx = 0 \quad (i = 1,\ldots,M)$$

$$\int_0^l \left(\frac{d^2\Phi_i}{dx^2} - m^2\Phi_i\right)\left[\sum_{k=1}^{M}\left(\frac{d^2\Phi_k}{dx^2} - m^2\Phi_k\right)\alpha_k - m^2g\right]dx = 0$$

Weighted Residual Method

$$\text{WRM: } \int_0^l \underbrace{\chi_i}_{\text{weighted}} \underbrace{r(U)}_{\text{residual}} dx = \int_0^l \chi_i \left[\sum_{k=1}^{M} \left(\frac{d^2\Phi_k}{dx^2} - m^2\Phi_k \right) \alpha_k - m^2 g \right] dx = 0$$

$$\sum_{k=1}^{M} A_{ik}\alpha_k = F_i \quad (i = 1, \ldots, M)$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ & & \ddots & \\ A_{M1} & A_{M2} & \cdots & A_{MM} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_M \end{Bmatrix}$$

$$A_{ik} = \int_0^l \chi_i \left(\frac{d^2\Phi_k}{dx^2} - m^2\Phi_k \right) dx$$

$$F_i = m^2 \int_0^l \chi_i g dx$$

Weighted Residual Method

χ_i	**Weighted Residual Method**
$\partial r(U)/\partial \alpha_i$	Least Square Method
$\delta(x - x_i)$	Collocation Method
$x^{i-1}\ (i \geq 1)$	Method of Moment
$\Phi_i\ (\text{basis function})$	Galerkin Method

Galerkin method:

- more accurate
- coefficient matrix is symmetric → used in FEM

Example

$$m=1,\ u_0=10,\ u_l=20,\ l=1,\ M=N-2=2$$

$$\begin{cases} \text{exact solution: } u(x)=\dfrac{u_0\sinh\left[m(l-x)\right]+u_l\sinh(mx)}{\sinh(ml)} \\ \text{least square method: } U(x)=10(1-x)+20x+\dfrac{109118}{24487}x(x-1)+\dfrac{854}{521}x\left(x^2-1\right) \\ \text{collocation method: } U(x)=10(1-x)+20x+\dfrac{460}{99}x(x-1)+\dfrac{400}{297}x\left(x^2-1\right) \\ \text{method of moment: } U(x)=10(1-x)+20x+\dfrac{3540}{793}x(x-1)+\dfrac{100}{61}x\left(x^2-1\right) \\ \text{Galerkin method: } U(x)=10(1-x)+20x+\dfrac{2070}{473}x(x-1)+\dfrac{70}{43}x\left(x^2-1\right) \end{cases}$$

x	exact	Least Square	Collocation	Moment	Galerkin
0.25	11.2963				
0.50	13.3023				
0.75	16.1440				

Generalization of WRM (1)

$$\chi_i(x)=\Phi_i(x)\ \ (i=1,\ldots,M)\rightarrow V(x)=\sum_i^M \Phi_i(x)\alpha_i^*$$

$$\Phi_k(0)=\Phi_k(l)=0\ \ (k=1,\ldots,M)\rightarrow V(0)=V(l)=0$$

$$\int_0^l V(x)r(U(x))dx=0$$

$$\int_0^l \sum_i^M \Phi_i(x)\alpha_i^*\left[\sum_{k=1}^M\left(\frac{d^2\Phi_k(x)}{dx^2}-m^2\Phi_k(x)\right)\alpha_k-m^2g(x)\right]dx=0$$

$$\sum_i^M \alpha_i^*\left(\sum_{k=1}^M\int_0^l \Phi_i(x)\left(\frac{d^2\Phi_k(x)}{dx^2}-m^2\Phi_k(x)\right)dx\alpha_k-m^2\int_0^l\Phi_i(x)g(x)dx\right)=0$$

$$\sum_i^M \alpha_i^*\left(\sum_{k=1}^M A_{ik}\alpha_k-F_i\right)=0\quad\text{where}\quad\begin{cases}A_{ik}=\int_0^l\Phi_i(x)\left(\frac{d^2\Phi_k(x)}{dx^2}-m^2\Phi_k(x)\right)dx\\ F_i=m^2\int_0^l\Phi_i(x)g(x)dx\end{cases}$$

$$\begin{Bmatrix}\alpha_1^* & \alpha_2^* & \cdots & \alpha_M^*\end{Bmatrix}\left(\begin{bmatrix}A_{11} & A_{12} & \cdots & A_{1M}\\ A_{21} & A_{22} & \cdots & A_{2M}\\ & & \ddots & \\ A_{M1} & A_{M2} & \cdots & A_{MM}\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_M\end{Bmatrix}-\begin{Bmatrix}F_1\\ F_2\\ \vdots\\ F_M\end{Bmatrix}\right)=0$$

Generalization of WRM (2)

$u(x)$: solution function $\rightarrow$ trial function

$v(x)$: arbitrary function (satisfying Dirichlet conditions) $\rightarrow$ test function

$$\left.\begin{array}{l}\text{equilibrium equation}\\ r\left(u(x)\right)=0\end{array}\right\} \Leftrightarrow \int_0^l v(x)\, r\left(u(x)\right) dx = 0$$

$$\xrightarrow[v \approx V = \sum_{i}^{M} \Phi_i(x)\alpha_i^*]{u \approx U = g(x) + \sum_{k=1}^{M} \Phi_k(x)\alpha_k} \int_0^l V(x)\, r\left(U(x)\right) dx = 0$$

Weak Formulation

$$\frac{d^2u(x)}{dx^2} = m^2u(x) \rightarrow \int_0^l v(x)\left(\frac{d^2u(x)}{dx^2} - m^2u(x)\right)dx = 0$$

$$\left.\frac{du}{dx}\right|_{x=l} v(l) - \left.\frac{du}{dx}\right|_{x=0} v(0) - \int_0^l \left(\frac{dv}{dx}\frac{du}{dx} + m^2vu\right)dx = 0$$

$$\begin{cases} \text{BVP I: } v(0) = v(l) = 0 \rightarrow \int_0^l \left(\frac{dv}{dx}\frac{du}{dx} + m^2vu\right)dx = 0 \\ \text{BVP II: } v(0) = 0,\ \frac{du}{dx} = Q_l \text{ on } x = l \rightarrow \int_0^l \left(\frac{dv}{dx}\frac{du}{dx} + m^2vu\right)dx = Q_l v(l) \end{cases}$$

* Dirichlet B.C.: constraint on the basis function of trial and test functions → essential B.C.
* Neumann B.C. automatically satisfied → natural B.C.

	equilibrium	Differential equation	solution
Strong form	all x	2nd order	twice differentiable
Weak form	average	1st order	once differentiable

* Neumann B.C. is satisfied in the average sense.

Example (1)

$$\begin{cases} \text{BVP I: } (\text{Result of Weak Formulation}) = (\text{Result of WRM}) \\ \text{BVP II: } u(0) \approx U(0) = u_0 \to c_0 = u_0,\ c_k \to \alpha_k \ (k = 1, \ldots, M\ (= N-1)) \end{cases}$$

$$\left.\begin{aligned} U(x) &= u_0 + \sum_{k=1}^{N-1} x^k \alpha_k = u_0 + \sum_{k=1}^{M} \Phi_k(x)\alpha_k \\ V(x) &= \sum_{i=1}^{N-1} x^i \alpha_i^* = u_0 + \sum_{i=1}^{M} \Phi_i(x)\alpha_i^* \end{aligned}\right\} \xrightarrow{\Phi_k(0)=0 \to v(0) \approx V(0)=0} \int_0^l \left(\frac{dv}{dx}\frac{du}{dx} + m^2 vu \right) dx = Q_l v(l)$$

$$\to \int_0^l \left[\left(\sum_{i=1}^{M} \frac{d\Phi_i}{dx} \alpha_i^* \right) \left(\sum_{k=1}^{M} \frac{d\Phi_k}{dx} \alpha_k \right) + m^2 \left(\sum_{i=1}^{M} \Phi_i \alpha_i^* \right) \left(u_0 + \sum_{k=1}^{M} \Phi_k \alpha_k \right) \right] dx = Q_l \left(\sum_{i=1}^{M} \Phi_i(l) \alpha_i^* \right)$$

$$\sum_{i=1}^{M} \alpha_i^* \left\{ \sum_{k=1}^{M} \int_0^l \left[\left(\frac{d\Phi_i}{dx}\frac{d\Phi_k}{dx} + m^2 \Phi_i \Phi_k \right) \alpha_k + m^2 u_0 \Phi_i \right] dx - Q_l \Phi_i(l) \right\} = 0$$

$$\sum_{k=1}^{M} \left[\int_0^l \left(\frac{d\Phi_i}{dx}\frac{d\Phi_k}{dx} + m^2 \Phi_i \Phi_k \right) dx \right] \alpha_k = Q_l \Phi_i(l) - m^2 u_0 \int_0^l \Phi_i \, dx$$

$$\sum_{k=1}^{M} K_{ik} \alpha_k = F_i \quad (i = 1, \ldots, M) \quad \text{where} \quad \begin{cases} K_{ik} = \int_0^l \left(\frac{d\Phi_i}{dx}\frac{d\Phi_k}{dx} + m^2 \Phi_i \Phi_k \right) dx \\ F_i = Q_l \Phi_i(l) - m^2 u_0 \int_0^l \Phi_i \, dx \end{cases}$$

Example (2)

$$M = 3,\ m = 1,\ u_0 = 10,\ l = 1,\ Q_l = 0$$

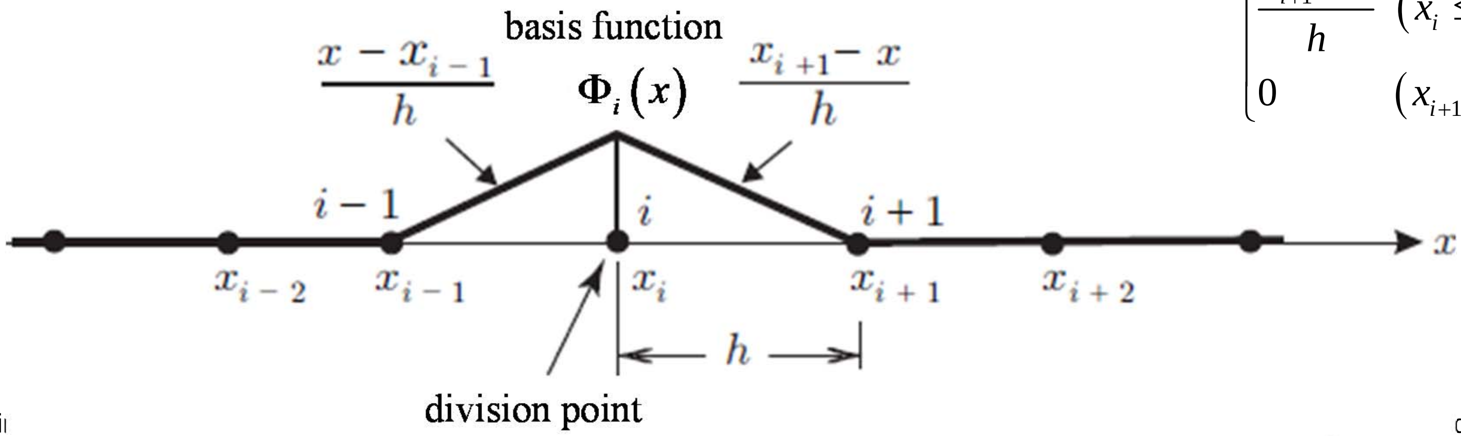
$$U(x) = 10 - \frac{35175}{4658}x + \frac{10725}{2329}x^2 - \frac{10675}{18632}x^3 \rightarrow \left.\frac{dU}{dx}\right|_{x=1} = -\frac{1125}{18632} \neq 0 \left(= Q_l\right)$$

x	u(x)	U(x)		
		M=2	M=3	M=4
0.25	8.3903			
0.50	7.3076			
0.75	6.6841			

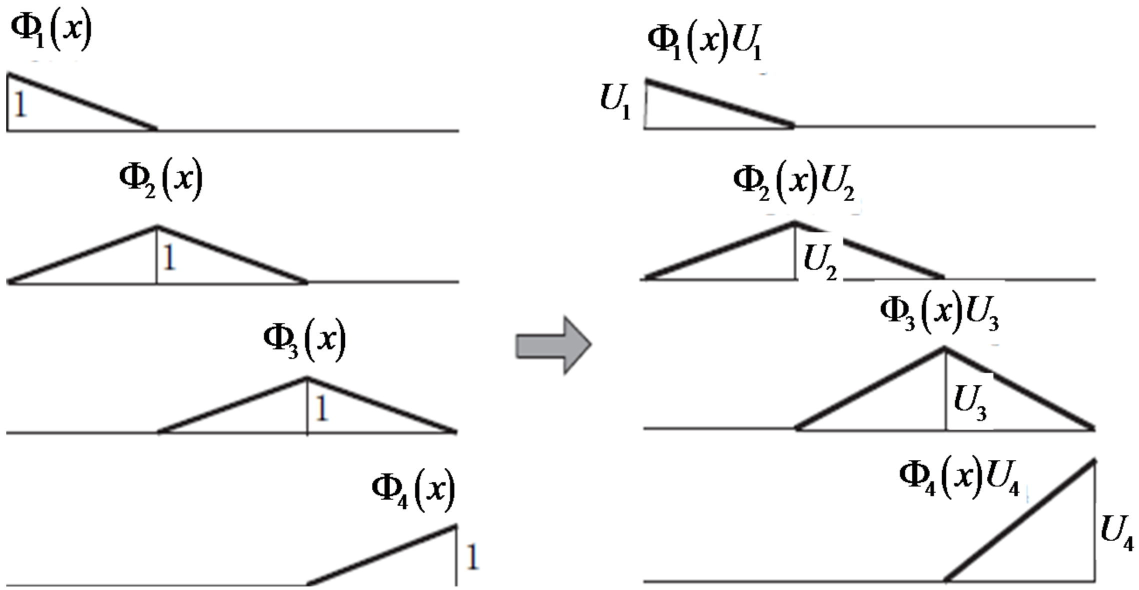
Galerkin Finite Element Method

- Problem of Galerkin Method
  - Simple → complex domain: how to choose basis function? Difficult to integrate the domain, defined over whole domain
  - Unknown parameters: no physical meaning, need combination to obtain the approximate solution
- Finite Element Method
  - Basis function: piecewise function
  - Unknown parameters: discretized points

$$\Phi_i(x) = \begin{cases} 0 & (x \le x_{i-1}) \\ \dfrac{x - x_{i-1}}{h} & (x_{i-1} \le x \le x_i) \\ \dfrac{x_{i+1} - x}{h} & (x_i \le x \le x_{i+1}) \\ 0 & (x_{i+1} \le x) \end{cases}$$

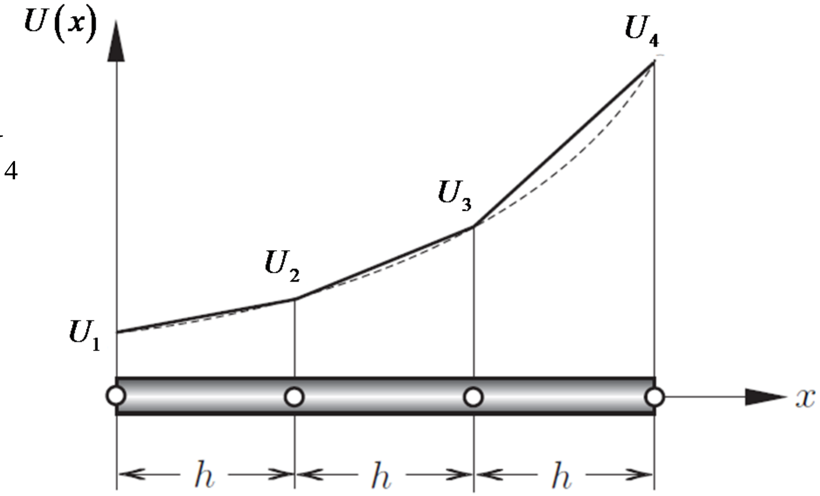


Galerkin Method Using Piecewise Functions

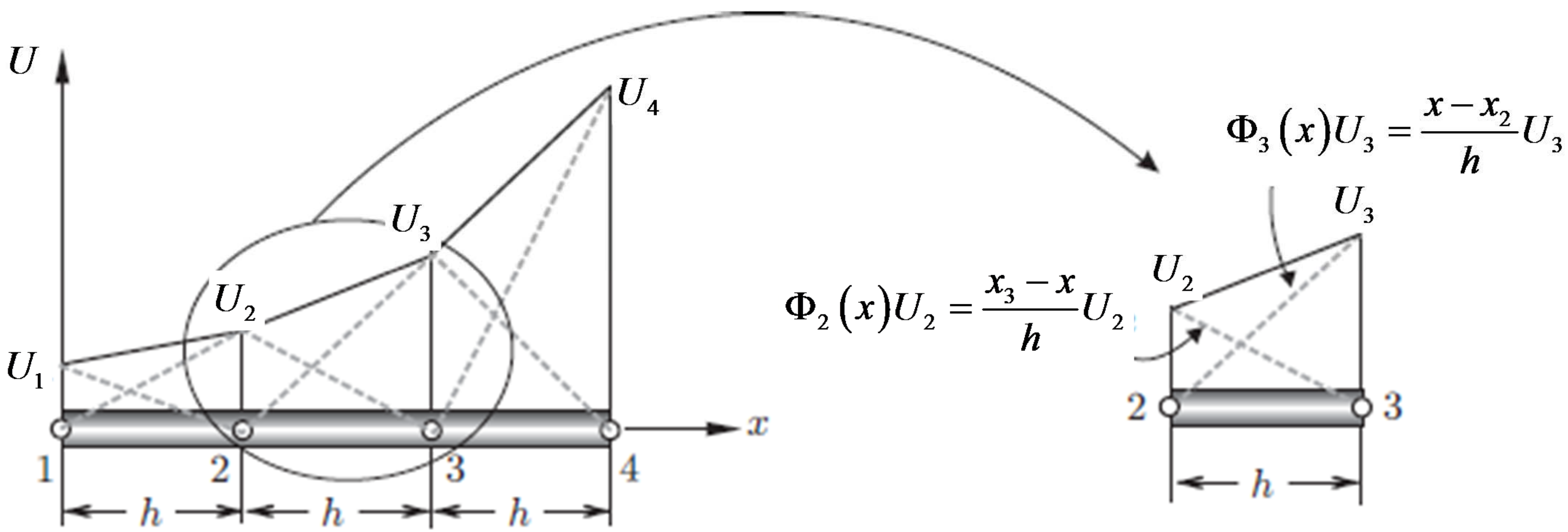


$$U(x) = \Phi_1(x)U_1 + \Phi_2(x)U_2 + \Phi_3(x)U_3 + \Phi_4(x)U_4$$

$$= \sum_{i=1}^{4} \Phi_i(x) \underbrace{U_i}_{\substack{\text{temperature} \\ \text{@ node } i}}$$

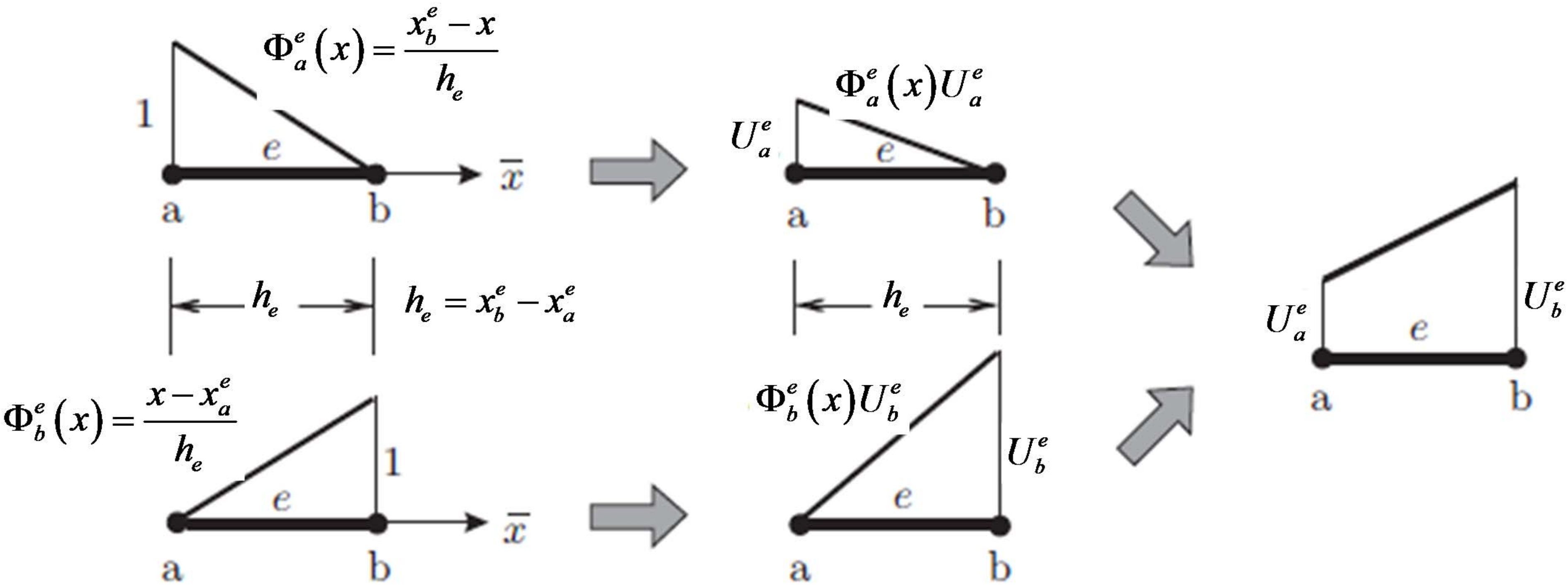


Node / Element / Shape Function



$$U(x) = \Phi_2(x)U_2 + \Phi_3(x)U_3$$
$$= \frac{x_3 - x}{h}U_2 + \frac{x - x_2}{h}U_3 \quad (x_2 \le x \le x_3)$$

Interpolation & Approximate Solution



$$U_a^e = U_i,\ U_b^e = U_{i+1} \rightarrow U(x) = \Phi_a^e(x)U_a^e + \Phi_b^e(x)U_b^e$$

$$\Phi_a^e(x) = \frac{x_b^e - x}{h_e},\ \Phi_b^e(x) = \frac{x - x_a^e}{h_e} \quad \text{where } h_e = x_b^e - x_a^e$$

$$\begin{cases} \Phi_a^e(x_a^e) = 1 \\ \Phi_a^e(x_b^e) = 0 \end{cases},\ \begin{cases} \Phi_b^e(x_a^e) = 0 \\ \Phi_b^e(x_b^e) = 1 \end{cases}$$

FEM: Finite Element Method

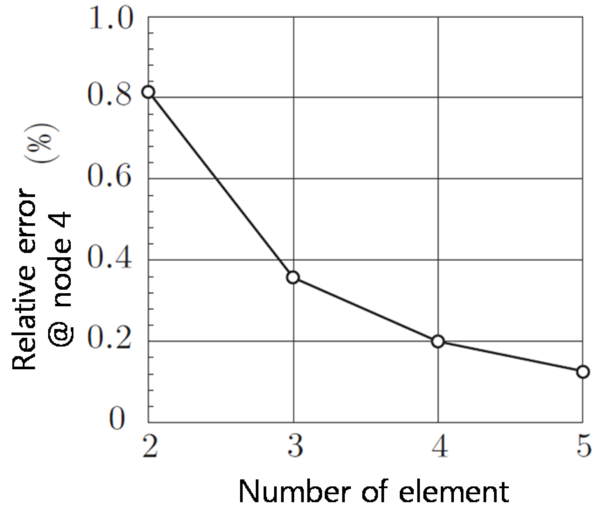
- Shape function
- Element
- Node

Example: FEM (1)

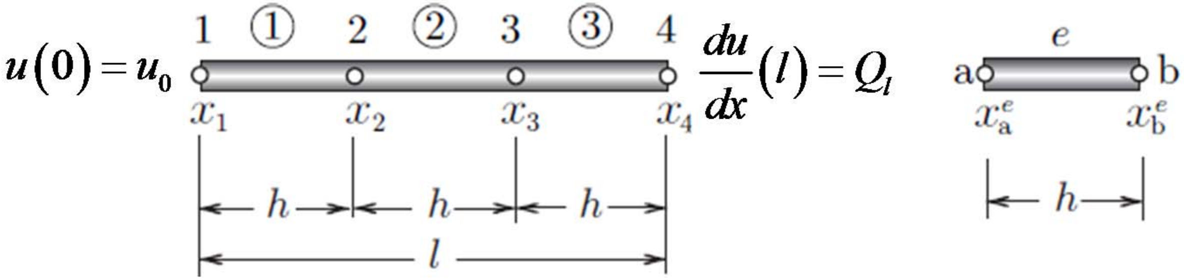
$$\begin{cases} \text{equilibrium equation: } \dfrac{d^2u}{dx^2} = u \\ \text{essential B.C.: } u(x) = u_0 \\ \text{natural B.C.: } \left.\dfrac{du}{dx}\right|_{x=l} = Q_l \end{cases}$$

$T_0 = 0,\ m = 1,\ k = 1$

$l = 1,\ u_0 = 10,\ Q_l = 0$



Temperature @node	U_1	U_2	U_3	U_4	U_a^e	U_b^e
Equivalent Heat Flow	Q_1	Q_2	Q_3	Q_4	Q_a^e	Q_b^e



Example: FEM (2)

$$\sum_{e=1}^{3}\left\{\int_{x_a^e}^{x_b^e} v\left(\frac{d^2u}{dx^2}-u\right)dx\right\}=0 \rightarrow \int_{x_a^e}^{x_b^e} v\left(\frac{d^2u}{dx^2}-u\right)dx=0$$

$$\begin{cases} u(x)\approx U(x)=N_a^e(x)U_a^e+N_b^e(x)U_b^e=\{N_a^e \quad N_b^e\}\begin{Bmatrix}U_a^e\\U_b^e\end{Bmatrix}=\boldsymbol{N}_e\boldsymbol{U}_e \\ v(x)\approx V(x)=N_a^e(x)V_a^e+N_b^e(x)V_b^e=\{N_a^e \quad N_b^e\}\begin{Bmatrix}V_a^e\\V_b^e\end{Bmatrix}=\boldsymbol{N}_e\boldsymbol{V}_e \end{cases}$$

$$\begin{cases} \dfrac{du}{dx}\approx\dfrac{dU}{dx}=\dfrac{dN_a^e}{dx}U_a^e+\dfrac{dN_b^e}{dx}U_b^e=\left\{\dfrac{dN_a^e}{dx} \quad \dfrac{dN_b^e}{dx}\right\}\begin{Bmatrix}U_a^e\\U_b^e\end{Bmatrix}=\boldsymbol{B}_e\boldsymbol{U}_e \\ \dfrac{dv}{dx}\approx\dfrac{dV}{dx}=\dfrac{dN_a^e}{dx}V_a^e+\dfrac{dN_b^e}{dx}V_b^e=\left\{\dfrac{dN_a^e}{dx} \quad \dfrac{dN_b^e}{dx}\right\}\begin{Bmatrix}V_a^e\\V_b^e\end{Bmatrix}=\boldsymbol{B}_e\boldsymbol{V}_e \end{cases}$$

$$\boldsymbol{V}_e^T\left\{\int_{x_a^e}^{x_b^e}\left(\boldsymbol{B}_e^T\boldsymbol{B}_e+\boldsymbol{N}_e^T\boldsymbol{N}_e\right)dx\boldsymbol{U}_e-\left[Q_a^e\boldsymbol{N}_e^T\left(x_a^e\right)+Q_b^e\boldsymbol{N}_e^T\left(x_b^e\right)\right]\right\}=0$$

$$\{V_a^e \quad V_b^e\}\left[\int_{x_a^e}^{x_b^e}\left(\begin{bmatrix}\dfrac{dN_a^e}{dx}\dfrac{dN_a^e}{dx} & \dfrac{dN_a^e}{dx}\dfrac{dN_b^e}{dx}\\ \dfrac{dN_b^e}{dx}\dfrac{dN_a^e}{dx} & \dfrac{dN_b^e}{dx}\dfrac{dN_b^e}{dx}\end{bmatrix}+\begin{bmatrix}N_a^eN_a^e & N_a^eN_b^e\\ N_b^eN_a^e & N_b^eN_b^e\end{bmatrix}\right)dx\begin{Bmatrix}U_a^e\\U_b^e\end{Bmatrix}-\begin{Bmatrix}Q_a^e\\Q_b^e\end{Bmatrix}\right]=0$$

Example: FEM (3)

$$N_a^e(x)=\frac{x_b^e-x}{h_e},\ N_b^e(x)=\frac{x-x_a^e}{h_e},\ \frac{dN_a^e(x)}{dx}=-\frac{1}{h_e},\ \frac{dN_b^e(x)}{dx}=\frac{1}{h_e}$$

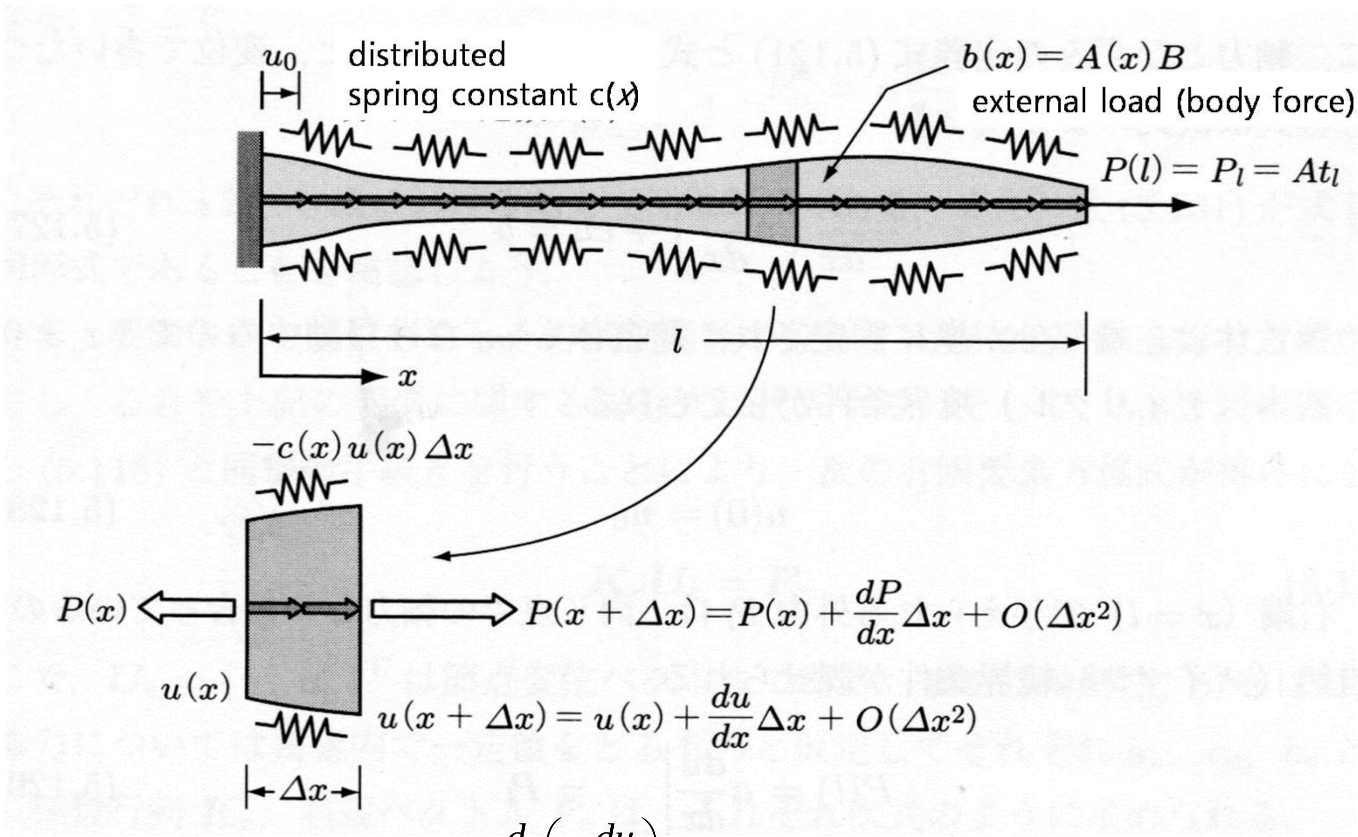
$$\begin{Bmatrix}V_a^e & V_b^e\end{Bmatrix}\left(\frac{1}{6h_e}\begin{bmatrix}2\left(h_e^2+3\right) & h_e^2-6\\ h_e^2-6 & 2\left(h_e^2+3\right)\end{bmatrix}\begin{Bmatrix}U_a^e\\U_b^e\end{Bmatrix}-\begin{Bmatrix}Q_a^e\\Q_b^e\end{Bmatrix}\right)=0$$

$$\begin{Bmatrix}V_1 & V_2 & V_3 & V_4\end{Bmatrix}\left(\frac{1}{6h}\begin{bmatrix}2\left(h^2+3\right) & h^2-6 & 0 & 0\\ h^2-6 & 4\left(h^2+3\right) & h^2-6 & 0\\ 0 & h^2-6 & 4\left(h^2+3\right) & h^2-6\\ 0 & 0 & h^2-6 & 2\left(h^2+3\right)\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\end{Bmatrix}-\begin{Bmatrix}Q_a^{①}\\Q_b^{①}+Q_a^{②}\\Q_b^{②}+Q_a^{③}\\Q_b^{③}\end{Bmatrix}\right)=0$$

$$\frac{1}{6h}\begin{bmatrix}4\left(h^2+3\right) & h^2-6 & 0\\ h^2-6 & 4\left(h^2+3\right) & h^2-6\\ 0 & h^2-6 & 2\left(h^2+3\right)\end{bmatrix}\begin{Bmatrix}U_2\\U_3\\U_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\Q_l\end{Bmatrix}-\frac{u_0}{6h}\begin{Bmatrix}h^2-6\\0\\0\end{Bmatrix}$$

$$\begin{Bmatrix}U_2\\U_3\\U_4\end{Bmatrix}\xrightarrow{\text{post-processing}}\begin{Bmatrix}Q_a^{①}\\Q_b^{①}\end{Bmatrix},\begin{Bmatrix}Q_a^{②}\\Q_b^{②}\end{Bmatrix},\left.\frac{du}{dx}\right|_{①},\left.\frac{du}{dx}\right|_{②}$$

1D Elasticity: Governing Equation



$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu = b$$

$$u(x) = u_0 \ @\ x = 0$$

$$P(l) = a\frac{du}{dx}\bigg|_{x=l} = P_l \ @\ x = l$$

1D Elasticity: Weak Form + Discretization

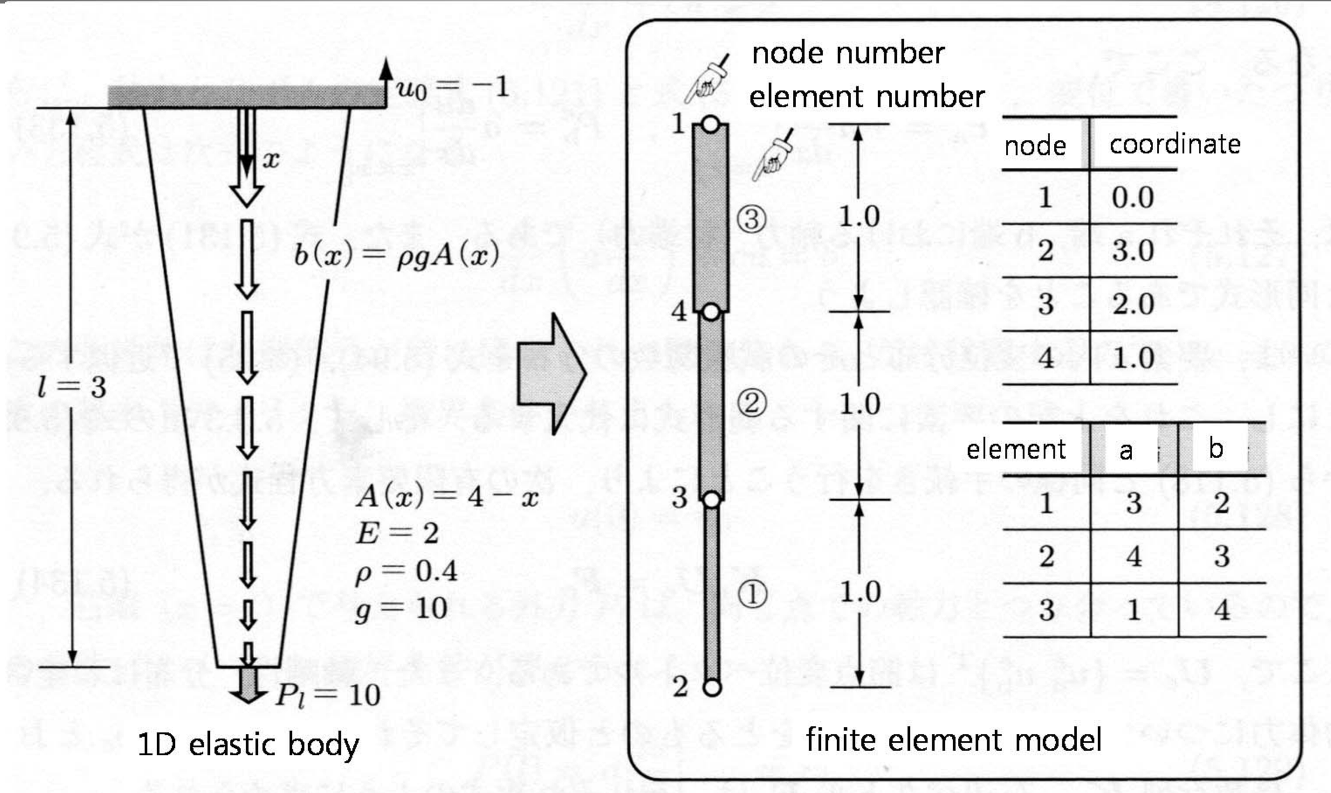
$$\int_0^l v\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-b\right]dx=0\rightarrow\int_0^l\left(a\frac{dv}{dx}\frac{du}{dx}+cvu\right)dx=\int_0^l vbdx+v(l)P_l$$

$$\text{element } e:\int_{x_a^e}^{x_b^e}\left(a\frac{dv}{dx}\frac{du}{dx}+cvu\right)dx=\int_{x_a^e}^{x_b^e}vbdx+v\left(x_a^e\right)P_a^e+v\left(x_b^e\right)P_b^e\quad\text{where }\ P_a^e=-a\frac{du}{dx}\bigg|_{x=x_a^e},\ P_b^e=a\frac{du}{dx}\bigg|_{x=x_b^e}$$

$$\begin{cases}u(x)\approx U(x)=N_a^e(x)U_a^e+N_b^e(x)U_b^e=\left\{N_a^e\quad N_b^e\right\}\begin{Bmatrix}U_a^e\\U_b^e\end{Bmatrix}=\boldsymbol{N}_e\boldsymbol{U}_e\\ v(x)\approx V(x)=N_a^e(x)V_a^e+N_b^e(x)V_b^e=\left\{N_a^e\quad N_b^e\right\}\begin{Bmatrix}V_a^e\\V_b^e\end{Bmatrix}=\boldsymbol{N}_e\boldsymbol{V}_e\end{cases}$$

$$\rightarrow\boldsymbol{K}_e\boldsymbol{U}_e=\boldsymbol{F}_e\quad\text{where}\quad\begin{cases}\boldsymbol{K}_e=a_e\begin{bmatrix}\frac{1}{h_e}&-\frac{1}{h_e}\\-\frac{1}{h_e}&\frac{1}{h_e}\end{bmatrix}+c_e\begin{bmatrix}\frac{h_e}{3}&\frac{h_e}{6}\\\frac{h_e}{6}&\frac{h_e}{3}\end{bmatrix}\\ \boldsymbol{F}_e=\frac{b_eh_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix}+\begin{Bmatrix}P_a^e\\P_b^e\end{Bmatrix}\end{cases}$$

1D Elasticity: Example



element	x_a	x_b	h	A	E	a = EA	b = ρgA
①	2.0	3.0	1.0	3/2	2.0	3.0	6.0
②	1.0	2.0	1.0	5/2	2.0	5.0	10.0
③	0.0	1.0	1.0	7/2	2.0	7.0	14.0