FEM for Steady Problems

  - Governing equation, weak form
  - Discretization, shape function, interpolation
  - Finite element equation, assembly
  - B.C., solution
  - Post-processing
- Potential Flow Problem
- Elasticity Problem
- Characteristics
  - Singularity of coefficient matrix
  - Compatibility and continuity
  - Convergence

Potential Flow Problem

- Governing equation: strong form

$$\underbrace{\left.\begin{array}{l} v_x = -\dfrac{\partial \phi}{\partial x} \\ v_y = -\dfrac{\partial \phi}{\partial y} \end{array}\right\} \left(\text{or } \boldsymbol{v} = -\nabla \phi\right)}_{\text{velocity potential}} \oplus \underbrace{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \left(\text{or } \nabla^T \boldsymbol{v} = 0\right)}_{\text{continuity: incompressible}}$$

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0 \ \left(\text{or } \Delta \phi = 0\right) \ \text{ in } \Omega \\ \text{Dirichlet B.C.: } \phi = \overline{\phi} \ \text{ on } \Gamma_\phi \\ \text{Neumann B.C.: } q = \overline{q} \ \text{ on } \Gamma_q \ \text{ where } q = v_x n_x + v_y n_y = -\dfrac{\partial \phi}{\partial x} n_x - \dfrac{\partial \phi}{\partial y} n_y \end{cases}$$

- Governing equation: weak form

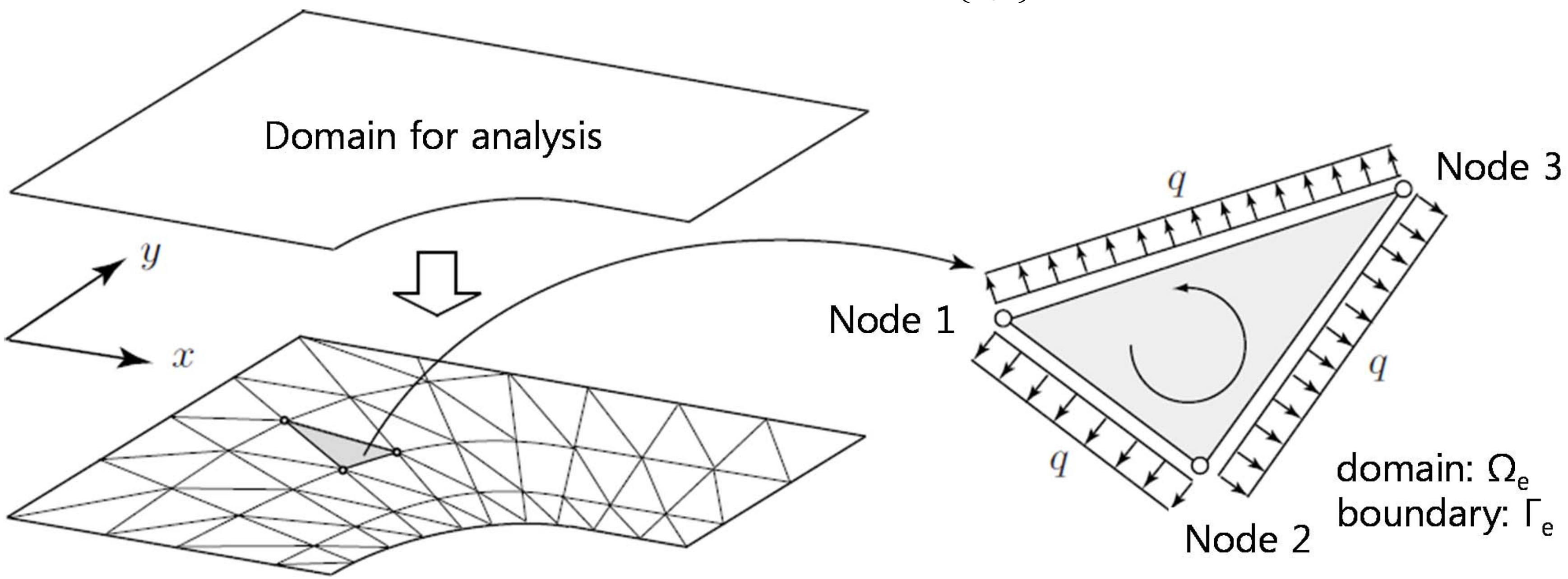
$$\xrightarrow{\phi^* = 0 \text{ on } \Gamma_\phi} \int_\Omega \phi^* \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) dV = 0 \xrightarrow{\text{Gauss-Green Theorem}} \int_\Gamma \phi^* \left(\frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y \right) dS - \int_\Omega \left(\frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi^*}{\partial y} \frac{\partial \phi}{\partial y} \right) dV = 0$$

$$\xrightarrow{\text{Boundary}} -\int_{\Gamma_\phi} \phi^* q dS - \int_{\Gamma_q} \phi^* q dS - \int_\Omega \left(\frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi^*}{\partial y} \frac{\partial \phi}{\partial y} \right) dV = 0 \xrightarrow{\text{B.C.}} \int_\Omega \left(\frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi^*}{\partial y} \frac{\partial \phi}{\partial y} \right) dV = -\int_{\Gamma_q} \phi^* \overline{q} dS$$

Discretization

$$\int_{\Omega}\left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}+\frac{\partial\phi^*}{\partial y}\frac{\partial\phi}{\partial y}\right)dV=-\int_{\Gamma_q}\phi^*\overline{q}dS\xrightarrow[\substack{dV=h_e dA\\ dS=h_e ds}]{\Omega\approx\bigcup_{e=1}^{M}\Omega_e}\sum_{e=1}^{M}\left\{\int_{\Omega_e}\left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}+\frac{\partial\phi^*}{\partial y}\frac{\partial\phi}{\partial y}\right)dV+\int_{\Gamma_e}\phi^*qdS\right\}=0$$

$$\int_{\Omega_e}\left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x}+\frac{\partial\phi^*}{\partial y}\frac{\partial\phi}{\partial y}\right)dV=-\int_{\Gamma_e}\phi^*qdS\rightarrow\int_{\Omega_e}\left\{\frac{\partial\phi^*}{\partial x}\quad\frac{\partial\phi^*}{\partial y}\right\}\left\{\begin{matrix}\dfrac{\partial\phi}{\partial x}\\ \dfrac{\partial\phi}{\partial y}\end{matrix}\right\}dV=-\int_{\Gamma_e}\phi^*qdS$$

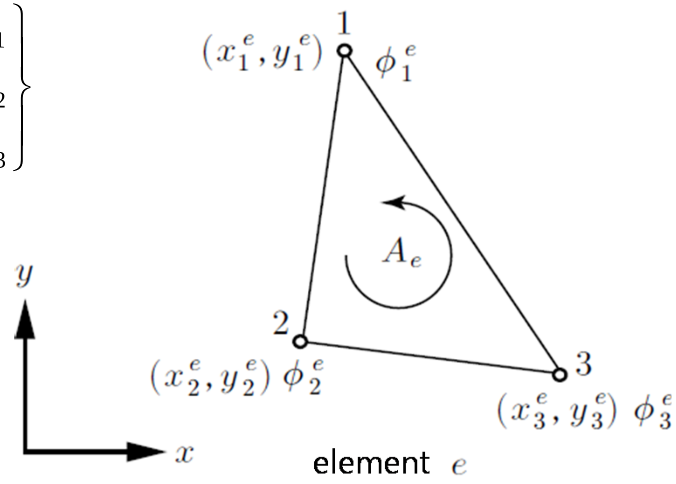


Assumed Potential

$$\phi(x,y) \approx \alpha_1 + \alpha_2 x + \alpha_3 y = \{1 \quad x \quad y\}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix}\phi_1^e\\ \phi_2^e\\ \phi_3^e\end{Bmatrix} = \begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}$$

$\alpha_i \;(i = 1,2,3)$: generalized coordinate

$$\phi(x,y) \approx \{1 \quad x \quad y\}\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}^{-1}\begin{Bmatrix}\phi_1^e\\ \phi_2^e\\ \phi_3^e\end{Bmatrix} = \{1 \quad x \quad y\}\begin{bmatrix}a_1^e & a_2^e & a_3^e\\ b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}\begin{Bmatrix}\phi_1^e\\ \phi_2^e\\ \phi_3^e\end{Bmatrix}$$



$$= \left(a_1^e + b_1^e x + c_1^e y\right)\phi_1^e + \left(a_2^e + b_2^e x + c_2^e y\right)\phi_2^e + \left(a_3^e + b_3^e x + c_3^e y\right)\phi_3^e$$

$$\text{where} \begin{cases} a_1^e = \dfrac{1}{2A_e}\left(x_2^e y_3^e - x_3^e y_2^e\right)\\ b_1^e = \dfrac{1}{2A_e}\left(y_2^e - y_3^e\right)\\ c_1^e = \dfrac{1}{2A_e}\left(x_3^e - x_2^e\right)\end{cases} \begin{cases} a_2^e = \dfrac{1}{2A_e}\left(x_3^e y_1^e - x_1^e y_3^e\right)\\ b_2^e = \dfrac{1}{2A_e}\left(y_3^e - y_1^e\right)\\ c_2^e = \dfrac{1}{2A_e}\left(x_1^e - x_3^e\right)\end{cases} \begin{cases} a_3^e = \dfrac{1}{2A_e}\left(x_1^e y_2^e - x_2^e y_1^e\right)\\ b_3^e = \dfrac{1}{2A_e}\left(y_1^e - y_2^e\right)\\ c_3^e = \dfrac{1}{2A_e}\left(x_2^e - x_1^e\right)\end{cases}$$

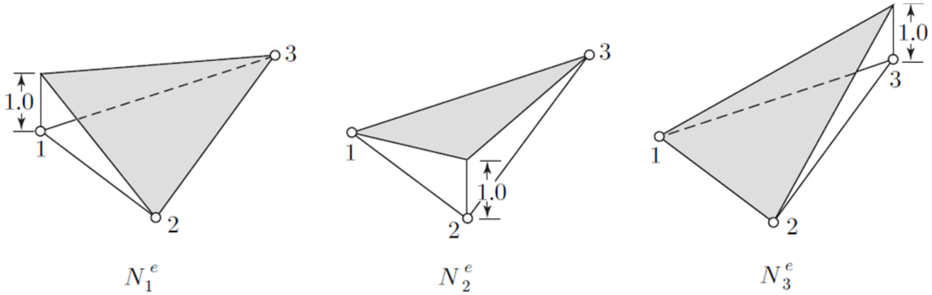
$$A_e = \frac{1}{2}\left[x_1^e\left(y_2^e - y_3^e\right) + x_2^e\left(y_3^e - y_1^e\right) + x_3^e\left(y_1^e - y_2^e\right)\right] = \frac{1}{2}\begin{vmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{vmatrix}$$

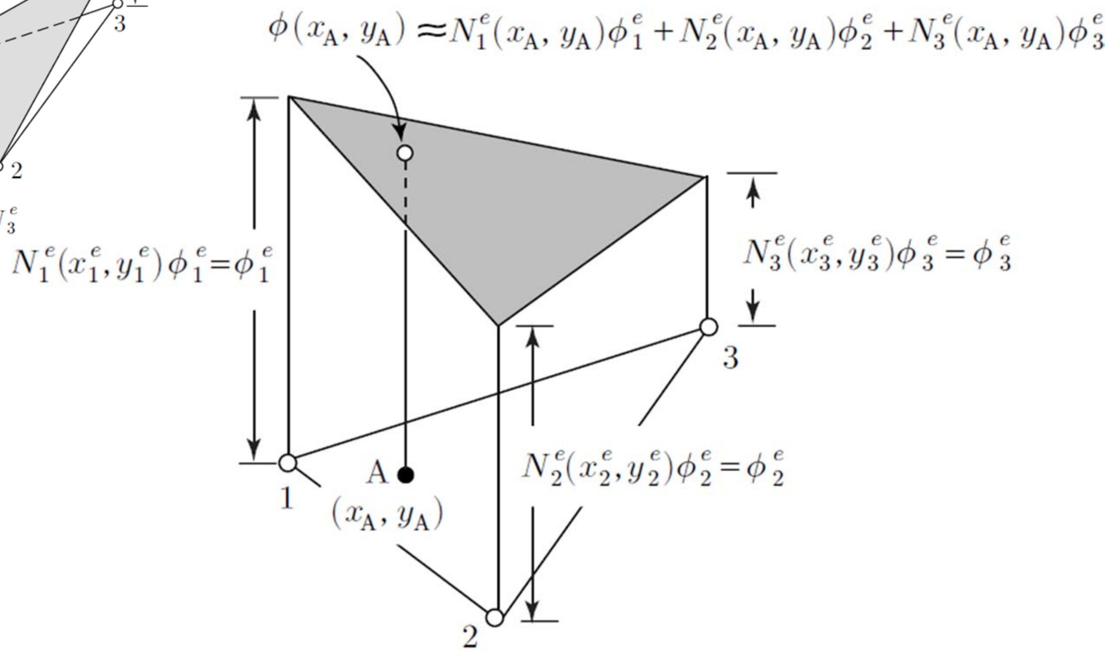
Shape Function, Interpolation

$$\phi(x,y) \approx \left\{ N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y) \right\} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = \sum_{\alpha=1}^{3} N_\alpha^e(x,y)\,\phi_\alpha^e$$

$\begin{cases} \text{linear triangle element} \\ \text{3 node triangle} \end{cases}$

$$N_\alpha^e(x,y) = a_\alpha^e + b_\alpha^e x + c_\alpha^e y \quad (\alpha = 1,2,3): \text{shape function} \rightarrow \begin{cases} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \ [\text{Kronecker } \delta] \text{ continuity} \\ \sum_{\alpha=1}^{3} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = 1 \ [\text{partition of unity}] \text{ convergence} \end{cases}$$





Finite Element Equation (1)

$$\phi \approx \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix}, \phi^* \approx \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} \phi_1^{*e} \\ \phi_2^{*e} \\ \phi_3^{*e} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} \approx \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{bmatrix} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = \begin{bmatrix} b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{bmatrix} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix}$$

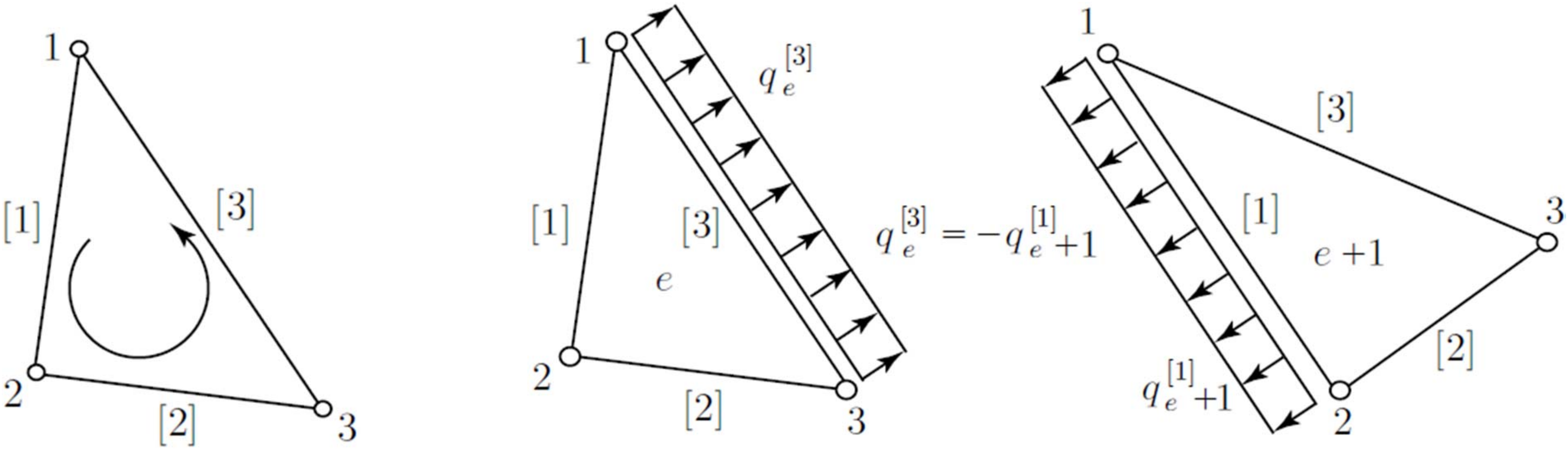
$$\int_{\Omega_e} \begin{Bmatrix} \dfrac{\partial \phi^*}{\partial x} & \dfrac{\partial \phi^*}{\partial y} \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} dV = -\int_{\Gamma_e} \phi^* q dS$$

$$\rightarrow \begin{Bmatrix} \phi_1^{*e} & \phi_2^{*e} & \phi_3^{*e} \end{Bmatrix} \int_{\Omega_e} \begin{bmatrix} b_1^e & c_1^e \\ b_2^e & c_2^e \\ b_3^e & c_3^e \end{bmatrix} \begin{bmatrix} b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{bmatrix} dV \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = -\begin{Bmatrix} \phi_1^{*e} & \phi_2^{*e} & \phi_3^{*e} \end{Bmatrix} \int_{\Gamma_e} \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} q dS$$

$$\rightarrow \int_{\Omega_e} \begin{bmatrix} b_1^e & c_1^e \\ b_2^e & c_2^e \\ b_3^e & c_3^e \end{bmatrix} \begin{bmatrix} b_1^e & b_2^e & b_3^e \\ c_1^e & c_2^e & c_3^e \end{bmatrix} dV \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = -\int_{\Gamma_e} \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} q dS$$

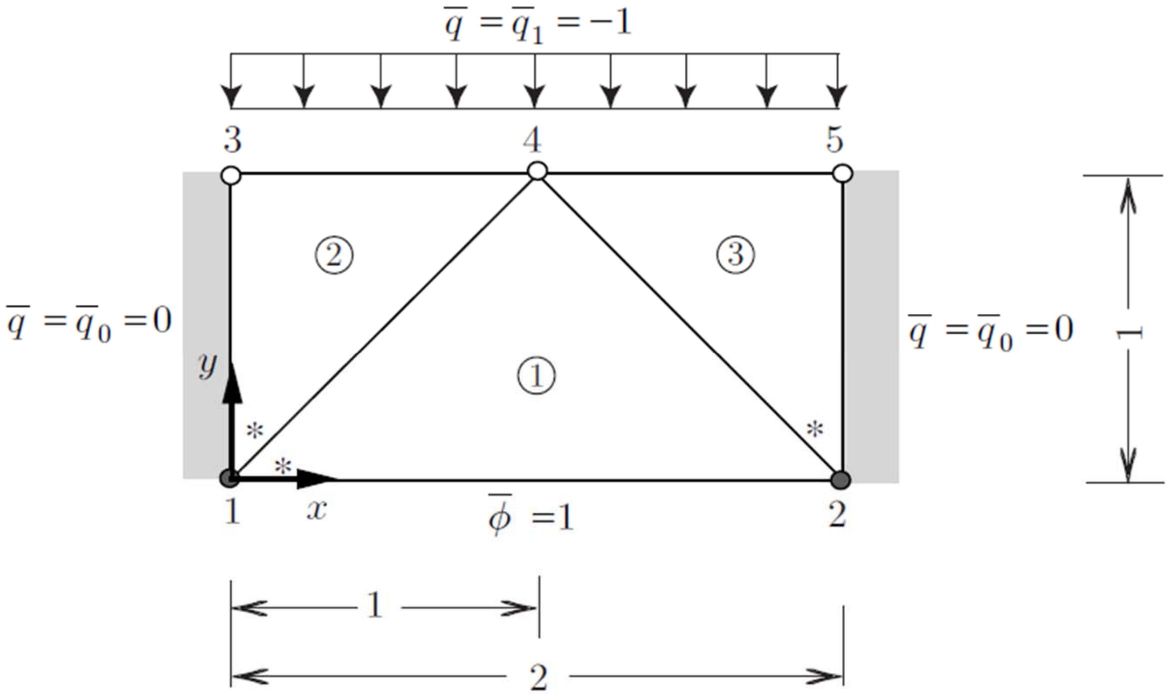
Finite Element Equation (2)

$$\xrightarrow[\text{along the boundary of element}]{\text{constant flow velocity}} -\int_{\Gamma_e}\begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} q\,dS = -\frac{q_e^{[1]}L_e^{[1]}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} - \frac{q_e^{[2]}L_e^{[2]}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} - \frac{q_e^{[3]}L_e^{[3]}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$



$$A_e\begin{bmatrix} b_1^e b_1^e & b_1^e b_2^e + c_1^e c_2^e & b_1^e b_3^e + c_1^e c_3^e \\ & b_2^e b_2^e + c_2^e c_2^e & b_2^e b_3^e + c_2^e c_3^e \\ \text{sym} & & b_3^e b_3^e + c_3^e c_3^e \end{bmatrix}\begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} = -\frac{1}{2}\begin{Bmatrix} q_e^{[1]}L_e^{[1]} + q_e^{[3]}L_e^{[3]} \\ q_e^{[1]}L_e^{[1]} + q_e^{[2]}L_e^{[2]} \\ q_e^{[2]}L_e^{[2]} + q_e^{[3]}L_e^{[3]} \end{Bmatrix}$$

Example



node	coordinates	
	x	y
1	0.0	0.0
2	2.0	0.0
3	0.0	1.0
4	1.0	1.0
5	2.0	1.0

node	potential
1	1.0
2	1.0

element	node			area	thickness
	1	2	3		
①	1	2	4	1	1
②	1	4	3	1/2	1
③	2	5	4	1/2	1

element	edge	velocity
②	[2]	-1
③	[2]	-1
②	[3]	0
③	[1]	0

Example: Finite Element Equation

element ① $\left(\text{node: } 1-2-4\right),\ \left\{\phi_1^{\textcircled{1}} \quad \phi_2^{\textcircled{1}} \quad \phi_3^{\textcircled{1}}\right\}^T \Leftrightarrow \left\{\phi_1 \quad \phi_2 \quad \phi_4\right\}^T$

$$\begin{bmatrix} k_{11}^{\textcircled{1}} & k_{12}^{\textcircled{1}} & k_{13}^{\textcircled{1}} \\ & k_{22}^{\textcircled{1}} & k_{23}^{\textcircled{1}} \\ \text{sym} & & k_{33}^{\textcircled{1}} \end{bmatrix} \begin{Bmatrix} \phi_1^{\textcircled{1}} \\ \phi_2^{\textcircled{1}} \\ \phi_3^{\textcircled{1}} \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{1}}^{[1]} L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[3]} L_{\textcircled{1}}^{[3]} \\ q_{\textcircled{1}}^{[1]} L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[2]} L_{\textcircled{1}}^{[2]} \\ q_{\textcircled{1}}^{[2]} L_{\textcircled{1}}^{[2]} + q_{\textcircled{1}}^{[3]} L_{\textcircled{1}}^{[3]} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_{11}^{\textcircled{1}} & k_{12}^{\textcircled{1}} & 0 & k_{13}^{\textcircled{1}} & 0 \\ & k_{22}^{\textcircled{1}} & 0 & k_{23}^{\textcircled{1}} & 0 \\ & & 0 & 0 & 0 \\ & \text{sym} & & k_{33}^{\textcircled{1}} & 0 \\ & & & & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{1}}^{[1]} L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[3]} L_{\textcircled{1}}^{[3]} \\ q_{\textcircled{1}}^{[1]} L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[2]} L_{\textcircled{1}}^{[2]} \\ 0 \\ q_{\textcircled{1}}^{[1]} L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[2]} L_{\textcircled{1}}^{[2]} \\ 0 \end{Bmatrix}$$

element ② $\left(\text{node: } 1-4-3\right) \left(q_{\textcircled{2}}^{[2]} = \bar{q}_1,\ q_{\textcircled{2}}^{[3]} = \bar{q}_0\right) \left\{\phi_1^{\textcircled{2}} \quad \phi_2^{\textcircled{2}} \quad \phi_3^{\textcircled{2}}\right\}^T \Leftrightarrow \left\{\phi_1 \quad \phi_4 \quad \phi_3\right\}^T$

$$\begin{bmatrix} k_{11}^{\textcircled{2}} & k_{12}^{\textcircled{2}} & k_{13}^{\textcircled{2}} \\ & k_{22}^{\textcircled{2}} & k_{23}^{\textcircled{2}} \\ \text{sym} & & k_{33}^{\textcircled{2}} \end{bmatrix} \begin{Bmatrix} \phi_1^{\textcircled{2}} \\ \phi_2^{\textcircled{2}} \\ \phi_3^{\textcircled{2}} \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{2}}^{[1]} L_{\textcircled{2}}^{[1]} + q_{\textcircled{2}}^{[3]} L_{\textcircled{2}}^{[3]} \\ q_{\textcircled{2}}^{[1]} L_{\textcircled{2}}^{[1]} + q_{\textcircled{2}}^{[2]} L_{\textcircled{2}}^{[2]} \\ q_{\textcircled{2}}^{[2]} L_{\textcircled{2}}^{[2]} + q_{\textcircled{2}}^{[3]} L_{\textcircled{2}}^{[3]} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_{11}^{\textcircled{2}} & 0 & k_{13}^{\textcircled{2}} & k_{12}^{\textcircled{2}} & 0 \\ & 0 & 0 & 0 & 0 \\ & & k_{33}^{\textcircled{2}} & k_{23}^{\textcircled{2}} & 0 \\ & \text{sym} & & k_{22}^{\textcircled{2}} & 0 \\ & & & & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{2}}^{[1]} L_{\textcircled{2}}^{[1]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ 0 \\ \bar{q}_1 L_{\textcircled{2}}^{[2]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ q_{\textcircled{2}}^{[1]} L_{\textcircled{2}}^{[1]} + \bar{q}_1 L_{\textcircled{2}}^{[2]} \\ 0 \end{Bmatrix}$$

element ③ $\left(\text{node: } 2-5-4\right) \left(q_{\textcircled{3}}^{[1]} = \bar{q}_0,\ q_{\textcircled{3}}^{[2]} = \bar{q}_1\right) \left\{\phi_1^{\textcircled{3}} \quad \phi_2^{\textcircled{3}} \quad \phi_3^{\textcircled{3}}\right\}^T \Leftrightarrow \left\{\phi_2 \quad \phi_5 \quad \phi_4\right\}^T$

$$\begin{bmatrix} k_{11}^{\textcircled{3}} & k_{12}^{\textcircled{3}} & k_{13}^{\textcircled{3}} \\ & k_{22}^{\textcircled{3}} & k_{23}^{\textcircled{3}} \\ \text{sym} & & k_{33}^{\textcircled{3}} \end{bmatrix} \begin{Bmatrix} \phi_1^{\textcircled{3}} \\ \phi_2^{\textcircled{3}} \\ \phi_3^{\textcircled{3}} \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{3}}^{[1]} L_{\textcircled{3}}^{[1]} + q_{\textcircled{3}}^{[3]} L_{\textcircled{3}}^{[3]} \\ q_{\textcircled{3}}^{[1]} L_{\textcircled{3}}^{[1]} + q_{\textcircled{3}}^{[2]} L_{\textcircled{3}}^{[2]} \\ q_{\textcircled{3}}^{[2]} L_{\textcircled{3}}^{[2]} + q_{\textcircled{3}}^{[3]} L_{\textcircled{3}}^{[3]} \end{Bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ & k_{11}^{\textcircled{3}} & 0 & k_{13}^{\textcircled{3}} & k_{12}^{\textcircled{3}} \\ & & 0 & 0 & 0 \\ & \text{sym} & & k_{33}^{\textcircled{3}} & k_{23}^{\textcircled{3}} \\ & & & & k_{22}^{\textcircled{3}} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} 0 \\ \bar{q}_0 L_{\textcircled{3}}^{[1]} + q_{\textcircled{3}}^{[3]} L_{\textcircled{3}}^{[3]} \\ 0 \\ \bar{q}_1 L_{\textcircled{3}}^{[2]} + q_{\textcircled{3}}^{[3]} L_{\textcircled{3}}^{[3]} \\ \bar{q}_0 L_{\textcircled{3}}^{[1]} + \bar{q}_1 L_{\textcircled{3}}^{[2]} \end{Bmatrix}$$

Example: Assembly

$$\begin{bmatrix} k_{11}^{\textcircled{1}}+k_{11}^{\textcircled{2}} & k_{12}^{\textcircled{1}} & k_{13}^{\textcircled{2}} & k_{13}^{\textcircled{1}}+k_{12}^{\textcircled{2}} & 0 \\ & k_{22}^{\textcircled{1}}+k_{11}^{\textcircled{3}} & 0 & k_{23}^{\textcircled{1}}+k_{13}^{\textcircled{3}} & k_{12}^{\textcircled{3}} \\ & & k_{33}^{\textcircled{2}} & k_{23}^{\textcircled{2}} & 0 \\ & sym & & k_{33}^{\textcircled{1}}+k_{22}^{\textcircled{2}}+k_{33}^{\textcircled{3}} & k_{23}^{\textcircled{3}} \\ & & & & k_{22}^{\textcircled{3}} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{1}}^{[1]}L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[3]}L_{\textcircled{1}}^{[3]} + q_{\textcircled{2}}^{[1]}L_{\textcircled{2}}^{[1]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ q_{\textcircled{1}}^{[1]}L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[2]}L_{\textcircled{1}}^{[2]} + \bar{q}_0 L_{\textcircled{3}}^{[1]} + q_{\textcircled{3}}^{[3]}L_{\textcircled{3}}^{[3]} \\ \bar{q}_1 L_{\textcircled{2}}^{[2]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ q_{\textcircled{1}}^{[1]}L_{\textcircled{1}}^{[1]} + q_{\textcircled{1}}^{[2]}L_{\textcircled{1}}^{[2]} + q_{\textcircled{2}}^{[1]}L_{\textcircled{2}}^{[1]} + \bar{q}_1 L_{\textcircled{2}}^{[2]} + \bar{q}_1 L_{\textcircled{3}}^{[2]} + q_{\textcircled{3}}^{[3]}L_{\textcircled{3}}^{[3]} \\ \bar{q}_0 L_{\textcircled{3}}^{[1]} + \bar{q}_1 L_{\textcircled{3}}^{[2]} \end{Bmatrix}$$

$$(\text{RHS}) = -\frac{1}{2} \begin{Bmatrix} q_{\textcircled{1}}^{[1]}L_{\textcircled{1}}^{[1]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ q_{\textcircled{1}}^{[1]}L_{\textcircled{1}}^{[1]} + \bar{q}_0 L_{\textcircled{3}}^{[1]} \\ \bar{q}_1 L_{\textcircled{2}}^{[2]} + \bar{q}_0 L_{\textcircled{2}}^{[3]} \\ \bar{q}_1 \left(L_{\textcircled{2}}^{[2]} + L_{\textcircled{3}}^{[2]} \right) \\ \bar{q}_0 L_{\textcircled{3}}^{[1]} + \bar{q}_1 L_{\textcircled{3}}^{[2]} \end{Bmatrix} \begin{array}{l} \Leftarrow q_{\textcircled{1}}^{[3]} + q_{\textcircled{2}}^{[1]} = 0,\ L_{\textcircled{1}}^{[3]} = L_{\textcircled{2}}^{[1]} \\ \Leftarrow q_{\textcircled{1}}^{[2]} + q_{\textcircled{3}}^{[3]} = 0,\ L_{\textcircled{1}}^{[2]} = L_{\textcircled{3}}^{[3]} \\ \Leftarrow \text{no change} \\ \Leftarrow q_{\textcircled{1}}^{[3]} + q_{\textcircled{2}}^{[1]} = 0,\ L_{\textcircled{1}}^{[3]} = L_{\textcircled{2}}^{[1]},\ q_{\textcircled{1}}^{[2]} + q_{\textcircled{3}}^{[3]} = 0,\ L_{\textcircled{1}}^{[2]} = L_{\textcircled{3}}^{[3]} \\ \Leftarrow \text{no change} \end{array}$$

Example: B.C. + Solution

$$\begin{bmatrix} k_{11}^{①}+k_{11}^{②} & k_{12}^{①} & k_{13}^{②} & k_{13}^{①}+k_{12}^{②} & 0 \\ & k_{22}^{①}+k_{11}^{③} & 0 & k_{23}^{①}+k_{13}^{③} & k_{12}^{③} \\ & & k_{33}^{②} & k_{23}^{②} & 0 \\ & sym & & k_{33}^{①}+k_{22}^{②}+k_{33}^{③} & k_{23}^{③} \\ & & & & k_{22}^{③} \end{bmatrix} \begin{Bmatrix} \phi_1 = \bar{\phi} \\ \phi_2 = \bar{\phi} \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} q_{①}^{[1]} L_{①}^{[1]} + \bar{q}_0 L_{②}^{[3]} \\ q_{①}^{[1]} L_{①}^{[1]} + \bar{q}_0 L_{③}^{[1]} \\ \bar{q}_1 L_{②}^{[2]} + \bar{q}_0 L_{②}^{[3]} \\ \bar{q}_1 \left(L_{②}^{[2]} + L_{③}^{[2]} \right) \\ \bar{q}_0 L_{③}^{[1]} + \bar{q}_1 L_{③}^{[2]} \end{Bmatrix}$$

$$\begin{bmatrix} k_{33}^{②} & k_{23}^{②} & 0 \\ & k_{33}^{①}+k_{22}^{②}+k_{33}^{③} & k_{23}^{③} \\ sym & & k_{22}^{③} \end{bmatrix} \begin{Bmatrix} \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} \bar{q}_1 L_{②}^{[2]} + \bar{q}_0 L_{②}^{[3]} \\ \bar{q}_1 \left(L_{②}^{[2]} + L_{③}^{[2]} \right) \\ \bar{q}_0 L_{③}^{[1]} + \bar{q}_1 L_{③}^{[2]} \end{Bmatrix} - \begin{Bmatrix} k_{13}^{②} \\ k_{13}^{①}+k_{12}^{②} \\ 0 \end{Bmatrix} \bar{\phi} - \begin{Bmatrix} 0 \\ k_{23}^{①}+k_{13}^{③} \\ k_{12}^{③} \end{Bmatrix} \bar{\phi}$$

$$\rightarrow \frac{1}{2} \begin{bmatrix} 1+1 & -1 & 0 \\ & 2+1+1 & -1 \\ sym & & 2 \end{bmatrix} \begin{Bmatrix} \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -1 \\ -1 \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} 0 \\ -1 \\ -1 \end{Bmatrix} \rightarrow \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ & 4 & -1 \\ sym & & 2 \end{bmatrix} \begin{Bmatrix} \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix}$$

$$\{\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \quad \phi_5\}^T = \{1 \quad 1 \quad 2 \quad 2 \quad 2\}^T$$

Example: Each Element

element ①

$$\left.\begin{array}{l}\left(x_1^{①}, y_1^{①}\right) = \left(x_1, y_1\right) = (0,0)\\ \left(x_2^{①}, y_2^{①}\right) = \left(x_2, y_2\right) = (2,0)\\ \left(x_3^{①}, y_3^{①}\right) = \left(x_4, y_4\right) = (1,1)\end{array}\right\}$$

$$A_{①} = 1/2$$

$$\begin{Bmatrix} a_1^{①} \\ b_1^{①} \\ c_1^{①} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1/2 \\ -1/2 \end{Bmatrix}, \quad \begin{Bmatrix} a_2^{①} \\ b_2^{①} \\ c_2^{①} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/2 \\ -1/2 \end{Bmatrix}, \quad \begin{Bmatrix} a_3^{①} \\ b_3^{①} \\ c_3^{①} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\left.\begin{array}{l} N_1^{①} = 1 - \frac{1}{2}x - \frac{1}{2}y \\ N_2^{①} = \frac{1}{2}x - \frac{1}{2}y \\ N_3^{①} = y \end{array}\right\}$$

$$\boldsymbol{K}_{①} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

element ②

$$\left.\begin{array}{l}\left(x_1^{②}, y_1^{②}\right) = \left(x_1, y_1\right) = (0,0)\\ \left(x_2^{②}, y_2^{②}\right) = \left(x_4, y_4\right) = (1,1)\\ \left(x_3^{②}, y_3^{②}\right) = \left(x_3, y_3\right) = (0,1)\end{array}\right\}$$

$$A_{②} = 1/2$$

$$\begin{Bmatrix} a_1^{②} \\ b_1^{②} \\ c_1^{②} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \begin{Bmatrix} a_2^{②} \\ b_2^{②} \\ c_2^{②} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} a_3^{②} \\ b_3^{②} \\ c_3^{②} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

$$\left.\begin{array}{l} N_1^{②} = 1 - y \\ N_2^{②} = x \\ N_3^{②} = -x + y \end{array}\right\}$$

$$\boldsymbol{K}_{②} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

element ③

$$\left.\begin{array}{l}\left(x_1^{③}, y_1^{③}\right) = \left(x_2, y_2\right) = (2,0)\\ \left(x_2^{③}, y_2^{③}\right) = \left(x_5, y_5\right) = (2,1)\\ \left(x_3^{③}, y_3^{③}\right) = \left(x_4, y_4\right) = (1,1)\end{array}\right\}$$

$$A_{③} = 1/2$$

$$\begin{Bmatrix} a_1^{③} \\ b_1^{③} \\ c_1^{③} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \begin{Bmatrix} a_2^{③} \\ b_2^{③} \\ c_2^{③} \end{Bmatrix} = \begin{Bmatrix} -2 \\ 1 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} a_3^{③} \\ b_3^{③} \\ c_3^{③} \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix}$$

$$\left.\begin{array}{l} N_1^{③} = 1 - y \\ N_2^{③} = -2 + x + y \\ N_3^{③} = 2 - x \end{array}\right\}$$

$$\boldsymbol{K}_{③} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

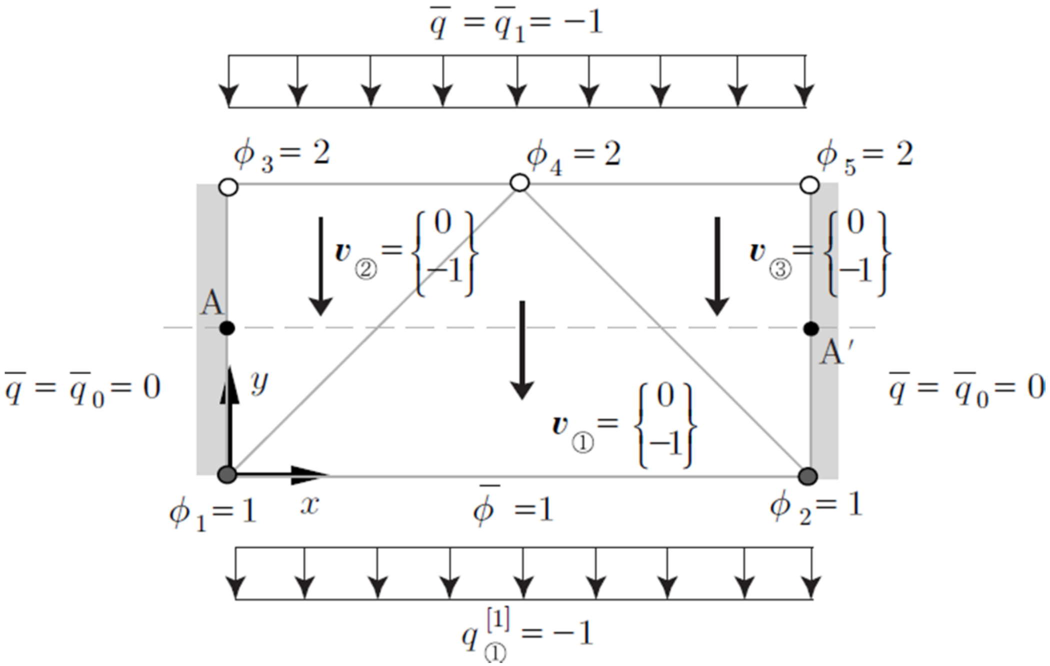
Example: Post-Process (1)

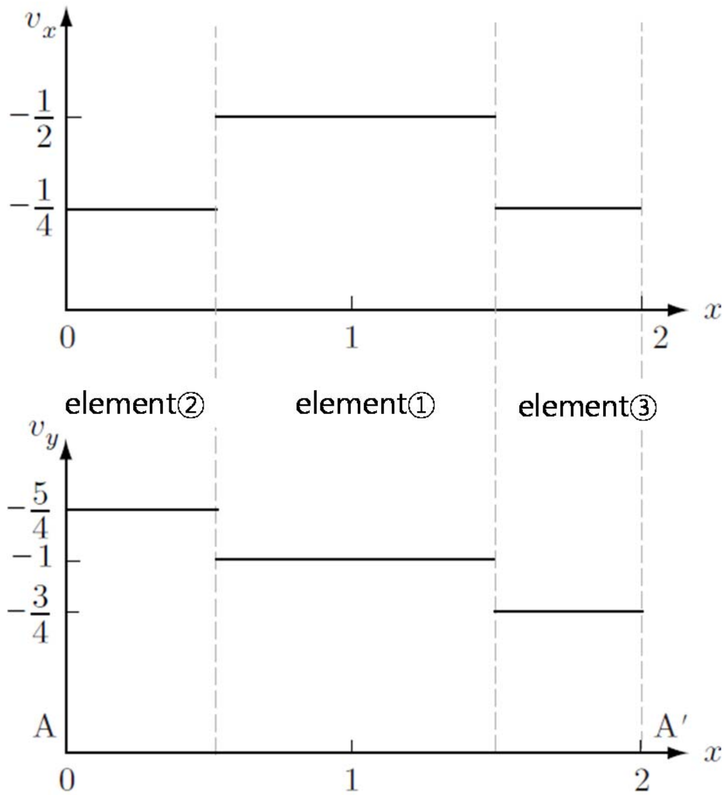
- Potential @ arbitrary point in the element: A(1, 0.5)
- Flow velocity

$$\left\{\phi_1^{①} \quad \phi_2^{①} \quad \phi_3^{①}\right\}^T = \left\{\phi_1 \quad \phi_2 \quad \phi_4\right\}^T = \left\{1 \quad 1 \quad 2\right\}^T$$

$$\phi^{①}(1,0.5) \approx N_1^{①}(1,0.5)\phi_1^{①} + N_2^{①}(1,0.5)\phi_2^{①} + N_3^{①}(1,0.5)\phi_3^{①}$$

$$= 0.25\times1.0 + 0.25\times1.0 + 0.5\times2.0 = 1.5$$





Example: Post-Process (2)

$$\boldsymbol{v}^{①}=\begin{Bmatrix} v_x^{①} \\ v_y^{①} \end{Bmatrix}=-\begin{Bmatrix} \dfrac{\partial\phi^{①}}{\partial x} \\ \dfrac{\partial\phi^{①}}{\partial y} \end{Bmatrix}\approx-\begin{bmatrix} \dfrac{\partial N_1^{①}}{\partial x} & \dfrac{\partial N_2^{①}}{\partial x} & \dfrac{\partial N_3^{①}}{\partial x} \\ \dfrac{\partial N_1^{①}}{\partial y} & \dfrac{\partial N_2^{①}}{\partial y} & \dfrac{\partial N_3^{①}}{\partial y} \end{bmatrix}\begin{Bmatrix} \phi_1^{①} \\ \phi_2^{①} \\ \phi_3^{①} \end{Bmatrix}=-\begin{bmatrix} b_1^{①} & b_2^{①} & b_3^{①} \\ c_1^{①} & c_2^{①} & c_3^{①} \end{bmatrix}\begin{Bmatrix} \phi_1^{①} \\ \phi_2^{①} \\ \phi_3^{①} \end{Bmatrix}=-\begin{bmatrix} -1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\boldsymbol{v}^{②}=\begin{Bmatrix} v_x^{②} \\ v_y^{②} \end{Bmatrix}\approx-\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ -1 \end{Bmatrix},\ \boldsymbol{v}^{③}=\begin{Bmatrix} v_x^{③} \\ v_y^{③} \end{Bmatrix}\approx-\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\text{if } \phi_2=2\overline{\phi}=2\rightarrow\{\phi_1\quad\phi_2\quad\phi_3\quad\phi_4\quad\phi_5\}^T=\left\{1\quad 2\quad \frac{9}{4}\quad \frac{5}{2}\quad \frac{11}{4}\right\}^T$$

$$\{\phi_1^{①}\quad\phi_2^{①}\quad\phi_3^{①}\}^T=\{\phi_1\quad\phi_2\quad\phi_4\}^T=\left\{1\quad 2\quad \frac{5}{2}\right\}^T$$

$$\{\phi_1^{②}\quad\phi_2^{②}\quad\phi_3^{②}\}^T=\{\phi_1\quad\phi_4\quad\phi_3\}^T=\left\{1\quad \frac{5}{2}\quad \frac{9}{4}\right\}^T$$

$$\{\phi_1^{③}\quad\phi_2^{③}\quad\phi_3^{③}\}^T=\{\phi_2\quad\phi_5\quad\phi_4\}^T=\left\{2\quad \frac{11}{4}\quad \frac{5}{2}\right\}^T$$

$$\Rightarrow$$

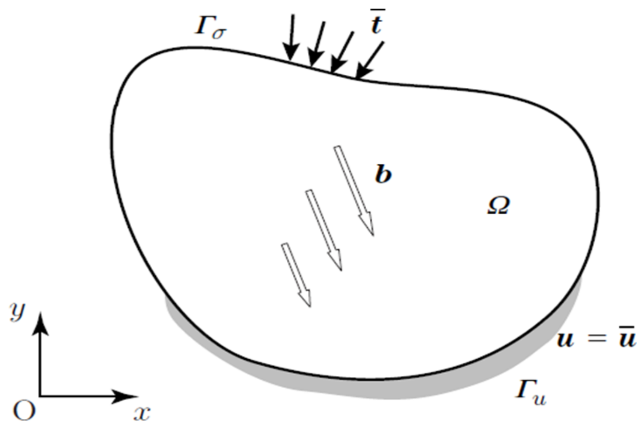
$$\boldsymbol{v}^{①}=\begin{Bmatrix} v_x^{①} \\ v_y^{①} \end{Bmatrix}\approx-\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 5/2 \end{Bmatrix}=\begin{Bmatrix} -1/2 \\ -1 \end{Bmatrix}$$

$$\boldsymbol{v}^{②}=\begin{Bmatrix} v_x^{②} \\ v_y^{②} \end{Bmatrix}\approx-\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 5/2 \\ 9/4 \end{Bmatrix}=\begin{Bmatrix} -1/4 \\ -5/4 \end{Bmatrix}$$

$$\boldsymbol{v}^{③}=\begin{Bmatrix} v_x^{③} \\ v_y^{③} \end{Bmatrix}\approx-\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 11/4 \\ 5/2 \end{Bmatrix}=\begin{Bmatrix} -1/4 \\ -3/4 \end{Bmatrix}$$

2D Elasticity Problem

- Governing equation: strong form
  - Equilibrium equation
  - Strain-displacement relation
  - Stress-strain relation (constitutive equation)
    - Plane stress / Plane strain
  - Boundary conditions
    - Geometrical or Kinematic B.C.: Displacement B.C., support condition
    - Kinetic B.C.: Load B.C., load condition



Governing Equation: Matrix Form (1)

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \end{Bmatrix},\ \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix},\ \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix},\ \partial = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

– Equilibrium equation

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = 0 \quad \left(\text{or } \partial^T \boldsymbol{\sigma} + \boldsymbol{b} = 0\right)$$

– Strain-displacement relation

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} \\ \varepsilon_y = \frac{\partial v}{\partial y} \\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{cases} \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \left(\text{or } \boldsymbol{\varepsilon} = \partial \boldsymbol{u}\right)$$

Governing Equation: Matrix Form (2)

– Stress-strain relation (constitutive equation)

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases} \rightarrow \begin{cases} \sigma_x = \dfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right] \\ \sigma_y = \dfrac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu^2\varepsilon_z\right] \\ \tau_{xy} = \dfrac{E}{2(1+\nu)}\gamma_{xy} \end{cases}$$

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ sym & & D_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (\text{or } \boldsymbol{\sigma} = \boldsymbol{D\varepsilon})$$

$$\text{plane stress}: \boldsymbol{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ & 1 & 0 \\ sym & & \dfrac{1-\nu}{2} \end{bmatrix}, \quad \text{plane strain}: \boldsymbol{D} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ & 1 & 0 \\ sym & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Governing Equation: Matrix Form (3)

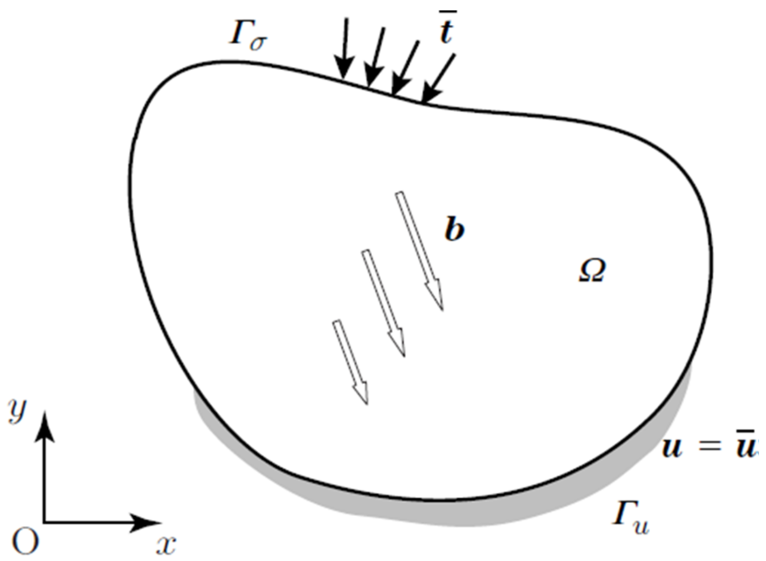
– Boundary Conditions

$$\left.\begin{array}{l} u=\bar{u} \\ v=\bar{v} \end{array}\right\} \text{ on } \Gamma_u \Rightarrow \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} \quad (\text{or } \boldsymbol{u}=\bar{\boldsymbol{u}}) \text{ on } \Gamma_u$$

$$\boldsymbol{t}=\{t_x \quad t_y\}^T \ (\text{surface force}), \boldsymbol{n}=\{n_x \quad n_y\}^T \ (\text{normal vector})$$

$$\left\{\begin{array}{l} t_x=\sigma_x n_x+\tau_{xy} n_y \\ t_y=\tau_{xy} n_x+\sigma_y n_y \end{array}\right. \Rightarrow \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} \quad (\boldsymbol{m\sigma}=\boldsymbol{t}=\bar{\boldsymbol{t}}) \text{ on } \Gamma_\sigma$$

$$\text{where } \bar{\boldsymbol{u}}=\begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix}, \ \bar{\boldsymbol{t}}=\begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix}, \ \boldsymbol{m}=\begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$$



Weak Form

Using test function $\begin{Bmatrix} u^* \\ v^* \end{Bmatrix}$ satisfying $\begin{Bmatrix} u^* \\ v^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ $\left(\text{or } \boldsymbol{u}^* = 0\right)$ on Γ_u

$$\int_\Omega \left[u^* \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x \right) + v^* \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y \right) \right] dV = 0$$

$$\int_\Omega \begin{Bmatrix} u^* & v^* \end{Bmatrix} \left(\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \right) dV = 0$$

$$\int_\Omega \boldsymbol{u}^{*T} \left(\partial^T \boldsymbol{\sigma} + \boldsymbol{b} \right) dV = 0 \xrightarrow{\text{Gauss-Green Theorem}} \underbrace{\int_\Gamma \boldsymbol{u}^{*T} \left(\boldsymbol{m\sigma} \right) dS - \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{\sigma} dV}_{} + \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV = 0$$

$$\xrightarrow{\text{kinetic B.C.}} \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{\sigma} dV = \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV + \int_{\Gamma_\sigma} \boldsymbol{u}^{*T} \bar{\boldsymbol{t}} dS \xrightarrow{\boldsymbol{\sigma} = \boldsymbol{D\varepsilon}} \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{D} \left(\partial \boldsymbol{u} \right) dV = \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV + \int_{\Gamma_\sigma} \boldsymbol{u}^{*T} \bar{\boldsymbol{t}} dS$$

$$\underbrace{\text{Weak form}\left(u^*\right)}_{\text{based on Galerkin's method}} = \underbrace{\text{Equation of Virtual Work}\left(\delta u\right)}_{\text{based on the displacement method}}$$

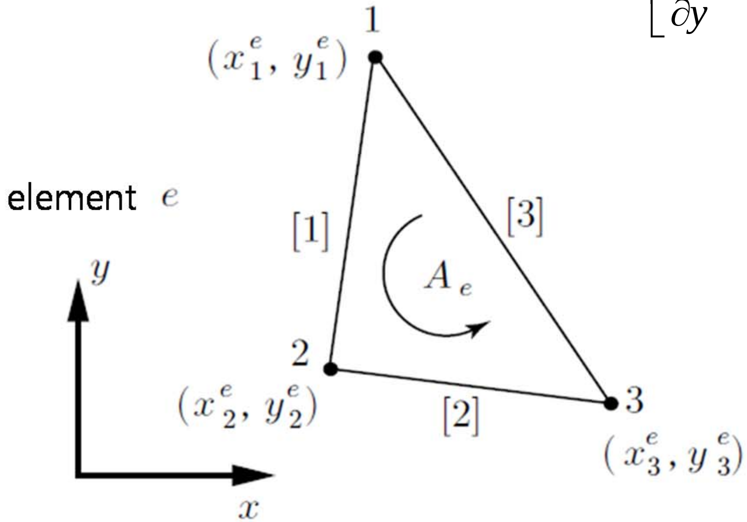
Discretization, Assumed Displacement

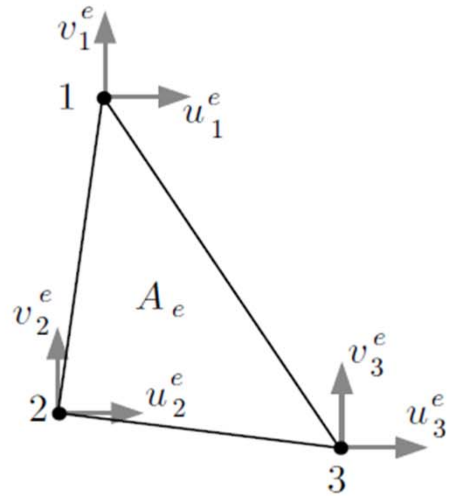
$$\int_{\Omega}\left(\partial \boldsymbol{u}^*\right)^T \boldsymbol{D}\left(\partial \boldsymbol{u}\right)dV=\int_{\Omega}\boldsymbol{u}^{*T}\boldsymbol{b}dV+\int_{\Gamma_\sigma}\boldsymbol{u}^{*T}\overline{\boldsymbol{t}}dS \xrightarrow[\substack{dV=h_e dA\\ dS=h_e ds}]{\Omega\approx\bigcup_{e=1}^{M}\Omega_e} \int_{\Omega_e}\left(\partial \boldsymbol{u}^*\right)^T \boldsymbol{D}_e\left(\partial \boldsymbol{u}\right)h_e dA=\int_{\Omega_e}\boldsymbol{u}^{*T}\boldsymbol{b}h_e dA+\int_{\Gamma_e}\boldsymbol{u}^{*T}\overline{\boldsymbol{t}}h_e ds$$

linear triangle element

CST(Constant Stain Triangle): edges remains straight after the deformation

$$\boldsymbol{u}^T=\{u \quad v\} \quad \text{where} \quad \begin{cases} u\approx\alpha_1+\alpha_2 x+\alpha_3 y \\ v\approx\beta_1+\beta_2 x+\beta_3 y\end{cases} \rightarrow \boldsymbol{\varepsilon}=\partial\boldsymbol{u}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}\alpha_1+\alpha_2 x+\alpha_3 y\\ \beta_1+\beta_2 x+\beta_3 y\end{Bmatrix}=\begin{Bmatrix}\alpha_2\\ \beta_3\\ \alpha_3+\beta_2\end{Bmatrix}$$





Shape Function

$$u \approx \{1 \quad x \quad y\}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix},\ v \approx \{1 \quad x \quad y\}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} \rightarrow \begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix},\ \begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix}$$

$$u \approx \{1 \quad x \quad y\}\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}^{-1}\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \{1 \quad x \quad y\}\begin{bmatrix}a_1^e & a_2^e & a_3^e\\ b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix}$$

$$v \approx \{1 \quad x \quad y\}\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}^{-1}\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \{1 \quad x \quad y\}\begin{bmatrix}a_1^e & a_2^e & a_3^e\\ b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix}$$

$$N_\alpha^e(x,y) = a_\alpha^e + b_\alpha^e x + c_\alpha^e y \ \ (\alpha = 1,2,3) \rightarrow \begin{cases} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = \begin{cases}1 \text{ if } \alpha = \beta\\ 0 \text{ if } \alpha \neq \beta\end{cases} \text{[Kronecker } \delta\text{] continuity}\\ \displaystyle\sum_{\alpha=1}^{3} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = 1 \text{ [partition of unity] convergence}\end{cases}$$

$$\text{where} \begin{cases} a_1^e = \dfrac{1}{2A_e}\left(x_2^e y_3^e - x_3^e y_2^e\right)\\ b_1^e = \dfrac{1}{2A_e}\left(y_2^e - y_3^e\right)\\ c_1^e = \dfrac{1}{2A_e}\left(x_3^e - x_2^e\right)\end{cases} \begin{cases} a_2^e = \dfrac{1}{2A_e}\left(x_3^e y_1^e - x_1^e y_3^e\right)\\ b_2^e = \dfrac{1}{2A_e}\left(y_3^e - y_1^e\right)\\ c_2^e = \dfrac{1}{2A_e}\left(x_1^e - x_3^e\right)\end{cases} \begin{cases} a_3^e = \dfrac{1}{2A_e}\left(x_1^e y_2^e - x_2^e y_1^e\right)\\ b_3^e = \dfrac{1}{2A_e}\left(y_1^e - y_2^e\right)\\ c_3^e = \dfrac{1}{2A_e}\left(x_2^e - x_1^e\right)\end{cases}$$

Interpolation

$$\boldsymbol{u}=\begin{Bmatrix}u\\v\end{Bmatrix}\approx\begin{bmatrix}N_1^e & 0 & N_2^e & 0 & N_3^e & 0\\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e\end{bmatrix}\begin{Bmatrix}u_1^e\\v_1^e\\u_2^e\\v_2^e\\u_3^e\\v_3^e\end{Bmatrix}=\boldsymbol{N}_e\boldsymbol{d}_e \quad \text{where} \quad \begin{cases}\boldsymbol{N}_e: \text{shape function matrix}\\ \boldsymbol{d}_e: \text{element nodal displacement vector}\end{cases}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\v\end{Bmatrix}\approx\begin{bmatrix}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}N_1^e & 0 & N_2^e & 0 & N_3^e & 0\\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e\end{bmatrix}\begin{Bmatrix}u_1^e\\v_1^e\\u_2^e\\v_2^e\\u_3^e\\v_3^e\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial N_1^e}{\partial x} & 0 & \dfrac{\partial N_2^e}{\partial x} & 0 & \dfrac{\partial N_3^e}{\partial x} & 0\\ 0 & \dfrac{\partial N_1^e}{\partial y} & 0 & \dfrac{\partial N_2^e}{\partial y} & 0 & \dfrac{\partial N_3^e}{\partial y}\\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial y} & \dfrac{\partial N_3^e}{\partial x}\end{bmatrix}\begin{Bmatrix}u_1^e\\v_1^e\\u_2^e\\v_2^e\\u_3^e\\v_3^e\end{Bmatrix}$$

$$\boldsymbol{\varepsilon}=\partial\boldsymbol{u}\approx\partial\boldsymbol{N}_e\boldsymbol{d}_e=\boldsymbol{B}_e\boldsymbol{d}_e$$

$$\xrightarrow{\text{CST}}\boldsymbol{B}_e=\begin{bmatrix}b_1^e & 0 & b_2^e & 0 & b_3^e & 0\\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e\\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e\end{bmatrix}=\frac{1}{2A_e}\begin{bmatrix}y_2^e-y_3^e & 0 & y_3^e-y_1^e & 0 & y_1^e-y_2^e & 0\\ 0 & x_3^e-x_2^e & 0 & x_1^e-x_3^e & 0 & x_2^e-x_1^e\\ x_3^e-x_2^e & y_2^e-y_3^e & x_1^e-x_3^e & y_3^e-y_1^e & x_2^e-x_1^e & y_1^e-y_2^e\end{bmatrix}$$

Element Stiffness Equation (1)

FEM based on $\begin{cases} \text{Galerkin method: weak form / test function} \left(\boldsymbol{u}^* \approx \boldsymbol{N}_e \boldsymbol{d}_e^* \right) \text{ / virtual strain} \left(\boldsymbol{\varepsilon}^* \approx \boldsymbol{B}_e \boldsymbol{d}_e^* \right) \\ \text{Displacement method: equation of vitual work / variation of displacement / strain from variation} \end{cases}$

$$\int_{\Omega_e} \left(\partial \boldsymbol{u}^*\right)^T \boldsymbol{D}_e \left(\partial \boldsymbol{u}\right) h_e dA = \int_{\Omega_e} \boldsymbol{u}^{*T} \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{u}^{*T} \boldsymbol{t} h_e ds \rightarrow \boldsymbol{d}_e^{*T} \int_{\Omega_e} \boldsymbol{B}_e^{\ T} \boldsymbol{D}_e \boldsymbol{B}_e h_e dA \boldsymbol{d}_e = \boldsymbol{d}_e^{*T} \int_{\Omega_e} \boldsymbol{N}_e^{\ T} \boldsymbol{b} h_e dA + \boldsymbol{d}_e^{*T} \int_{\Gamma_e} \boldsymbol{N}_e^{\ T} \boldsymbol{t} h_e ds$$

$$\rightarrow \boldsymbol{K}_e \boldsymbol{d}_e = \boldsymbol{F}_e \quad \text{where} \quad \begin{cases} \boldsymbol{K}_e : \text{element stiffness matrix} \\ \boldsymbol{F}_e : \text{element nodal load vector} \end{cases}$$

$$\boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}_e^{\ T} \boldsymbol{D}_e \boldsymbol{B}_e h_e dA = \int_{\Omega_e} \begin{bmatrix} b_1^e & 0 & c_1^e \\ 0 & c_1^e & b_1^e \\ b_2^e & 0 & c_2^e \\ 0 & c_2^e & b_2^e \\ b_3^e & 0 & c_3^e \\ 0 & c_3^e & b_3^e \end{bmatrix} \begin{bmatrix} D_{11}^e & D_{12}^e & D_{13}^e \\ & D_{22}^e & D_{23}^e \\ sym & & D_{33}^e \end{bmatrix} \begin{bmatrix} b_1^e & 0 & b_2^e & 0 & b_3^e & 0 \\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e \\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e \end{bmatrix} h_e dA$$

$$\boldsymbol{F}_e = \boldsymbol{F}_e^b + \boldsymbol{F}_e^t = \int_{\Omega_e} \boldsymbol{N}_e^{\ T} \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{N}_e^{\ T} \boldsymbol{t} h_e ds = \int_{\Omega_e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} h_e dA + \int_{\Gamma_e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} h_e ds$$

Element Stiffness Equation (2)

$$\left.\begin{array}{l}\text{TRI3: node increase}\\ \text{QUAD4: isoparametric}\end{array}\right\}\rightarrow \text{numerical integration!}$$

$$\xrightarrow[\text{uniform thickness}]{\text{homogeneous material}} \boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e h_e dA = A_e h_e \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e = A_e h_e \begin{bmatrix} b_1^e & 0 & c_1^e \\ 0 & c_1^e & b_1^e \\ b_2^e & 0 & c_2^e \\ 0 & c_2^e & b_2^e \\ b_3^e & 0 & c_3^e \\ 0 & c_3^e & b_3^e \end{bmatrix} \begin{bmatrix} D_{11}^e & D_{12}^e & D_{13}^e \\ & D_{22}^e & D_{23}^e \\ sym & & D_{33}^e \end{bmatrix} \begin{bmatrix} b_1^e & 0 & b_2^e & 0 & b_3^e & 0 \\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e \\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e \end{bmatrix}$$

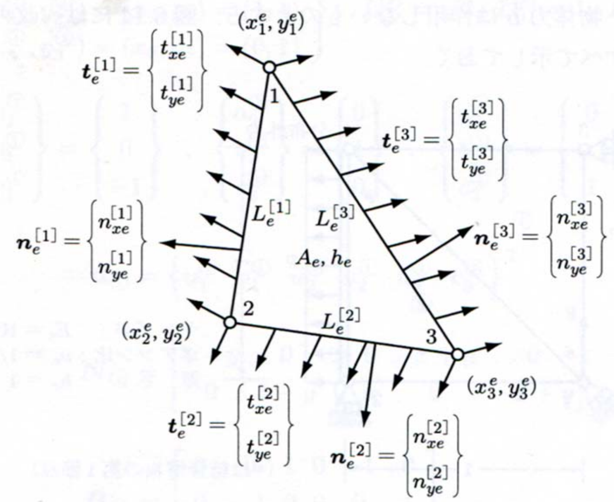
$$\xrightarrow[\text{in the element}]{\text{constant body force}} \boldsymbol{F}_e^b = \frac{A_e h_e}{3} \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix}$$

surface force along the edge neighboring elements: internal force (unknown)

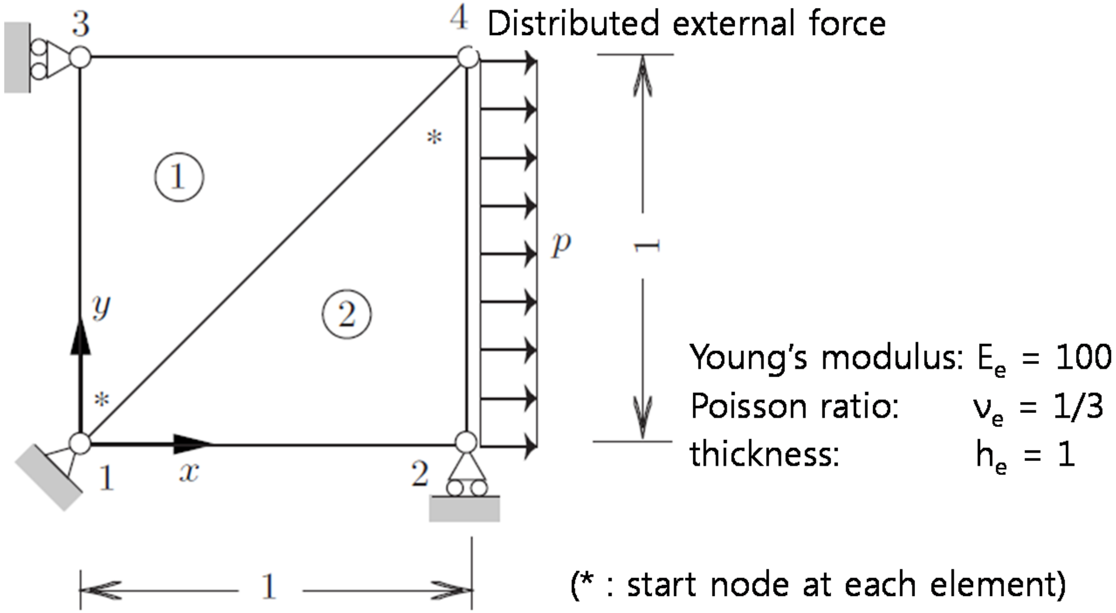
$$\begin{cases} \text{natural boundary: surface force (known)} \\ \text{essential boundary: reaction force (unknown)} \end{cases}$$

physical quantities distributed in the element $\xrightarrow[\text{based on weak form}]{\text{FEM}}$ equivalent nodal force

$$\xrightarrow[\text{along the edge}]{\text{constant surface force}} \boldsymbol{F}_e^t = \sum_{i=1}^{3} \left(\int_{\Gamma_e^{[i]}} \boldsymbol{N}_e^T ds \right) \boldsymbol{t}_e^{[i]} h_e = \frac{L_e^{[1]} h_e}{2} \begin{Bmatrix} t_{xe}^{[1]} \\ t_{ye}^{[1]} \\ t_{xe}^{[1]} \\ t_{ye}^{[1]} \\ 0 \\ 0 \end{Bmatrix} + \frac{L_e^{[2]} h_e}{2} \begin{Bmatrix} 0 \\ 0 \\ t_{xe}^{[2]} \\ t_{ye}^{[2]} \\ t_{xe}^{[2]} \\ t_{ye}^{[2]} \end{Bmatrix} + \frac{L_e^{[3]} h_e}{2} \begin{Bmatrix} t_{xe}^{[3]} \\ t_{ye}^{[3]} \\ 0 \\ 0 \\ t_{xe}^{[3]} \\ t_{ye}^{[3]} \end{Bmatrix}$$



Example



$$\boldsymbol{D}_e = \frac{E_e}{1-\nu_e^2}\begin{bmatrix} 1 & \nu_e & 0 \\ & 1 & 0 \\ sym & & \frac{1-\nu_e}{2} \end{bmatrix} \rightarrow \boldsymbol{D}_{①} = \boldsymbol{D}_{②} = \frac{100}{1-(1/3)^2}\begin{bmatrix} 1 & 1/3 & 0 \\ & 1 & 0 \\ sym & & \frac{1-1/3}{2} \end{bmatrix} = \frac{75}{2}\begin{bmatrix} 3 & 1 & 0 \\ & 3 & 0 \\ sym & & 1 \end{bmatrix}$$

element	node			area	thickness
	1	2	3		
①	1	4	3	1/2	1
②	4	1	2	1/2	1

node	coordinates	
	x	y
1	0.0	0.0
2	1.0	0.0
3	0.0	1.0
4	1.0	1.0

node	displacement	
	u	v
1	0.0	0.0
2	-	0.0
3	0.0	-

element	edge	force	
		tx	ty
①	[2]	0	0
①	[3]	-	0
②	[2]	0	-
②	[3]	p	0

Example: Element ①

$\left(\text{node: } 1-4-3\right)$ $\boldsymbol{u} = \boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{①}\boldsymbol{d}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①}$

$\boldsymbol{d}_{①} = \left\{ u_1^{①} \quad v_1^{①} \quad u_2^{①} \quad v_2^{①} \quad u_3^{①} \quad v_3^{①} \right\}^T,$

$\boldsymbol{N}_{①} =$

$\boldsymbol{B}_{①} =$

$\boldsymbol{K}_{①} =$

$\boldsymbol{F}_{①} =$

Example: Element ②

$\left(\text{node: } 4-1-2\right) \quad \boldsymbol{u} = \boldsymbol{N}_{②}\boldsymbol{d}_{②}, \ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{②}\boldsymbol{d}_{②} = \boldsymbol{B}_{②}\boldsymbol{d}_{②}$

$\boldsymbol{d}_{②} = \left\{ u_1^{②} \quad v_1^{②} \quad u_2^{②} \quad v_2^{②} \quad u_3^{②} \quad v_3^{②} \right\}^T$

$\boldsymbol{N}_{②} =$

$\boldsymbol{B}_{②} =$

$\boldsymbol{K}_{②} =$

$\boldsymbol{F}_{②} =$

Example: Assembly

$$\boldsymbol{d} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T$$

$$\boldsymbol{K} \Leftarrow \boldsymbol{K}_{①} \quad \boldsymbol{K} \Leftarrow \boldsymbol{K}_{②}$$

$$\Rightarrow \boldsymbol{K} =, \quad \boldsymbol{F} =$$

Example: Solution

$\boldsymbol{Kd} = \boldsymbol{F}$ with B.C. $u_1 = 0,\ v_1 = 0,\ v_2 = 0,\ u_3 = 0$

$$\boldsymbol{d} = \left\{ 0 \quad 0 \quad \frac{p}{100} \quad 0 \quad 0 \quad -\frac{p}{300} \quad \frac{p}{100} \quad -\frac{p}{300} \right\}^T$$

$\boldsymbol{F} =$

Example: Post-process (stress/strain)

element ① $\left(\text{node: } 1-4-3\right)$ $\boldsymbol{u}=\boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon}=\partial\boldsymbol{N}_{①}\boldsymbol{d}_{①}=\boldsymbol{B}_{①}\boldsymbol{d}_{①}$

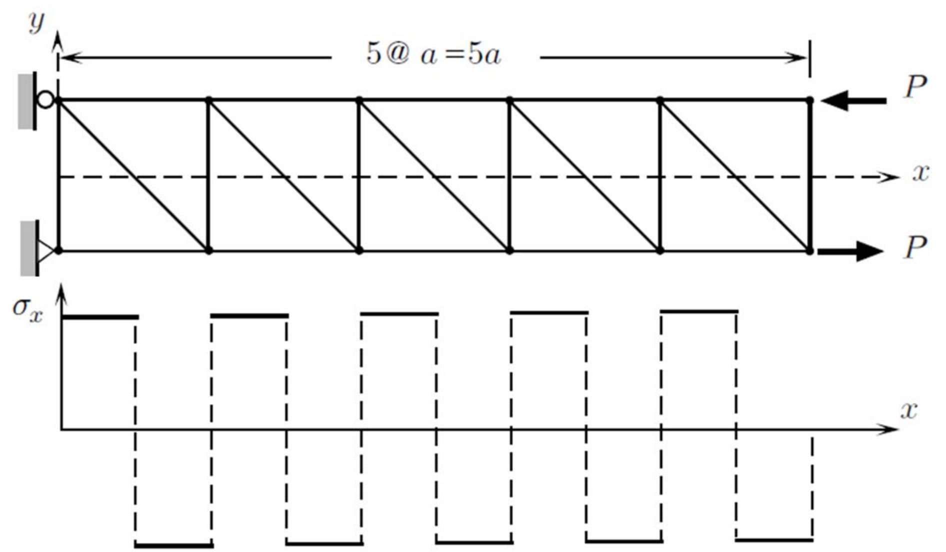
$\boldsymbol{\sigma}_{①}=\boldsymbol{D}_{①}\boldsymbol{\varepsilon}_{①}$

element ② $\left(\text{node: } 4-1-2\right)$ $\boldsymbol{u}=\boldsymbol{N}_{②}\boldsymbol{d}_{②},\ \boldsymbol{\varepsilon}=\partial\boldsymbol{N}_{②}\boldsymbol{d}_{②}=\boldsymbol{B}_{②}\boldsymbol{d}_{②}$

$\boldsymbol{\sigma}_{②}=\boldsymbol{D}_{②}\boldsymbol{\varepsilon}_{②}$

Characteristics (1)

- Singularity of stiffness matrix
  - Check with eigenvalues
  - No energy
  - Need physical constraints to prevent rigid body motion (displacement B.C., support condition)
  - Assembled matrix (singular) → apply B.C. → reduced matrix (regular)
- Approximate solution
  - Equilibrium only at nodes
  - at arbitrary point in elements?

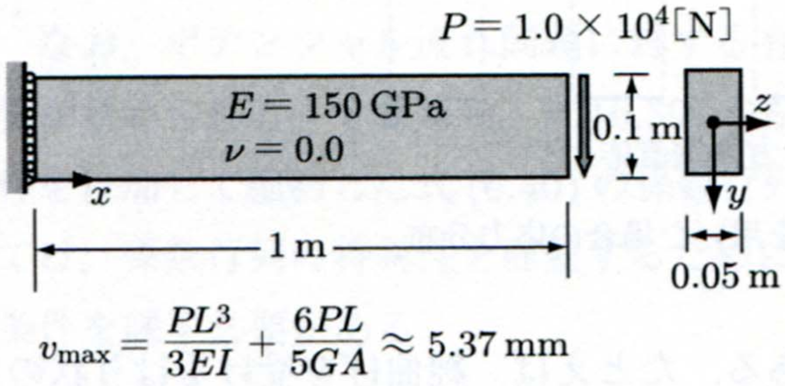


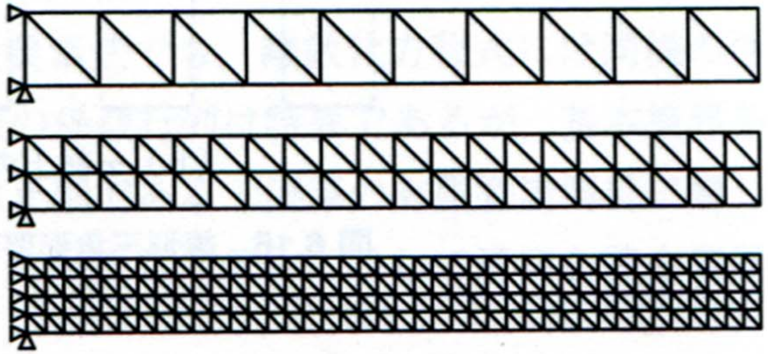
Characteristics (2)

- Compatibility and Continuity
  - Node values including edges are continuous among elements → conforming
  - Non-conforming element?
  - Linear triangular element: (displacement) linear (stress/strain) constant → C^0 continuous
  - Multi-material elastic body: strain?
- Convergence
  - The accuracy of an approximate solution improves as long as the number of nodes or elements is increased.
  - Element-type, mesh pattern

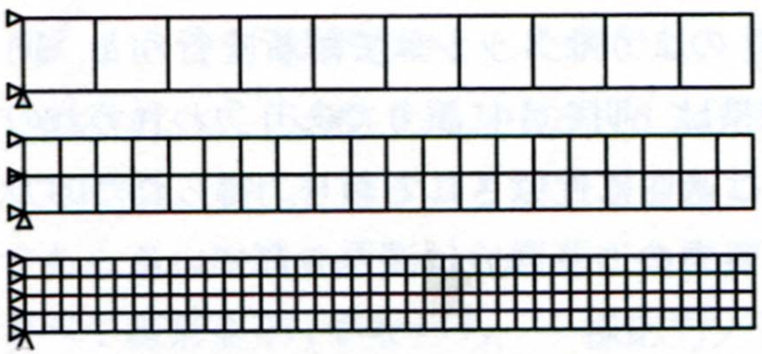
Convergence: Example (1)

- 2D elasticity (plane stress)

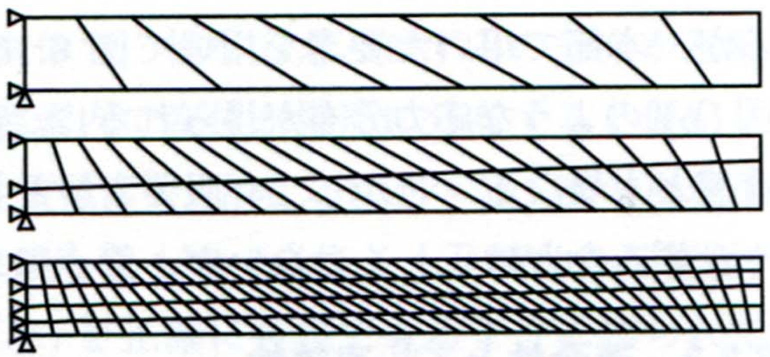




(a) Linear Triangular Element

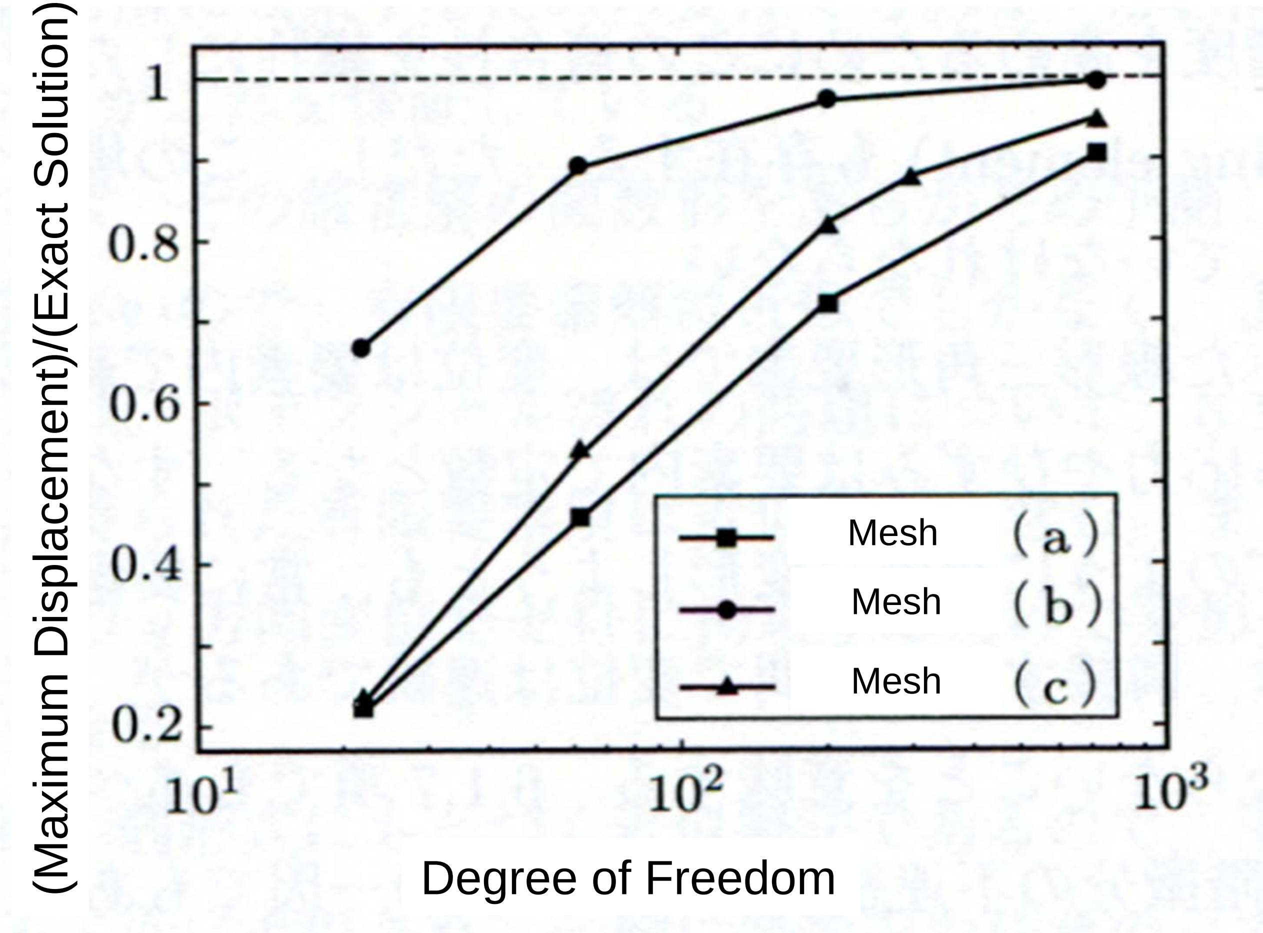


(b) Linear Quadrilateral Element



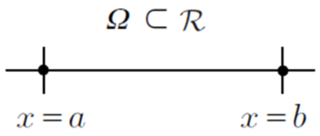
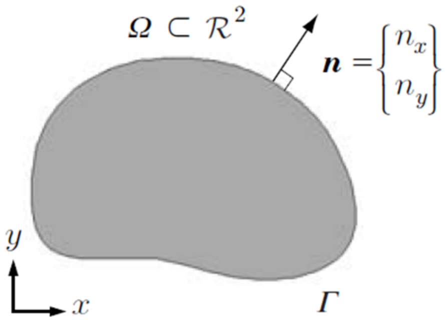
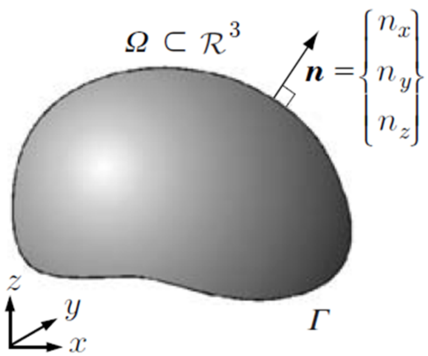
(c) Linear Quadrilateral Element (distorted)

Convergence: Example (2)



Appendix

Integral

domain	infinitesimal domain	domain integral	boundary integral
	dx	$\int_\Omega \square \, dx = \int_a^b \square \, dx$	$-f(a)$ and $f(b)$
	dA 	$\int_\Omega \square \, dA$ (multiple integral $\int_c^d \int_a^b \square \, dxdy$)	$\int_\Gamma \square \boldsymbol{n} \, ds$
	dV 	$\int_\Omega \square \, dV$ (multiple integral $\int_e^f \int_c^d \int_a^b \square \, dxdydz$)	$\int_\Gamma \square \boldsymbol{n} \, dS$

Integral Theorem

- Gauss theorem: scalar f

$$\int_{\Omega} \nabla f dV = \int_{\Gamma} f \mathbf{n} dS \quad \text{where } \mathbf{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} \cos(x,\mathbf{n}) \\ \cos(y,\mathbf{n}) \\ \cos(z,\mathbf{n}) \end{Bmatrix}$$

- Gauss's divergence theorem: vector w

$$\int_{\Omega} \nabla^T \mathbf{w} dV = \int_{\Gamma} \mathbf{w}^T \mathbf{n} dS$$

- Green-Gauss's theorem

$$\mathbf{w} = v\nabla u \rightarrow \nabla^T \mathbf{w} = \nabla^T (v\nabla u) = (\nabla^T v)(\nabla u) + v(\nabla^T \nabla u) = (\nabla^T v)(\nabla u) + v\Delta u$$

$$\Rightarrow v\Delta u = \nabla^T (v\nabla u) - (\nabla^T v)(\nabla u)$$

$$\int_{\Omega} v\Delta u dV = \int_{\Gamma} (v\nabla u)^T \mathbf{n} dS - \int_{\Omega} (\nabla^T v)(\nabla u) dV$$

$$\text{1D}: \int_a^b v \frac{d^2 u}{dx^2} dx = \left(v \frac{du}{dx}\bigg|_{x=b} - v \frac{du}{dx}\bigg|_{x=a} \right) - \int_a^b \frac{dv}{dx}\frac{du}{dx} dx \quad (\text{integration by parts})$$

$$\text{2D}: \int_{\Omega} v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \right) dA = \int_{\Gamma} \left(v \frac{\partial u}{\partial x} n_x + v \frac{\partial u}{\partial x} n_y \right) ds - \int_{\Omega} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) dA$$