# FEM for Unsteady Problems

- Discretization: space / time
- Parabolic problem
- Hyperbolic problem
- Vibration problem

# Discretization: Time

- Finite difference method
- Finite element method: space-time FEM
  - Moving boundary problem w.r.t time-varying analysis domain
  - Dimension of element: increased including time

- PDE → Discretization in space → unknown: time → ODE

# Finite Difference Method: Time

Taylor's expansion @$t \pm \Delta t$

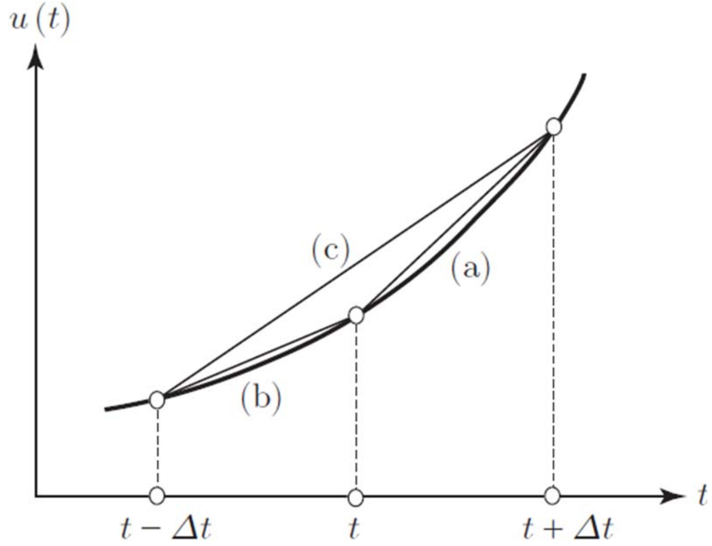
$$u(t+\Delta t) = u(t) + \Delta t \frac{du(t)}{dt} + \frac{1}{2}(\Delta t)^2 \frac{d^2u(t)}{dt^2} + O\left((\Delta t)^3\right)$$

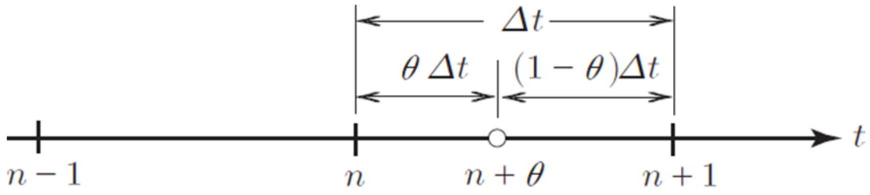
$$(1) \xrightarrow{\text{forward difference}} \frac{du(t)}{dt} = \dot{u}(t) = \frac{1}{\Delta t}\left\{u(t+\Delta t) - u(t)\right\} + O(\Delta t)$$

$$u(t-\Delta t) = u(t) - \Delta t \frac{du(t)}{dt} + \frac{1}{2}(\Delta t)^2 \frac{d^2u(t)}{dt^2} + O\left((\Delta t)^3\right)$$

$$(2) \xrightarrow{\text{backward difference}} \frac{du(t)}{dt} = \dot{u}(t) = \frac{1}{\Delta t}\left\{u(t) - u(t-\Delta t)\right\} + O(\Delta t)$$

$$(1)-(2) \xrightarrow{\text{central difference}} \frac{du(t)}{dt} = \dot{u}(t) = \frac{1}{2\Delta t}\left\{u(t+\Delta t) - u(t-\Delta t)\right\} + O\left((\Delta t)^2\right)$$






$$\frac{d}{dt}u^{n+\theta} = ?,\quad u^{n+\theta} = ?$$

$$\begin{cases} u^{n} = u^{n+\theta} - \theta\Delta t \dfrac{d}{dt}u^{n+\theta} + \dfrac{1}{2}(\theta\Delta t)^{2}\dfrac{d^{2}}{dt^{2}}u^{n+\theta} - \dfrac{1}{6}(\theta\Delta t)^{3}\dfrac{d^{3}}{dt^{3}}u^{n+\theta} + \cdots \\ u^{n+1} = u^{n+\theta} + (1-\theta)\Delta t\dfrac{d}{dt}u^{n+\theta} + \dfrac{1}{2}\left\{(1-\theta)\Delta t\right\}^{2}\dfrac{d^{2}}{dt^{2}}u^{n+\theta} + \dfrac{1}{6}\left\{(1-\theta)\Delta t\right\}^{3}\dfrac{d^{3}}{dt^{3}}u^{n+\theta} + \cdots \end{cases}$$

$$\begin{cases} (1-\theta)u^{n} = (1-\theta)u^{n+\theta} - (1-\theta)\theta\Delta t\dfrac{d}{dt}u^{n+\theta} + \dfrac{1}{2}(1-\theta)(\theta\Delta t)^{2}\dfrac{d^{2}}{dt^{2}}u^{n+\theta} + \dfrac{1}{6}(1-\theta)(\theta\Delta t)^{3}\dfrac{d^{3}}{dt^{3}}u^{n+\theta} + \cdots \\ \theta u^{n+1} = \theta u^{n+\theta} + \theta(1-\theta)\Delta t\dfrac{d}{dt}u^{n+\theta} + \dfrac{1}{2}\theta\left\{(1-\theta)\Delta t\right\}^{2}\dfrac{d^{2}}{dt^{2}}u^{n+\theta} + \dfrac{1}{6}\theta\left\{(1-\theta)\Delta t\right\}^{3}\dfrac{d^{3}}{dt^{3}}u^{n+\theta} + \cdots \end{cases}$$

# Parabolic Problem

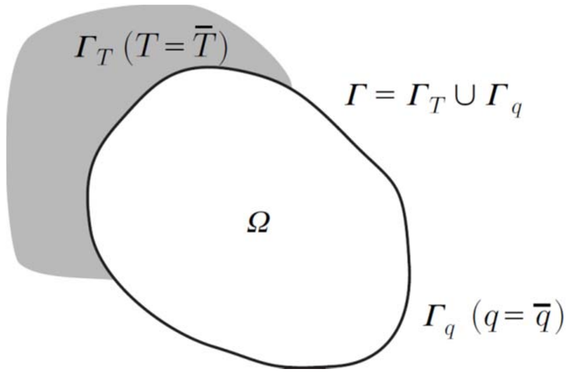
- 2D unsteady heat conduction
  - Governing equation and Initial/Boundary Conditions

$$\rho c \frac{\partial T}{\partial t} = k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} \quad \text{in } \Omega$$

$$T = \overline{T} \quad \text{on } \Gamma_T$$

$$q = k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y = \overline{q} \quad \text{on } \Gamma_q$$

$$T = T^0 \ @\, t = 0 \left( n = 0 \right)$$




  - Weak form and discretization in space

$$\text{weak form} \xrightarrow{T^*=0 \text{ on } \Gamma_T} \int_\Omega T^* \left( \rho c \frac{\partial T}{\partial t} \right) dV = \int_\Omega T^* \left( k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} \right) dV$$

$$\text{discretization} \xrightarrow{\Omega \approx \bigcup_{e=1}^{M} \Omega_e} \int_{\Omega_e} T^* \left( \rho c \frac{\partial T}{\partial t} \right) dV = \int_{\Omega_e} T^* \left( k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} \right) dV$$

$$\int_{\Omega_e} T^* \left( \rho c \frac{\partial T}{\partial t} \right) dV + \int_{\Omega_e} \left( k_x \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} + k_y \frac{\partial T^*}{\partial y} \frac{\partial T}{\partial y} \right) dV = \int_{\Gamma_e} T^* q dS$$

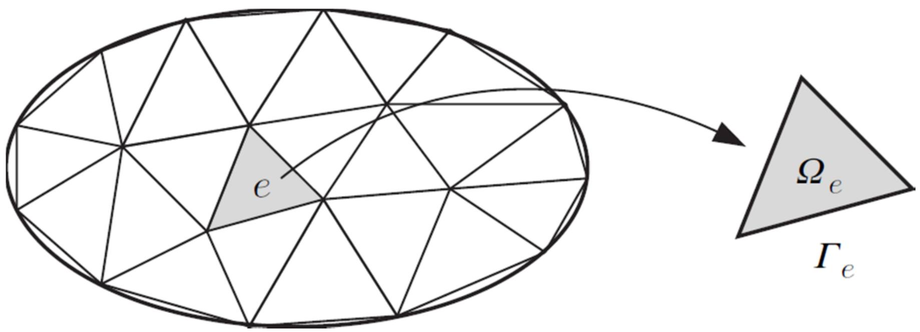
# Shape Function & Interpolation (1)

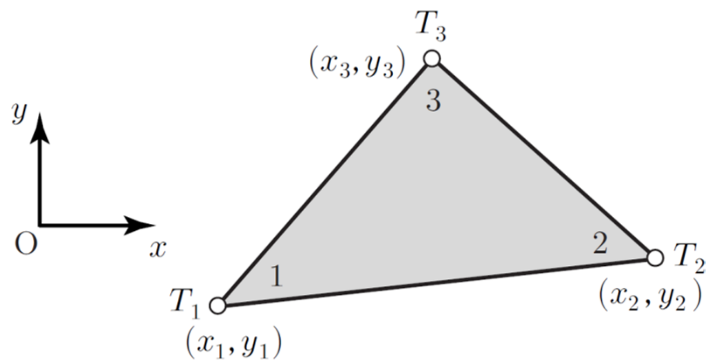
$$T(x,y,t) \approx N_1^e(x,y)T_1^e(t) + N_2^e(x,y)T_2^e(t) + N_3^e(x,y)T_3^e(t) = \sum_{\alpha=1}^{3} N_\alpha^e(x,y)T_\alpha^e(t)$$

$$N_\alpha^e(x,y) = a_\alpha^e + b_\alpha^e x + c_\alpha^e y \quad (\alpha = 1,\ 2,\ 3)$$

$$\left.\begin{aligned} a_\alpha^e &= \frac{1}{2A_e}\left(x_\beta^e y_\gamma^e - x_\gamma^e y_\beta^e\right) \\ b_\alpha^e &= \frac{1}{2A_e}\left(y_\beta^e - y_\gamma^e\right) \\ c_\alpha^e &= \frac{1}{2A_e}\left(x_\gamma^e - x_\beta^e\right) \end{aligned}\right\}(\alpha,\beta,\gamma) = (1,2,3),(2,3,1),(3,1,2)$$

$$A_e = \frac{1}{2}\left[\left(x_1^e - x_3^e\right)\left(y_2^e - y_3^e\right) - \left(y_3^e - y_1^e\right)\left(x_3^e - x_2^e\right)\right]$$



# Shape Function & Interpolation (2)

$$T \approx N_1^e T_1^e + N_2^e T_2^e + N_3^e T_3^e = \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \end{Bmatrix} = \mathbf{N}_e \mathbf{T}_e$$

$$T^* \approx N_1^e T_1^{*e} + N_2^e T_2^{*e} + N_3^e T_3^{*e} = \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} T_1^{*e} \\ T_2^{*e} \\ T_3^{*e} \end{Bmatrix} = \mathbf{N}_e \mathbf{T}_e^*$$

$$\frac{\partial T}{\partial t} \approx \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial T_1^e}{\partial t} \\ \dfrac{\partial T_2^e}{\partial t} \\ \dfrac{\partial T_3^e}{\partial t} \end{Bmatrix} = \mathbf{N}_e \dot{\mathbf{T}}_e$$

$$\frac{\partial T}{\partial t} \approx \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial T_1^e}{\partial t} \\ \dfrac{\partial T_2^e}{\partial t} \\ \dfrac{\partial T_3^e}{\partial t} \end{Bmatrix} = \mathbf{N}_e \dot{\mathbf{T}}_e$$

$$\frac{\partial T}{\partial x} \approx \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \end{Bmatrix} = \begin{Bmatrix} b_1^e & b_2^e & b_3^e \end{Bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \end{Bmatrix} = \mathbf{B}_e \mathbf{T}_e$$

$$\frac{\partial T^*}{\partial x} \approx \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} T_1^{*e} \\ T_2^{*e} \\ T_3^{*e} \end{Bmatrix} = \begin{Bmatrix} b_1^e & b_2^e & b_3^e \end{Bmatrix} \begin{Bmatrix} T_1^{*e} \\ T_2^{*e} \\ T_3^{*e} \end{Bmatrix} = \mathbf{B}_e \mathbf{T}_e^*$$

$$\frac{\partial T}{\partial y} \approx \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{Bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \end{Bmatrix} = \begin{Bmatrix} c_1^e & c_2^e & c_3^e \end{Bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \end{Bmatrix} = \mathbf{C}_e \mathbf{T}_e$$

$$\frac{\partial T^*}{\partial y} \approx \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{Bmatrix} \begin{Bmatrix} T_1^{*e} \\ T_2^{*e} \\ T_3^{*e} \end{Bmatrix} = \begin{Bmatrix} c_1^e & c_2^e & c_3^e \end{Bmatrix} \begin{Bmatrix} T_1^{*e} \\ T_2^{*e} \\ T_3^{*e} \end{Bmatrix} = \mathbf{C}_e \mathbf{T}_e^*$$

# Finite Element Equation

$$\int_{\Omega_e} T^* \left( \rho c \frac{\partial T}{\partial t} \right) dV + \int_{\Omega_e} \left( k_x \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} + k_y \frac{\partial T^*}{\partial y} \frac{\partial T}{\partial y} \right) dV = \int_{\Gamma_e} T^* q dS$$

$\rightarrow$

$$\mathbf{M}_e \dot{\mathbf{T}}_e + \mathbf{K}_e \mathbf{T}_e = \mathbf{F}_e \quad \text{where} \quad \begin{cases} \mathbf{M}_e = \int_{\Omega_e} \rho c \mathbf{N}_e{}^T \mathbf{N}_e dV & \text{(heat capacity matrix / mass matrix)} \\ \mathbf{K}_e = \int_{\Omega_e} \left( k_x \mathbf{B}_e{}^T \mathbf{B}_e + k_y \mathbf{C}_e{}^T \mathbf{C}_e \right) dV & \text{(heat conductivity matrix)} \\ \mathbf{F}_e = \int_{\Gamma_e} \mathbf{N}_e{}^T q dS & \text{(heat flux vector)} \end{cases}$$

# Area Coordinate

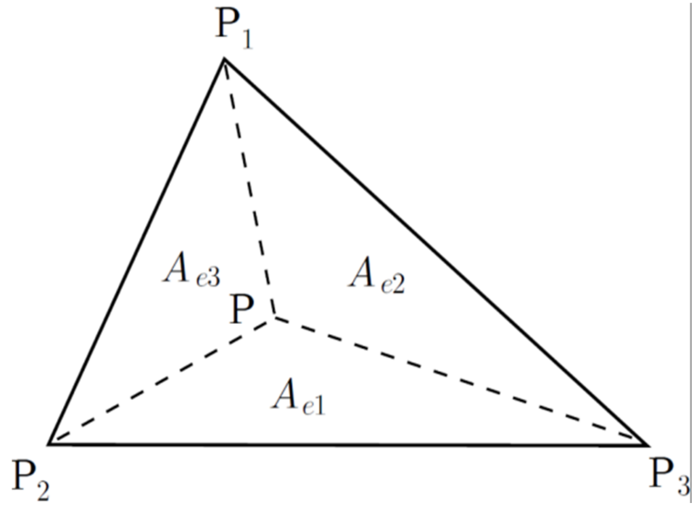
$$L_1^e = \frac{A_{e1}}{A_e},\ L_2^e = \frac{A_{e2}}{A_e},\ L_3^e = \frac{A_{e3}}{A_e}$$

$$L_1^e + L_2^e + L_3^e = 1$$

$$L_1^e = N_1^e,\ L_2^e = N_2^e,\ L_3^e = N_3^e$$

$$\frac{\partial L_i^e}{\partial x} = b_i^e,\quad \frac{\partial L_i^e}{\partial y} = c_i^e$$

$$\int_{\Omega_e} \left(L_1^e\right)^l \left(L_2^e\right)^m \left(L_3^e\right)^n dV = \frac{l!m!n!}{\left(l+m+n+2\right)!}\left(2A_e\right)$$



# Element Matrix (1)

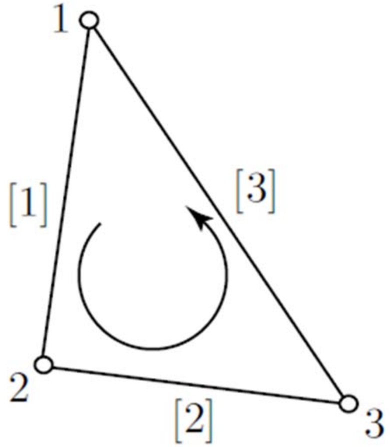
$$\mathbf{M}_e = \int_{\Omega_e} \rho c \mathbf{N}_e^T \mathbf{N}_e dV \rightarrow$$

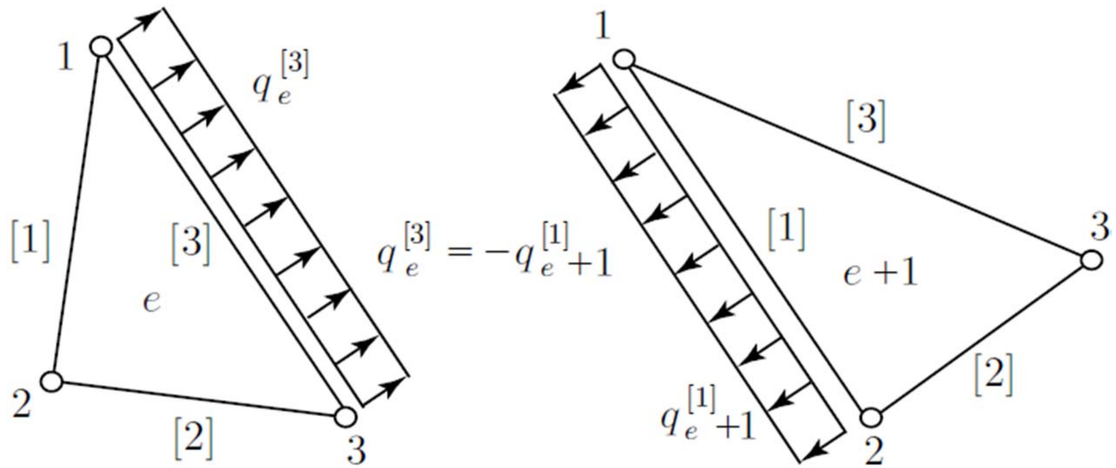
$$\mathbf{K}_e = \int_{\Omega_e} \left( k_x \mathbf{B}_e^T \mathbf{B}_e + k_y \mathbf{C}_e^T \mathbf{C}_e \right) dV \rightarrow$$

# Element Matrix (2)

$$\boldsymbol{F}_e = \int_{\Gamma_e} \boldsymbol{N}_e^{\,T} q dS \quad (\text{heat flux vector})$$

$$\xrightarrow[\text{along the boundary of element}]{\text{constant heat flux}} \int_{\Gamma_e} \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} q dS = \frac{q_e^{[1]} L_e^{[1]}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \frac{q_e^{[2]} L_e^{[2]}}{2} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \frac{q_e^{[3]} L_e^{[3]}}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} q_e^{[1]} L_e^{[1]} + q_e^{[3]} L_e^{[3]} \\ q_e^{[1]} L_e^{[1]} + q_e^{[2]} L_e^{[2]} \\ q_e^{[2]} L_e^{[2]} + q_e^{[3]} L_e^{[3]} \end{Bmatrix}$$




$$\Rightarrow \mathbf{M}\dot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F}$$

# Discretization: Time

$$\mathbf{M}\dot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F} \xrightarrow[\text{time step}]{n+\theta(0\le\theta\le1)} \mathbf{M}\dot{\mathbf{T}}^{n+\theta} + \mathbf{K}\mathbf{T}^{n+\theta} = \mathbf{F}$$

$$\begin{cases} \mathbf{T}^{n+\theta} = (1-\theta)\mathbf{T}^{n} + \theta\mathbf{T}^{n+1} \\ \dot{\mathbf{T}}^{n+\theta} = \dfrac{1}{\Delta t}\left(\mathbf{T}^{n+1} - \mathbf{T}^{n}\right) \end{cases}$$

# Mass Matrix

[forward difference]

$$\frac{1}{\Delta t}\mathbf{M}\mathbf{T}^{n+1} = \mathbf{F} + \left(\frac{1}{\Delta t}\mathbf{M} - \mathbf{K}\right)\mathbf{T}^{n} \rightarrow \text{obtain } \mathbf{T}^{n+1} \text{ by solving the equation (implicit method)}$$

$\downarrow$ use lumped mass (diagonal only)

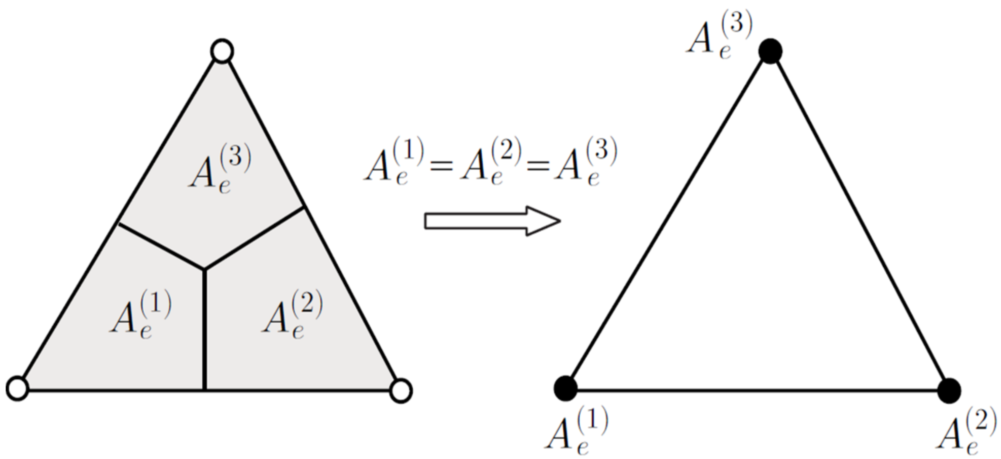
$$\frac{1}{\Delta t}\overline{\mathbf{M}}\mathbf{T}^{n+1} = \mathbf{F} + \left(\frac{1}{\Delta t}\mathbf{M} - \mathbf{K}\right)\mathbf{T}^{n}$$

$\downarrow$

$$\mathbf{T}^{n+1} = \overline{\mathbf{M}}^{-1}\Delta t\left\{\mathbf{F} + \left(\frac{1}{\Delta t}\mathbf{M} - \mathbf{K}\right)\mathbf{T}^{n}\right\} \rightarrow \text{obtain } \mathbf{T}^{n+1} \text{ by only multiplication (explicit method)}$$

$$\text{mass}\begin{cases}\text{lumped mass}\\ \text{consistent mass}\end{cases}$$

$$\overline{\mathbf{M}}_e = \rho c\begin{bmatrix} A_e/3 & 0 & 0 \\ 0 & A_e/3 & 0 \\ 0 & 0 & A_e/3 \end{bmatrix} = \rho c\frac{A_e}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Implicit vs. Explicit (1)

| | Implicit method | Explicit method |
|---|---|---|
| Solution method | Matrix operation | algebra |
| Time to solve (each time step) | Long | Short |
| Time step size | Not limited (can be large) | Stability condition (relatively small) |
| Memory | Large | Small |
| Total time elapsed | ? | ? |

$$\Delta t \le \frac{1}{2}\frac{h^2}{k} \quad \left(\Delta t \downarrow \text{ as } h \downarrow \text{ or } k \uparrow\right)$$

$h$ : element size

$k$ : heat conductivity / diffusion coefficient

For parabolic problems (heat conduction/diffusion),

- Physical quantity is propagated fast for overall region
- Phenomenon change w.r.t time is not that fast
- **Implicit method** is better than explicit method

# Hyperbolic Problem

- Governing equation
- Space discretization
- Time discretization
- Multi-step implicit method
- Stability condition for implicit method
- Stabilization method

# 1D Advection Problem

- Governing equation and Initial/Boundary Conditions

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \text{ in } \Omega \text{ where } 0 \le t \le T\left(\text{time}\right), 0 \le x \le L\left(\text{space}\right)$$

$$u\left(0,t\right) = u\left(L,t\right) = \overline{u}$$

$$u\left(x,0\right) = u^0\left(x\right)$$

- Weak form and discretization

$$\text{weak form} \xrightarrow{u^*=0 \text{ on } x=0,L} \int_0^L u^* \frac{\partial u}{\partial t} dx + \int_0^L u^* c \frac{\partial u}{\partial x} dx = 0$$

$$\text{discretization} \xrightarrow[\text{line element}]{\Omega \approx \bigcup_{e=1}^{M} \Omega_e} \sum_{e=1}^{M} \int_{x_1^e}^{x_2^e} u^* \frac{\partial u}{\partial t} dx + \sum_{e=1}^{M} \int_{x_1^e}^{x_2^e} u^* c \frac{\partial u}{\partial x} dx = 0$$

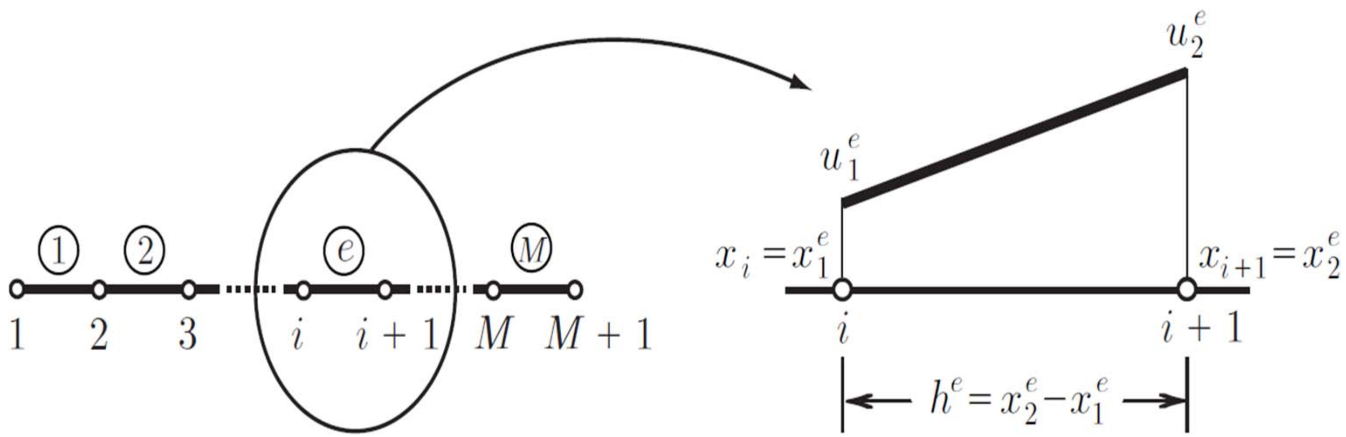
$$\int_{x_1^e}^{x_2^e} u^* \frac{\partial u}{\partial t} dx + \int_{x_1^e}^{x_2^e} u^* c \frac{\partial u}{\partial x} dx = 0 \text{ where } h_e \left(= x_2^e - x_1^e\right)$$

# Shape Function & Interpolation

$$u(x) \approx \frac{x_2^e - x}{h_e} u_1^e + \frac{x - x_1^e}{h_e} u_2^e = N_1^e(x) u_1^e + N_2^e(x) u_2^e = \begin{Bmatrix} N_1^e & N_2^e \end{Bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \mathbf{N}_e \mathbf{u}_e$$

$$N_1^e(x) = \frac{x_2^e - x}{h_e}, \; N_2^e(x) = \frac{x - x_1^e}{h_e}$$

$$u^*(x) \approx N_1^e(x) u_1^{*e} + N_2^e(x) u_2^{*e} = \begin{Bmatrix} N_1^e & N_2^e \end{Bmatrix} \begin{Bmatrix} u_1^{*e} \\ u_2^{*e} \end{Bmatrix} = \mathbf{N}_e \mathbf{u}_e^*$$



# Discretization: Space (1D)

$$\frac{\partial u}{\partial t} \approx N_1^e \frac{\partial u_1^e}{\partial t} + N_2^e \frac{\partial u_2^e}{\partial t} = \begin{Bmatrix} N_1^e & N_2^e \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial u_1^e}{\partial t} \\ \dfrac{\partial u_2^e}{\partial t} \end{Bmatrix} = \mathbf{N}_e \dot{\mathbf{u}}_e, \quad \frac{\partial u}{\partial x} \approx \frac{\partial N_1^e}{\partial x} u_1^e + \frac{\partial N_2^e}{\partial x} u_2^e = \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \mathbf{B}_e \mathbf{u}_e$$

$$\int_{x_1^e}^{x_2^e} u^* \frac{\partial u}{\partial t} dx + \int_{x_1^e}^{x_2^e} u^* c \frac{\partial u}{\partial x} dx = 0 \rightarrow \int_{x_1^e}^{x_2^e} \mathbf{N}_e \mathbf{u}_e^* \mathbf{N}_e \dot{\mathbf{u}}_e dx + \int_{x_1^e}^{x_2^e} c \mathbf{N}_e \mathbf{u}_e^* \mathbf{B}_e \mathbf{u}_e dx = 0$$

$$\xrightarrow{\mathbf{N}_e \mathbf{T}_e^* = \left(\mathbf{N}_e \mathbf{T}_e^*\right)^T = \mathbf{T}_e^{*T} \mathbf{N}_e^T} \mathbf{u}_e^{*T} \left[ \int_{x_1^e}^{x_2^e} \mathbf{N}_e \mathbf{N}_e dx \dot{\mathbf{u}}_e + \int_{x_1^e}^{x_2^e} c \mathbf{N}_e \mathbf{B}_e dx \mathbf{u}_e \right] = 0$$

$$\mathbf{M}_e \dot{\mathbf{u}}_e + \mathbf{S}_e \mathbf{u}_e = 0 \;\text{ where }\; \begin{cases} \mathbf{M}_e = \int_{x_1^e}^{x_2^e} \mathbf{N}_e^T \mathbf{N}_e dx \;\; (\text{mass matrix}) \\ \mathbf{S}_e = \int_{x_1^e}^{x_2^e} c \mathbf{N}_e^T \mathbf{B}_e dx \;\; (\text{advection matrix}) \end{cases} \Rightarrow \mathbf{M}\dot{\mathbf{u}} + \mathbf{S}\mathbf{u} = 0$$

$$\mathbf{M}_e = \int_{x_1^e}^{x_2^e} \mathbf{N}_e^T \mathbf{N}_e dx = \int_{x_1^e}^{x_2^e} \begin{Bmatrix} N_1^e \\ N_2^e \end{Bmatrix} \begin{Bmatrix} N_1^e & N_2^e \end{Bmatrix} dx = \begin{bmatrix} \dfrac{h_e}{3} & \dfrac{h_e}{6} \\ \dfrac{h_e}{6} & \dfrac{h_e}{3} \end{bmatrix}$$

$$\mathbf{S}_e = \int_{x_1^e}^{x_2^e} c \mathbf{N}_e^T \mathbf{B}_e dx = \int_{x_1^e}^{x_2^e} c \begin{Bmatrix} N_1^e \\ N_2^e \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} \end{Bmatrix} dx = c \begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}$$

# Discretization: Space (2D)

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t}+c_x\dfrac{\partial u}{\partial x}+c_y\dfrac{\partial u}{\partial y}=0 \text{ in } \Omega \\ u=\bar{u} \text{ on } \Gamma_u\end{array}\right\}\rightarrow \int_\Omega u^*\frac{\partial u}{\partial t}dV+\int_\Omega u^*\left(c_x\frac{\partial u}{\partial x}+c_y\frac{\partial u}{\partial y}\right)dV=0$$

$$\rightarrow \int_{\Omega_e} u^*\frac{\partial u}{\partial t}dV+\int_{\Omega_e} u^*\left(c_x\frac{\partial u}{\partial x}+c_y\frac{\partial u}{\partial y}\right)dV=0$$

$$\left.\begin{array}{l}u\approx N_1^e u_1^e+N_2^e u_2^e+N_3^e u_3^e=\mathbf{N}_e\mathbf{u}_e \\ u^*\approx N_1^e u_1^{*e}+N_2^e u_2^{*e}+N_3^e u_3^{*e}=\mathbf{N}_e\mathbf{u}_e^*\end{array}\right\}\rightarrow \mathbf{M}_e\dot{\mathbf{u}}_e+\mathbf{S}_e\mathbf{u}_e=0\Rightarrow \mathbf{M}\dot{\mathbf{u}}+\mathbf{S}\mathbf{u}=0$$

# Discretization: Time

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{S}\mathbf{u} = 0 \xrightarrow[\text{time step}]{n+\theta(0\le\theta\le1)} \mathbf{M}\dot{\mathbf{u}}^{n+\theta} + \mathbf{S}\mathbf{u}^{n+\theta} = 0$$

$$\begin{cases} \dot{\mathbf{u}}^{n+\theta} = \dfrac{1}{\Delta t}\left(\mathbf{u}^{n+1} - \mathbf{u}^{n}\right) \\ \mathbf{u}^{n+\theta} = (1-\theta)\mathbf{u}^{n} + \theta\mathbf{u}^{n+1} \end{cases}$$

$$\left(\frac{1}{\Delta t}\mathbf{M} + \theta\mathbf{S}\right)\mathbf{u}^{n+1} = \left\{\frac{1}{\Delta t}\mathbf{M} - (1-\theta)\mathbf{S}\right\}\mathbf{u}^{n}$$

$$\begin{cases} \theta = 1/2\left(\text{Crank} - \text{Nicolson method}\right): \left(\dfrac{1}{\Delta t}\mathbf{M} + \dfrac{1}{2}\mathbf{S}\right)\mathbf{u}^{n+1} = \left(\dfrac{1}{\Delta t}\mathbf{M} - \dfrac{1}{2}\mathbf{S}\right)\mathbf{u}^{n} \\ \theta = 0\left(\text{Euler's forward difference method}\right): \dfrac{1}{\Delta t}\mathbf{M}\mathbf{u}^{n+1} = \left(\dfrac{1}{\Delta t}\mathbf{M} - \mathbf{S}\right)\mathbf{u}^{n} \rightarrow \overline{\mathbf{M}}\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^{n} - \Delta t\mathbf{S}\mathbf{u}^{n} \;\; \left(\text{explicit method}\right) \\ \theta = 1\left(\text{Euler's backward difference method}\right): \left(\dfrac{1}{\Delta t}\mathbf{M} + \mathbf{S}\right)\mathbf{u}^{n+1} = \dfrac{1}{\Delta t}\mathbf{M}\mathbf{u}^{n} \end{cases}$$

[1D: 2 node linear element]

$$\overline{\mathbf{M}}_e = \begin{bmatrix} h_e/2 & 0 \\ 0 & h_e/2 \end{bmatrix} = \frac{h_e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[2D: linear triangle element]

$$\overline{\mathbf{M}}_e = \begin{bmatrix} A_e/3 & 0 & 0 \\ 0 & A_e/3 & 0 \\ 0 & 0 & A_e/3 \end{bmatrix} = \frac{A_e}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

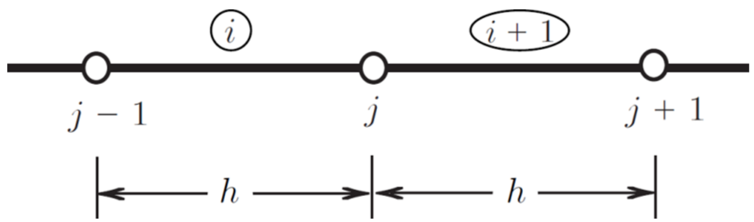
# Lumped Mass → Artificial Diffusion

$$\overline{\mathbf{M}}\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^{n} - \Delta t \mathbf{S}\mathbf{u}^{n}$$

$$\left[\text{element }(i)\right] \qquad\qquad \left[\text{element }(i+1)\right]$$

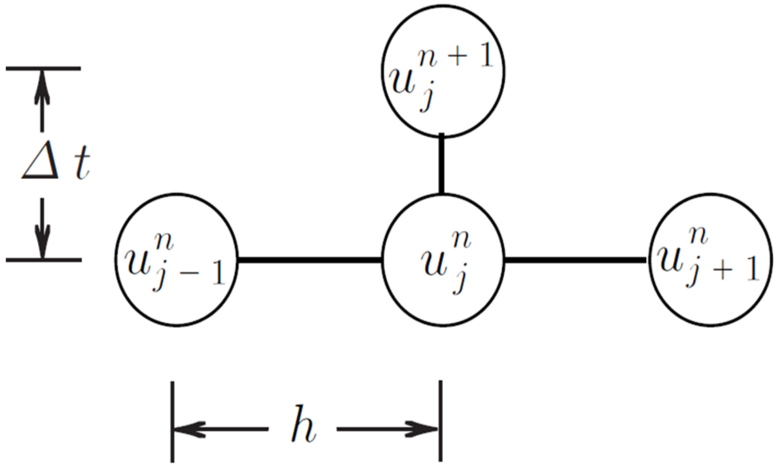
$$\begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix} \begin{Bmatrix} u_{j-1}^{n+1} \\ u_{j}^{n+1} \end{Bmatrix} = \begin{bmatrix} \frac{h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{h}{3} \end{bmatrix} \begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \end{Bmatrix} - c\Delta t \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \end{Bmatrix} \quad \begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix} \begin{Bmatrix} u_{j}^{n+1} \\ u_{j+1}^{n+1} \end{Bmatrix} = \begin{bmatrix} \frac{h}{3} & \frac{h}{6} \\ \frac{h}{6} & \frac{h}{3} \end{bmatrix} \begin{Bmatrix} u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix} - c\Delta t \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix}$$

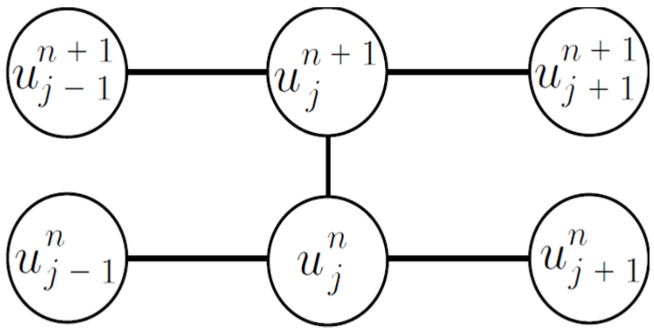
$$\Rightarrow$$



# Implicit vs. Explicit (2)

| Problem | Parabolic | Hyperbolic |
|---|---|---|
| Characteristics | Global | Local, directional |
| Propagation | Infinite | Finite, slow |
| Time discretization | Implicit | Explicit<br>• stability condition → small time step<br>• lumped mass → artificial diffusion |



# Example

$0 < x < 20$

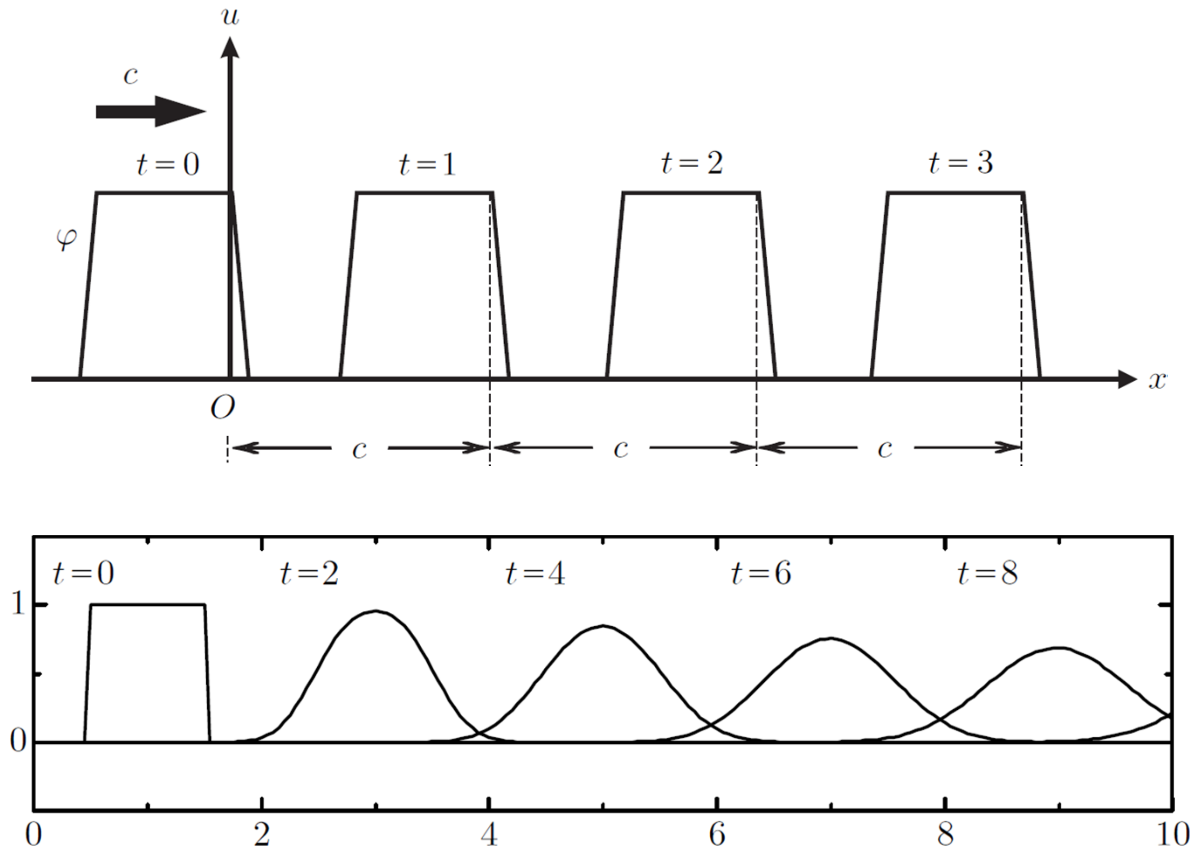
$h = 0.05$

$c = 1$

$\Delta t = 1/60$

B.C.: $u = 0$ at both ends

I.C.: $u = \begin{cases} 1, & 0.5 < x < 1.5 \\ 0, & \text{elsewhere} \end{cases}$



# How to Prevent Artificial Diffusion

$$\overline{\mathbf{M}}\mathbf{u}^{n+1} = \overline{\mathbf{M}}\mathbf{u}^{n} - \Delta t\mathbf{S}\mathbf{u}^{n}$$

$$\underbrace{\begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n+1} \\ u_{j}^{n+1} \end{Bmatrix} = \begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \end{Bmatrix} - c\Delta t\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \end{Bmatrix}}_{\left[\text{element } (i)\right]} \qquad \underbrace{\begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j}^{n+1} \\ u_{j+1}^{n+1} \end{Bmatrix} = \begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix} - c\Delta t\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix}}_{\left[\text{element } (i+1)\right]}$$

$$\Rightarrow \begin{bmatrix} \frac{h}{2} & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n+1} \\ u_{j}^{n+1} \\ u_{j+1}^{n+1} \end{Bmatrix} = \begin{bmatrix} \frac{h}{2} & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \frac{h}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix} - c\Delta t\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_{j-1}^{n} \\ u_{j}^{n} \\ u_{j+1}^{n} \end{Bmatrix}$$

node $j$ $\longrightarrow$

# FTCS → Upwind

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+c\frac{\left(-u_{j-1}^n+u_{j+1}^n\right)}{2h}=0 \rightarrow \frac{u_j^{n+1}-u_j^n}{\Delta t}+c\frac{\left(u_j^n-u_{j-1}^n\right)}{h}=0$$

$$\rightarrow$$

- Stabilization method
  - FDM: Upwind = FTCS + artificial diffusion
  - FEM: weak form (weighting function)

# Multi-step Explicit Method

$$\begin{cases} u^{n+1} \approx u^n + \Delta t \dot{u}^n + \dfrac{\Delta t^2}{2} \ddot{u}^n + \dfrac{\Delta t^3}{6} \dddot{u}^n \\ u^{n+1} \approx u^n + \Delta t \dot{u}^n \rightarrow \text{Euler's method } (\text{1st order accuracy}) \\ u^{n+1} \approx u^n + \Delta t \dot{u}^n + \dfrac{\Delta t^2}{2} \ddot{u}^n \rightarrow \text{Lax-Wendroff's method } (\text{2nd order accuracy}) \end{cases}$$

- Single step method
  - n-th known value → (n+1)-th unknown value
  - To increase accuracy, consider high order approximation of unknown variable
- Multi-step method
  - Several time steps between n-th and (n+1)-th → (n+1)-th unknown value
  - Higher accuracy without high order approximation

# Two-step Explicit Scheme (1)

$$\begin{cases} \bar{u}^{n+\alpha} = u^n + \Delta t\left(\alpha \dot{u}^n\right) \\ u^{n+1} = \beta_1 u^n + \beta_2 \bar{u}^{n+\alpha} + \Delta t\left(\beta_3 \dot{u}^n + \beta_4 \dot{\bar{u}}^{n+\alpha}\right) \end{cases}$$

$$\left.\begin{aligned} &\rightarrow u^{n+1} = \left(\beta_1 + \beta_2\right)u^n + \Delta t\left(\beta_2\alpha + \beta_3 + \beta_4\right)\dot{u}^n + \Delta t^2 \beta_4 \alpha \ddot{u}^n \\ &\Leftrightarrow u^{n+1} \approx u^n + \Delta t \dot{u}^n + \frac{\Delta t^2}{2}\ddot{u}^n \end{aligned}\right\} \rightarrow \begin{cases} \beta_1 + \beta_2 = 1 \\ \beta_2\alpha + \beta_3 + \beta_4 = 1 \\ \beta_4 \alpha = 1/2 \end{cases}$$

[two-step Lax-Wendroff's scheme]

$$\alpha = \frac{1}{2},\ \beta_1 = \beta_4 = 1,\ \beta_2 = \beta_3 = 0$$

# Two-step Explicit Scheme (2)

[two-step Runge Kutta scheme]

$$\alpha = \frac{2}{3},\ \beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = \frac{1}{4},\ \beta_4 = \frac{3}{4}$$

$$\begin{cases} \bar{u}^{n+\frac{2}{3}} = u^n + \dfrac{2\Delta t}{3}\dot{u}^n \\ u^{n+1} = u^n + \dfrac{\Delta t}{4}\left(\dot{u}^n + 3\dot{\bar{u}}^{n+\frac{2}{3}}\right) \end{cases} \xrightarrow{\mathbf{M}\dot{\mathbf{u}}+\mathbf{S}\mathbf{u}=0} \begin{cases} \overline{\mathbf{M}}\bar{\mathbf{u}}^{n+\frac{2}{3}} = \mathbf{M}\mathbf{u}^n - \dfrac{2\Delta t}{3}\mathbf{S}\mathbf{u}^n \\ \overline{\mathbf{M}}\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n - \dfrac{\Delta t}{4}\mathbf{S}\left(\mathbf{u}^n + 3\bar{\mathbf{u}}^{n+\frac{2}{3}}\right) \end{cases}$$

# Three-step Explicit Scheme (1)

$$
\begin{cases}
\overline{u}^{n+\alpha_1} = u^n + \alpha_1 \Delta t \dot{u}^n \\
\overline{u}^{n+\alpha_2} = u^n + \beta_0 \Delta t \dot{u}^n + \beta_1 \Delta t \dot{\overline{u}}^{n+\alpha_1} \\
u^{n+1} = u^n + \gamma_0 \Delta t \dot{u}^n + \gamma_1 \Delta t \dot{\overline{u}}^{n+\alpha_1} + \gamma_2 \Delta t \dot{\overline{u}}^{n+\alpha_2}
\end{cases}
$$

$$
\left.
\begin{array}{l}
\rightarrow u^{n+1} = u^n + \left(\gamma_0 + \gamma_1 + \gamma_2\right) \Delta t \dot{u}^n + \left(\gamma_1 \alpha_1 + \gamma_1 \beta_0 + \gamma_2 \beta_1\right) \Delta t^2 \ddot{u}^n + \gamma_2 \beta_1 \alpha_1 \Delta t^3 \dddot{u}^n \\
\Leftrightarrow u^{n+1} \approx u^n + \Delta t \dot{u}^n + \dfrac{\Delta t^2}{2} \ddot{u}^n + \dfrac{\Delta t^3}{6} \dddot{u}^n
\end{array}
\right\}
\rightarrow
\begin{cases}
\gamma_0 + \gamma_1 + \gamma_2 = 1 \\
\gamma_1 \alpha_1 + \gamma_1 \beta_0 + \gamma_2 \beta_1 = \dfrac{1}{2} \\
\gamma_2 \beta_1 \alpha_1 = \dfrac{1}{6}
\end{cases}
$$

# Three-step Explicit Scheme (2)

$$\left[\text{three-step Taylor-Galerkin scheme}\right]$$

$$\begin{cases} \alpha_1 = 1/3,\ \alpha_2 = 1/2 \\ \beta_0 = 0,\ \beta_1 = 1/2 \\ \gamma_0 = 0,\ \gamma_1 = 0,\ \gamma_2 = 1 \end{cases}$$

$$\begin{cases} \bar{u}^{n+\frac{1}{3}} = u^n + \dfrac{\Delta t}{3}\dot{u}^n \\ \bar{u}^{n+\frac{1}{2}} = u^n + \dfrac{\Delta t}{2}\dot{\bar{u}}^{n+\frac{1}{3}} \\ u^{n+1} = u^n + \Delta t \dot{\bar{u}}^{n+\frac{1}{2}} \end{cases}$$

$$\xrightarrow{\mathbf{M}\dot{\mathbf{u}}+\mathbf{S}\mathbf{u}=0} \begin{cases} \overline{\mathbf{M}\bar{\mathbf{u}}}^{n+\frac{1}{3}} = \mathbf{M}\mathbf{u}^n - \dfrac{\Delta t}{3}\mathbf{S}\mathbf{u}^n \\ \overline{\mathbf{M}\bar{\mathbf{u}}}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{u}^n - \dfrac{\Delta t}{2}\mathbf{S}\bar{\mathbf{u}}^{n+\frac{1}{3}} \\ \overline{\mathbf{M}}\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n - \Delta t\mathbf{S}\bar{\mathbf{u}}^{n+\frac{1}{2}} \end{cases}$$

$$\left[\text{three-step Runge-Kutta scheme}\right]$$

$$\begin{cases} \alpha_1 = 1/3,\ \alpha_2 = 1/2 \\ \beta_0 = 0,\ \beta_1 = 1/2 \\ \gamma_0 = 0,\ \gamma_1 = 0,\ \gamma_2 = 1 \end{cases}$$

$$\begin{cases} \bar{u}^{n+\frac{1}{2}} = u^n + \dfrac{\Delta t}{2}\dot{u}^n \\ \bar{u}^{n+1} = u^n + \Delta t\left(-\dot{u}^n + 2\dot{\bar{u}}^{n+\frac{1}{2}}\right) \\ u^{n+1} = u^n - \dfrac{\Delta t}{6}\left(\dot{u}^n + 4\dot{\bar{u}}^{n+\frac{1}{2}} + \dot{\bar{u}}^{n+1}\right) \end{cases}$$

$$\xrightarrow{\mathbf{M}\dot{\mathbf{u}}+\mathbf{S}\mathbf{u}=0} \begin{cases} \overline{\mathbf{M}\bar{\mathbf{u}}}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{u}^n - \dfrac{\Delta t}{2}\mathbf{S}\mathbf{u}^n \\ \overline{\mathbf{M}\bar{\mathbf{u}}}^{n+1} = \mathbf{M}\mathbf{u}^n - \Delta t\mathbf{S}\left(-\mathbf{u}^n + 2\bar{\mathbf{u}}^{n+\frac{1}{2}}\right) \\ \overline{\mathbf{M}}\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n + \dfrac{\Delta t}{6}\Delta t\mathbf{S}\left(\mathbf{u}^n + 4\bar{\mathbf{u}}^{n+\frac{1}{2}} + \bar{\mathbf{u}}^{n+1}\right) \end{cases}$$

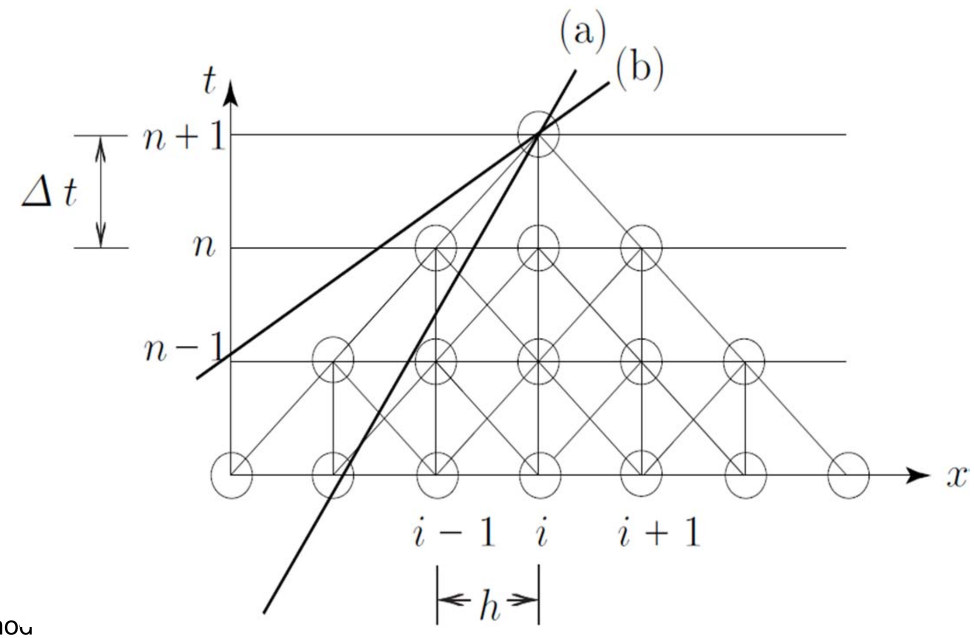
# Explicit Method: Stability Condition

$$\text{Courant number:} \; \nu = \frac{c}{\left(\dfrac{h}{\Delta t}\right)} = c\frac{\Delta t}{h}$$

$$\text{CFL}\left(\text{Courant, Friedrich, Lewy}\right) \text{ condition}: c \le \frac{h}{\Delta t} \Rightarrow \nu = c\frac{\Delta t}{h} \le 1 \Rightarrow \Delta t \le \frac{h}{c}$$

[heat conduction / diffusion]

$$\text{diffusion number}: d = k\frac{\Delta t}{h^2} \le \frac{1}{2} \Rightarrow \Delta t \le \frac{1}{2}\frac{h^2}{k}$$



# Stabilization Method

- Hyperbolic equation using Galerkin FEM → numerical instability
- Explicit method: lumped mass → over-damping
- Stabilized method: less damping + numerical stability

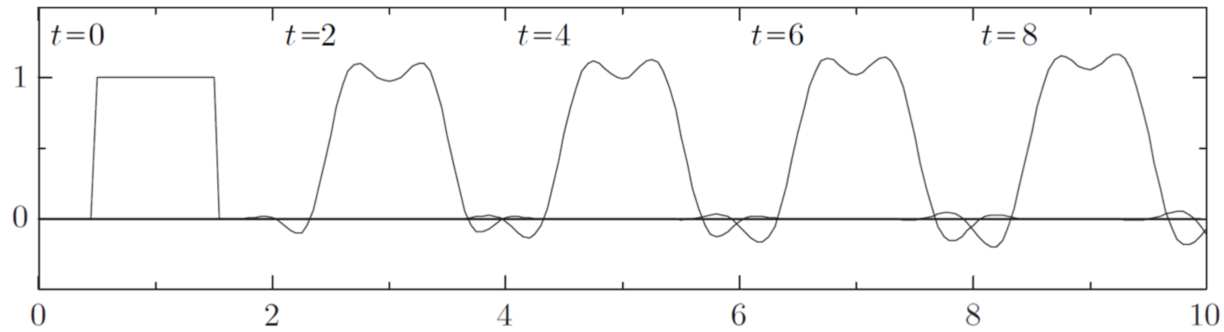
- (1) Multi pass method
- (2) Talyor-Galerkin method
- (3) Upwind finite element method + SUPG/GLS

# Multi Pass Method

- Lumped mass → damping + phase error
- How to deal with mass matrix?

$$\underbrace{\mathbf{M}\mathbf{u}^{n+1} = \underbrace{\mathbf{M}\mathbf{u}^{n} - \Delta t \mathbf{S}\mathbf{u}^{n}}_{\text{known vector}} = \mathbf{F}^{n}}_{\text{implicit method}} \Rightarrow \underbrace{\begin{cases} \overline{\mathbf{M}}\mathbf{u}_{r+1}^{n+1} = \left[\overline{\mathbf{M}} - \mathbf{M}\right]\mathbf{u}_{r}^{n+1} + \mathbf{F}^{n} \\ r : \text{number of repeat} \end{cases}}_{\text{explicit method}}$$

$$r = 2, \ \ \nu = 0.1\left(\text{Courant number}\right)$$



# Talyor-Galerkin Method (1)

- Time: 2nd order accuracy (Taylor)
- Space: Galerkin FEM

$$u^{n+1} = u^n + \Delta t \frac{\partial u^n}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u^n}{\partial t^2} + O\left((\Delta t)^3\right) \rightarrow \frac{u^{n+1} - u^n}{\Delta t} = \frac{\partial u^n}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u^n}{\partial t^2} + O\left((\Delta t)^2\right)$$

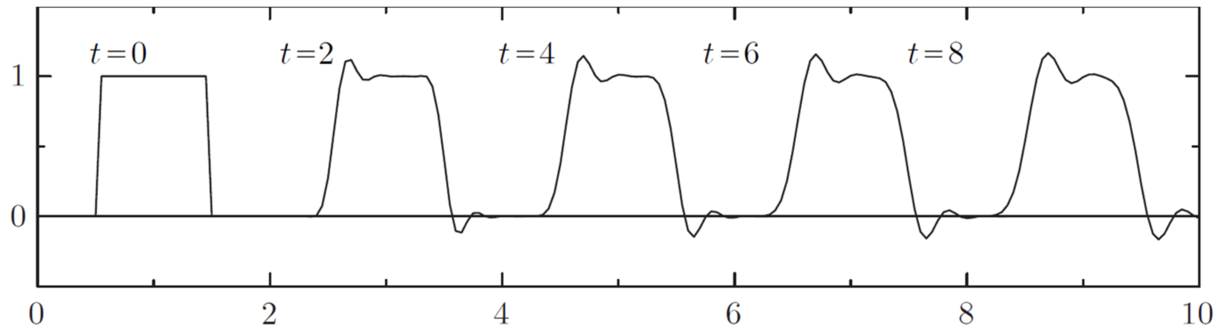
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \xrightarrow{\text{time step } n} \frac{\partial u^n}{\partial t} = -c\frac{\partial u^n}{\partial x} \xrightarrow[\text{wrt time}]{\text{differentiation}} \frac{\partial^2 u^n}{\partial t^2} = -c\frac{\partial}{\partial x}\left(\frac{\partial u^n}{\partial t}\right)$$

$$\frac{u^{n+1} - u^n}{\Delta t} = -c\frac{\partial u^n}{\partial x} + \frac{\Delta t}{2}\left\{-c\frac{\partial}{\partial x}\left(\frac{\partial u^n}{\partial t}\right)\right\} + O\left((\Delta t)^2\right) \approx -c\frac{\partial u^n}{\partial x} + \frac{\Delta t}{2}\left\{-c\frac{\partial}{\partial x}\left(-c\frac{\partial u^n}{\partial x}\right)\right\}$$

# Talyor-Galerkin Method (2)

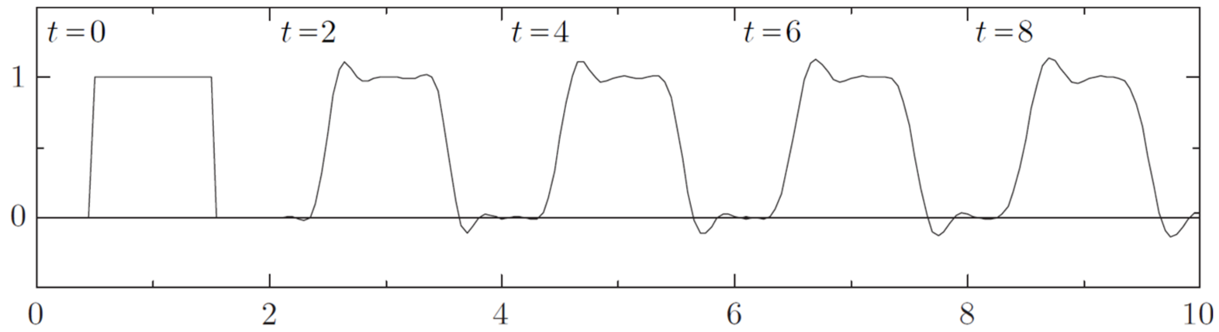
Talyor-Galerkin method (implicit method)

$\nu = 0.33(\text{Courant number})$




Talyor-Galerkin method using multi pass method

$r = 3, \;\; \nu = 0.33(\text{Courant number})$



# Upwind FEM

- Advection term: central difference → upwind difference (artificial diffusion)
- Central difference ↔ Galerkin FEM

[steady advection-diffusion problem]

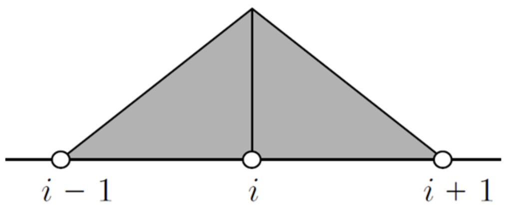
$$c\frac{\partial u}{\partial x}-k\frac{\partial^2 u}{\partial x^2}=0 \text{ in } \Omega \rightarrow c\frac{\partial u}{\partial x}-\left(k+k_\alpha\right)\frac{\partial^2 u}{\partial x^2}=0 \quad \left(0\le x\le 1\right)$$

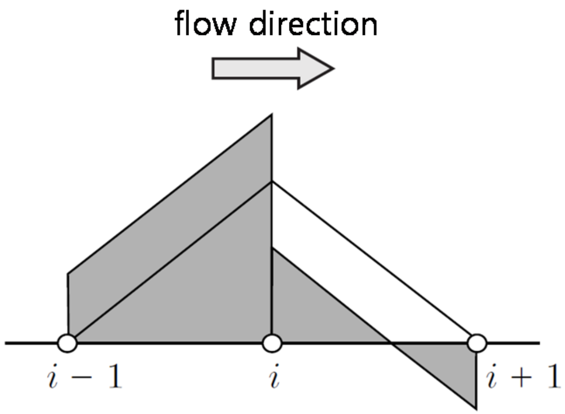
$k_\alpha = ch/2$ : coefficient of artificial diffusion

$$\int_\Omega wc\frac{\partial u}{\partial x}dx+\left(k+k_\alpha\right)\int_\Omega \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx=0$$

$$\int_\Omega\left(w+\frac{k_\alpha}{c}\frac{\partial w}{\partial x}\right)c\frac{\partial u}{\partial x}dx+k\int_\Omega \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx=0$$

$$\overline{w}=w+\frac{k_\alpha}{c}\frac{\partial w}{\partial x}=w+\delta$$



# Upwind / SUPG

- Upwind FEM → (extension to multi-dimension, upwind only for direction of mass transport) → Streamline-Upwind FEM
- Upwind ↔ artificial diffusion effect
- Test(w+δ)/Trial(w) function: Petrov-Galerkin method

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}-k\frac{\partial^2 u}{\partial x^2}=0$$

$$\int_\Omega w\left(\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}\right)dx+k\int_\Omega \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx+\underbrace{\sum_{e=1}^{M}\int_{\Omega_e}\delta\left(\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}-k\frac{\partial^2 u}{\partial x^2}\right)dx}_{\text{stabilization}}=0$$

# SUPG/GLS: How to determine δ ?

- SUPG(Streamline-Upwind/Petrov-Galerkin) Method

$$\delta = \tau c \frac{\partial w}{\partial x} \quad \text{where } \tau : \text{stabilized parameter} \begin{cases} \tau = \left[ \left( \frac{2|c|}{h} \right)^2 \right]^{-\frac{1}{2}} \text{(steady equation)} \\ \tau = \left[ \left( \frac{2}{\Delta t} \right)^2 + \left( \frac{2|c|}{h} \right)^2 \right]^{-\frac{1}{2}} \text{(unsteady equation)} \end{cases}$$

- GLS(Galerkin Least Square) Method: $\delta = \tau \left( \frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x} - k \frac{\partial^2 w}{\partial x^2} \right)$
  - Symmetric stabilization term
  - If space-time FEM: $w(x,t)$

# SUPG

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \xrightarrow[\text{SUPG method}]{\text{FEM based}} \left(\mathbf{M} + \mathbf{M}_{\delta}\right)\dot{\mathbf{u}} + \left(\mathbf{S} + \mathbf{S}_{\delta}\right)\mathbf{u} = 0$$

$$\mathbf{M}_{\delta e} = \int_0^h \tau c \mathbf{B}_e{}^T \mathbf{N}_e dx = \int_{x_1^e}^{x_2^e} \tau c \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} \\ \dfrac{\partial N_2^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} N_1^e & N_2^e \end{Bmatrix} dx = \tau c \begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}$$

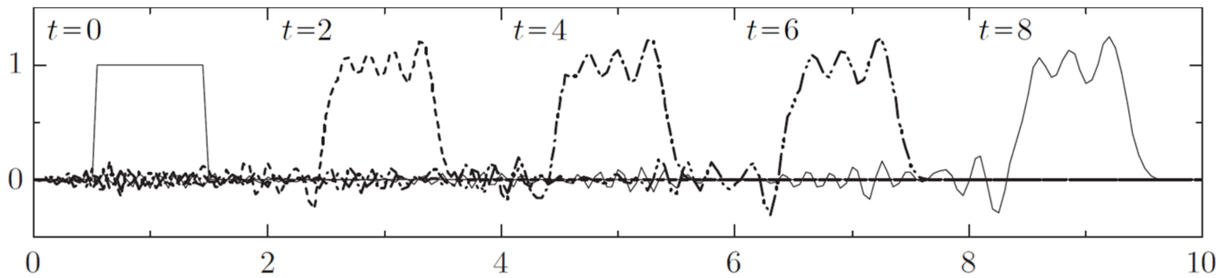
$$\mathbf{S}_{\delta e} = \int_0^h \tau c^2 \mathbf{B}_e{}^T \mathbf{B}_e dx = \int_{x_1^e}^{x_2^e} \tau c^2 \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} \\ \dfrac{\partial N_2^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} \end{Bmatrix} dx = \tau c^2 \begin{bmatrix} \dfrac{1}{h} & -\dfrac{1}{h} \\ -\dfrac{1}{h} & \dfrac{1}{h} \end{bmatrix}$$

SUPG in space / implicit method in time

# Example

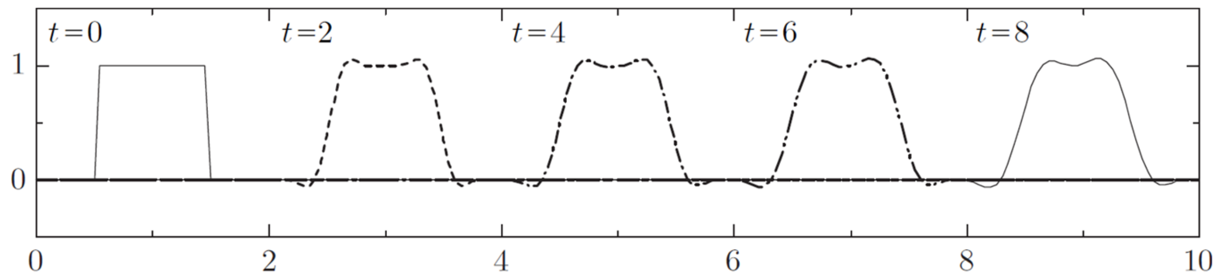
Galerkin method / implicit method(Crank-Nicolson)

$\nu = 0.33(\text{Courant number})$




SUPG method / $\begin{cases} \text{implicit method} \\ \text{explicit method with multi pass method} \end{cases}$

$\nu = 0.33(\text{Courant number})$




explicit method: ignore $\mathbf{M}_{\delta e}$? not SUPG method

$\tau = \Delta t/2 \rightarrow$ Taylor-Galerkin(BTD) method