FEM for Vibration Problem

- Governing equation and weak form
- Space discretization
- Mass matrix and damping matrix
- Direct integration

Governing Equation & Weak Form (1)

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases} \oplus (\text{inertia force}) \begin{cases} B_x = -\rho \dfrac{\partial^2 u}{\partial t^2} \\ B_y = -\rho \dfrac{\partial^2 v}{\partial t^2} \end{cases}$$

$$(\text{equations of motion}) \begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x + B_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + b_y + B_y = 0 \end{cases}$$

$$\int_\Omega \left\{ u^* \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x + B_x \right) + v^* \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y + B_y \right) \right\} h dA = 0$$

$$-\int_\Omega \left(u^* B_x + v^* B_y \right) h dA + \int_\Omega \left\{ \frac{\partial u^*}{\partial x} \sigma_x + \frac{\partial v^*}{\partial y} \sigma_y + \left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right) \tau_{xy} \right\} h dA$$

$$= \int_\Omega \left(u^* b_x + v^* b_y \right) h dA + \int_\Omega \left\{ \frac{\partial}{\partial x}\left(u^* \sigma_x \right) + \frac{\partial}{\partial y}\left(u^* \tau_{xy} \right) + \frac{\partial}{\partial y}\left(v^* \sigma_y \right) + \frac{\partial}{\partial x}\left(v^* \tau_{xy} \right) \right\} h dA$$

Governing Equation & Weak Form (2)

$$\left[\varepsilon_x^* = \frac{\partial u^*}{\partial x},\ \varepsilon_y^* = \frac{\partial v^*}{\partial y},\ \tau_{xy}^* = \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right]$$

$$-\int_\Omega \left(u^* B_x + v^* B_y \right) hdA + \int_\Omega \left(\varepsilon_x^* \sigma_x + \varepsilon_y^* \sigma_y + \tau_{xy}^* \tau_{xy} \right) hdA$$

$$= \int_\Omega \left(u^* b_x + v^* b_y \right) hdA + \int_\Gamma \left\{ u^* \left(\sigma_x n_x + \tau_{xy} n_y \right) + v^* \left(\sigma_y n_y + \tau_{xy} n_x \right) \right\} hds$$

$$\left[\begin{array}{l} \left. \begin{array}{l} t_x = \overline{t}_x \\ t_y = \overline{t}_y \end{array} \right\} \text{ on } \Gamma_\sigma \\ u^* = v^* = 0 \text{ on } \Gamma_u \end{array} \right]$$

$$-\int_\Omega \left(u^* B_x + v^* B_y \right) hdA + \int_\Omega \left(\varepsilon_x^* \sigma_x + \varepsilon_y^* \sigma_y + \tau_{xy}^* \tau_{xy} \right) hdA$$

$$= \int_\Omega \left(u^* b_x + v^* b_y \right) hdA + \int_{\Gamma_\sigma} \left(u^* \overline{t}_x + v^* \overline{t}_y \right) hds$$

Discretization: Space

$$-\int_{\Omega_e}\left(u^* B_x + v^* B_y\right)h_e dA = \int_{\Omega_e}\rho_e\left(u^*\frac{\partial^2 u}{\partial t^2} + v^*\frac{\partial^2 v}{\partial t^2}\right)h_e dA$$

$$\begin{cases} u \approx N_1^e u_1^e + N_2^e u_2^e + N_3^e u_3^e = \boldsymbol{N}_e \boldsymbol{u}_e \\ v \approx N_1^e v_1^e + N_2^e v_2^e + N_3^e v_3^e = \boldsymbol{N}_e \boldsymbol{v}_e \end{cases}$$

$$\begin{cases} u^* \approx N_1^e u_1^{*e} + N_2^e u_2^{*e} + N_3^e u_3^{*e} = \boldsymbol{N}_e \boldsymbol{u}_e^* \\ v^* \approx N_1^e v_1^{*e} + N_2^e v_2^{*e} + N_3^e v_3^{*e} = \boldsymbol{N}_e \boldsymbol{v}_e^* \end{cases}$$

$$\int_{\Omega_e}\rho_e\left(u^*\frac{\partial^2 u}{\partial t^2} + v^*\frac{\partial^2 v}{\partial t^2}\right)h_e dA \approx \boldsymbol{u}_e^{*T}\left[\int_{\Omega_e}\rho_e\left(\boldsymbol{N}_e^T\boldsymbol{N}_e\right)h_e dA\right]\ddot{\boldsymbol{u}}_e + \boldsymbol{v}_e^{*T}\left[\int_{\Omega_e}\rho_e\left(\boldsymbol{N}_e^T\boldsymbol{N}_e\right)h_e dA\right]\ddot{\boldsymbol{v}}_e$$

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} \approx \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix}\begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{Bmatrix} = \boldsymbol{N}_e\boldsymbol{d}_e \quad \text{where} \quad \begin{cases} \boldsymbol{N}_e : \text{shape function matrix} \\ \boldsymbol{d}_e : \text{element nodal displacement vector} \end{cases}$$

$$\int_{\Omega_e}\rho_e\left(u^*\frac{\partial^2 u}{\partial t^2} + v^*\frac{\partial^2 v}{\partial t^2}\right)h_e dA \approx \boldsymbol{d}_e^{*T}\boldsymbol{M}_e\ddot{\boldsymbol{d}}_e \quad \text{where } \boldsymbol{M}_e = \int_{\Omega_e}\rho_e\left(\boldsymbol{N}_e^T\boldsymbol{N}_e\right)h_e dA \ (\text{mass matrix})$$

$$\boldsymbol{M}_e\ddot{\boldsymbol{d}}_e + \boldsymbol{K}_e\boldsymbol{d}_e = \boldsymbol{F}_e \Rightarrow \boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \Rightarrow \boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{C}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$$

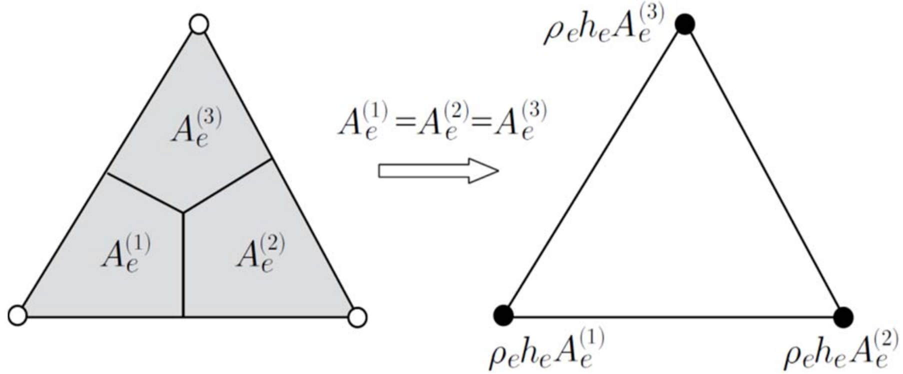
Mass Matrix

$$\boldsymbol{M}_e = \int_{\Omega_e} \rho_e \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} \begin{Bmatrix} N_1^e & N_2^e & N_3^e \end{Bmatrix} h_e dA = \frac{\rho_e V_e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{where} \quad \begin{cases} V_e = A_e h_e \\ \rho_e V_e \left(\text{element mass}\right) \end{cases}$$

$$\text{2D}: \boldsymbol{d}_e = \begin{Bmatrix} u_1^e & u_2^e & u_3^e & u_4^e & u_5^e & u_6^e \end{Bmatrix}^T$$

consistent mass matrix: (1st order element) lumped mass matrix: (0th order element)

$$\boldsymbol{M}_e = \frac{\rho_e V_e}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{M}_e = \frac{\rho_e V_e}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Damping Matrix

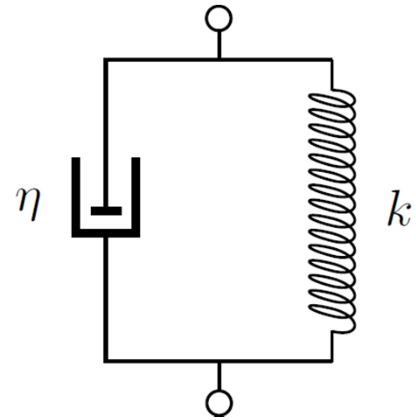
- Structural damping: inside the structure
- Viscous damping: outside the structure

if the damping coefficient η of the specimen is obtained through the dynamic materal test,

$$\sigma = \underbrace{\sigma_k}_{\text{elasticity}} + \underbrace{\sigma_\eta}_{\text{viscousity}} = k\varepsilon + \eta\dot{\varepsilon} \rightarrow \boldsymbol{\sigma} = \boldsymbol{D}_e\boldsymbol{\varepsilon} + \boldsymbol{H}_e\dot{\boldsymbol{\varepsilon}} \xrightarrow{\boldsymbol{H}_e = \alpha\boldsymbol{D}_e} \boldsymbol{\sigma} = \boldsymbol{D}_e\left(\boldsymbol{\varepsilon} + \alpha\dot{\boldsymbol{\varepsilon}}\right)$$

$$\xrightarrow{\text{FEM}} \left\{ \begin{array}{l} \boldsymbol{C}_e = \alpha\boldsymbol{K}_e \left(\text{proportional damping}\right) \\ \boldsymbol{C}_e = \beta\boldsymbol{M}_e \end{array} \right\} \rightarrow \boldsymbol{C}_e = \beta\boldsymbol{M}_e + \alpha\boldsymbol{K}_e \left(\text{Rayleigh damping}\right)$$

$$\begin{cases} \alpha = 2\dfrac{h_2\omega_2 - h_1\omega_1}{\omega_2^2 - \omega_1^2} \\ \beta = 2\omega_1\omega_2\dfrac{h_1\omega_2 - h_2\omega_1}{\omega_2^2 - \omega_1^2} \end{cases} \quad \text{where} \quad \begin{cases} \omega_i : i\text{-th natural frequency} \\ h_i : i\text{-th damping coefficient} \end{cases}$$



Solution Method

$$\begin{cases} M\ddot{d} + Kd = F & (\text{no damping}) \\ M\ddot{d} + C\dot{d} + Kd = F & (\text{damping}) \end{cases}$$

- Linear problem
  - Direction integration method (sequential integration)
  - Mode analysis method
  - Frequency response method
  - Response spectral method
- Nonlinear problem
  - Direction integration method

Direct Integration Method

- Central difference: O((Δt)^2)

$$\ddot{\boldsymbol{d}}^{n} \approx \frac{1}{2\Delta t}\left(\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n-1}\right)$$

$$\ddot{\boldsymbol{d}}^{n} \approx \frac{\dfrac{\left(\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}\right)}{\Delta t}-\dfrac{\left(\boldsymbol{d}^{n}-\boldsymbol{d}^{n-1}\right)}{\Delta t}}{\Delta t}=\frac{1}{\Delta t^{2}}\left(\boldsymbol{d}^{n+1}-2\boldsymbol{d}^{n}+\boldsymbol{d}^{n-1}\right)$$

$$\boldsymbol{M}\ddot{\boldsymbol{d}}+\boldsymbol{C}\dot{\boldsymbol{d}}+\boldsymbol{K}\boldsymbol{d}=\boldsymbol{F}\rightarrow\left(\frac{1}{\Delta t^{2}}\boldsymbol{M}+\frac{1}{2\Delta t}\boldsymbol{C}\right)\boldsymbol{d}^{n+1}=\boldsymbol{F}^{n}-\left(\boldsymbol{K}-\frac{2}{\Delta t^{2}}\boldsymbol{M}\right)\boldsymbol{d}^{n}-\left(\frac{1}{\Delta t^{2}}\boldsymbol{M}-\frac{1}{2\Delta t}\boldsymbol{C}\right)\boldsymbol{d}^{n-1}$$

$$\boldsymbol{d}^{n},\boldsymbol{d}^{n-1}\rightarrow\boldsymbol{d}^{n+1},\ \text{lumped mass + no damping}\rightarrow\text{explicit method}$$

$$\boldsymbol{d}^{n},\boldsymbol{d}^{n-1}\rightarrow\boldsymbol{d}^{n+1}\rightarrow\ddot{\boldsymbol{d}}^{n},\ddot{\boldsymbol{d}}^{n}:\text{sequential integration method (step-by-step method)}$$

- Linear acceleration method
- Newmark's β method
- Wilson θ method

Linear Acceleration Method

$$\begin{cases} \boldsymbol{d}^{n+1} = \boldsymbol{d}^{n} + \Delta t \dot{\boldsymbol{d}}^{n} + \dfrac{\Delta t^2}{3} \ddot{\boldsymbol{d}}^{n} + \dfrac{\Delta t^2}{6} \ddot{\boldsymbol{d}}^{n+1} \\ \dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^{n} + \dfrac{\Delta t}{2} \left(\ddot{\boldsymbol{d}}^{n} + \ddot{\boldsymbol{d}}^{n+1} \right) \end{cases}$$

[unknown: acceleration]

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+1} + \boldsymbol{C}\dot{\boldsymbol{d}}^{n+1} + \boldsymbol{K}\boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} \rightarrow$$

$$\left(\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{C} + \frac{\Delta t^2}{3}\boldsymbol{K} \right) \ddot{\boldsymbol{d}}^{n+1} = \boldsymbol{F}^{n+1} - \boldsymbol{C}\left(\dot{\boldsymbol{d}}^{n} + \frac{\Delta t}{2}\ddot{\boldsymbol{d}}^{n} \right) - \boldsymbol{K}\left(\boldsymbol{d}^{n} + \Delta t \dot{\boldsymbol{d}}^{n} + \frac{\Delta t^2}{3}\ddot{\boldsymbol{d}}^{n} \right)$$

$$\ddot{\boldsymbol{d}}^{n+1} \rightarrow \dot{\boldsymbol{d}}^{n+1}, \boldsymbol{d}^{n+1}$$

[unknown: displacement]

$$\begin{cases} \boldsymbol{d}^{n+1} = \boldsymbol{d}^{n} + \Delta t \dot{\boldsymbol{d}}^{n} + \dfrac{\Delta t^2}{3} \ddot{\boldsymbol{d}}^{n} + \dfrac{\Delta t^2}{6} \ddot{\boldsymbol{d}}^{n+1} \\ \dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^{n} + \dfrac{\Delta t}{2} \left(\ddot{\boldsymbol{d}}^{n} + \ddot{\boldsymbol{d}}^{n+1} \right) \end{cases} \rightarrow \begin{cases} \ddot{\boldsymbol{d}}^{n+1} = \dfrac{6}{\Delta t^2}\left(\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n} \right) - \dfrac{6}{\Delta t}\dot{\boldsymbol{d}}^{n} - 2\ddot{\boldsymbol{d}}^{n} \\ \dot{\boldsymbol{d}}^{n+1} = \dfrac{3}{\Delta t}\left(\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n} \right) - 2\dot{\boldsymbol{d}}^{n} - \dfrac{\Delta t}{2}\ddot{\boldsymbol{d}}^{n} \end{cases}$$

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+1} + \boldsymbol{C}\dot{\boldsymbol{d}}^{n+1} + \boldsymbol{K}\boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} \rightarrow$$

$$\left(\frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{C} + \boldsymbol{K} \right) \boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} + \boldsymbol{M}\left(2\ddot{\boldsymbol{d}}^{n} + \frac{6}{\Delta t}\dot{\boldsymbol{d}}^{n} + \frac{6}{\Delta t^2}\boldsymbol{d}^{n} \right) + \boldsymbol{C}\left(\frac{\Delta t}{2}\ddot{\boldsymbol{d}}^{n} + 2\dot{\boldsymbol{d}}^{n} + \frac{3}{\Delta t}\boldsymbol{d}^{n} \right)$$

$$\boldsymbol{d}^{n+1} \rightarrow \dot{\boldsymbol{d}}^{n+1}, \ddot{\boldsymbol{d}}^{n+1}$$

Newmark's β method (1)

$$\begin{cases} \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t \dot{\boldsymbol{d}}^n + \Delta t^2 \left(\frac{1}{2} - \beta \right) \ddot{\boldsymbol{d}}^n + \Delta t^2 \beta \ddot{\boldsymbol{d}}^{n+1} \\ \dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^n + \Delta t (1-\gamma) \ddot{\boldsymbol{d}}^n + \Delta t \gamma \ddot{\boldsymbol{d}}^{n+1} \end{cases}$$

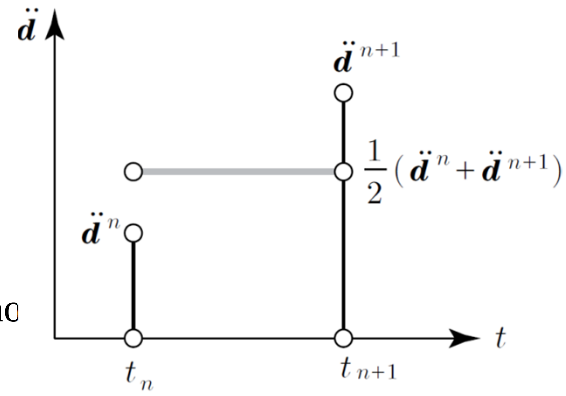
if $\gamma = \frac{1}{2}$, then $\dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^n + \frac{\Delta t}{2} \left(\ddot{\boldsymbol{d}}^n + \ddot{\boldsymbol{d}}^{n+1} \right) \rightarrow$ linear acceleration method

$0 \le \beta \le \frac{1}{2}$

(1) $\beta = 0 : \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t \dot{\boldsymbol{d}}^n + \frac{\Delta t^2}{2} \ddot{\boldsymbol{d}}^n \rightarrow$ unconditionally unstable

(2) $\beta = \frac{1}{4} : \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t \dot{\boldsymbol{d}}^n + \frac{\Delta t^2}{2} \left(\frac{\ddot{\boldsymbol{d}}^n + \ddot{\boldsymbol{d}}^{n+1}}{2} \right) \rightarrow$ average acceleration metho

(3) $\beta = \frac{1}{6} : \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t \dot{\boldsymbol{d}}^n + \frac{\Delta t^2}{3} \ddot{\boldsymbol{d}}^n + \frac{\Delta t^2}{6} \ddot{\boldsymbol{d}}^{n+1} \rightarrow$ linear acceleration method



Newmark's β method $\left(\gamma = \frac{1}{2}, \beta = \frac{1}{6} \right) \Leftrightarrow$ linear acceleration method

Newmark's β method is proposed for unconditionally stable integration scheme.

Newmark's β method (2)

$$\left[\text{unknown: acceleration}\right]\left(\gamma = 1/2\right)$$

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+1} + \boldsymbol{C}\dot{\boldsymbol{d}}^{n+1} + \boldsymbol{K}\boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} \rightarrow$$

$$\left(\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{C} + \beta\Delta t^2\boldsymbol{K}\right)\ddot{\boldsymbol{d}}^{n+1} = \boldsymbol{F}^{n+1} - \boldsymbol{C}\left(\dot{\boldsymbol{d}}^n + \frac{\Delta t}{2}\ddot{\boldsymbol{d}}^n\right) - \boldsymbol{K}\left\{\boldsymbol{d}^n + \Delta t\dot{\boldsymbol{d}}^n + \left(\frac{1}{2} - \beta\right)\Delta t^2\ddot{\boldsymbol{d}}^n\right\}$$

$$\ddot{\boldsymbol{d}}^{n+1} \rightarrow \dot{\boldsymbol{d}}^{n+1}, \boldsymbol{d}^{n+1}$$

$$\left[\text{unknown: displacement}\right]\left(\gamma = 1/2\right)$$

$$\begin{cases} \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t\dot{\boldsymbol{d}}^n + \Delta t^2\left(\frac{1}{2} - \beta\right)\ddot{\boldsymbol{d}}^n + \Delta t^2\beta\ddot{\boldsymbol{d}}^{n+1} \\ \dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^n + \Delta t\left(1-\gamma\right)\ddot{\boldsymbol{d}}^n + \Delta t\gamma\ddot{\boldsymbol{d}}^{n+1} \end{cases} \rightarrow \begin{cases} \ddot{\boldsymbol{d}}^{n+1} = \frac{1}{\beta\Delta t^2}\left(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{d}}^n - \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{d}}^n \\ \dot{\boldsymbol{d}}^{n+1} = \frac{1}{2\beta\Delta t}\left(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n\right) - \left(\frac{1}{2\beta} - 1\right)\dot{\boldsymbol{d}}^n - \left(\frac{1}{4\beta} - 1\right)\Delta t\ddot{\boldsymbol{d}}^n \end{cases}$$

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+1} + \boldsymbol{C}\dot{\boldsymbol{d}}^{n+1} + \boldsymbol{K}\boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} \rightarrow$$

$$\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{1}{2\beta\Delta t}\boldsymbol{C} + \boldsymbol{K}\right)\boldsymbol{d}^{n+1} = \boldsymbol{F}^{n+1} + \boldsymbol{M}\left\{\left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{d}}^n + \frac{1}{\beta\Delta t}\dot{\boldsymbol{d}}^n + \frac{1}{\beta\Delta t^2}\boldsymbol{d}^n\right\}$$

$$+\boldsymbol{C}\left\{\left(\frac{1}{4\beta} - 1\right)\Delta t\ddot{\boldsymbol{d}}^n + \left(\frac{1}{2\beta} - 1\right)\dot{\boldsymbol{d}}^n + \frac{1}{2\beta\Delta t}\boldsymbol{d}^n\right\}$$

$$\boldsymbol{d}^{n+1} \rightarrow \dot{\boldsymbol{d}}^{n+1}, \ddot{\boldsymbol{d}}^{n+1}$$

Wilson θ method (1)

- Extension of linear acceleration method
  - Newmark β method: linear between t_n and t_{n+1}
  - Wilson θ method: linear between t_n and $t_{n+\theta}$

$$\begin{cases} \boldsymbol{d}^{n+\theta} = \boldsymbol{d}^n + \theta\Delta t\dot{\boldsymbol{d}}^n + \dfrac{(\theta\Delta t)^2}{6}\left(\ddot{\boldsymbol{d}}^{n+\theta} + 2\ddot{\boldsymbol{d}}^n\right) \\ \dot{\boldsymbol{d}}^{n+\theta} = \dot{\boldsymbol{d}}^n + \dfrac{\theta\Delta t}{2}\left(\ddot{\boldsymbol{d}}^{n+\theta} + \ddot{\boldsymbol{d}}^n\right) \end{cases}$$

$\theta = 1 \rightarrow$ linear acceleration method

$\theta \geq 1.37 \rightarrow$ unconditionally stable, but inaccurate solution for too large θ

$$\begin{cases} \ddot{\boldsymbol{d}}^{n+1} = \dfrac{(\theta - 1)\ddot{\boldsymbol{d}}^n + \ddot{\boldsymbol{d}}^{n+\theta}}{\theta} \\ \dot{\boldsymbol{d}}^{n+1} = \dot{\boldsymbol{d}}^n + \dfrac{\Delta t}{2}\left(\ddot{\boldsymbol{d}}^{n+1} + \ddot{\boldsymbol{d}}^n\right) \\ \boldsymbol{d}^{n+1} = \boldsymbol{d}^n + \Delta t\dot{\boldsymbol{d}}^n + \dfrac{\Delta t^2}{6}\left(\ddot{\boldsymbol{d}}^{n+1} + 2\ddot{\boldsymbol{d}}^n\right) \end{cases}$$

Wilson θ method (2)

[unknown: acceleration]

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+\theta}+\boldsymbol{C}\dot{\boldsymbol{d}}^{n+\theta}+\boldsymbol{K}\boldsymbol{d}^{n+\theta}=\boldsymbol{F}^{n+\theta}\rightarrow$$

$$\left(\boldsymbol{M}+\frac{\theta\Delta t}{2}\boldsymbol{C}+\frac{(\theta\Delta t)^2}{6}\boldsymbol{K}\right)\ddot{\boldsymbol{d}}^{n+\theta}=\boldsymbol{F}^{n+\theta}-\boldsymbol{C}\left(\dot{\boldsymbol{d}}^{n}+\frac{\theta\Delta t}{2}\ddot{\boldsymbol{d}}^{n}\right)-\boldsymbol{K}\left\{\boldsymbol{d}^{n}+\theta\Delta t\dot{\boldsymbol{d}}^{n}+\frac{(\theta\Delta t)^2}{3}\ddot{\boldsymbol{d}}^{n}\right\}$$

$$\ddot{\boldsymbol{d}}^{n+\theta}\rightarrow\ddot{\boldsymbol{d}}^{n+1}\rightarrow\dot{\boldsymbol{d}}^{n+1},\boldsymbol{d}^{n+1}$$

[unknown: displacement]

$$\begin{cases}\boldsymbol{d}^{n+\theta}=\boldsymbol{d}^{n}+\theta\Delta t\dot{\boldsymbol{d}}^{n}+\frac{(\theta\Delta t)^2}{6}\left(\ddot{\boldsymbol{d}}^{n+\theta}+2\ddot{\boldsymbol{d}}^{n}\right)\\ \dot{\boldsymbol{d}}^{n+\theta}=\dot{\boldsymbol{d}}^{n}+\frac{\theta\Delta t}{2}\left(\ddot{\boldsymbol{d}}^{n+\theta}+\ddot{\boldsymbol{d}}^{n}\right)\end{cases}\rightarrow\begin{cases}\ddot{\boldsymbol{d}}^{n+\theta}=\frac{6}{(\theta\Delta t)^2}\left(\boldsymbol{d}^{n+\theta}-\boldsymbol{d}^{n}\right)-\frac{6}{\theta\Delta t}\dot{\boldsymbol{d}}^{n}-2\ddot{\boldsymbol{d}}^{n}\\ \dot{\boldsymbol{d}}^{n+\theta}=\frac{3}{\theta\Delta t}\left(\boldsymbol{d}^{n+\theta}-\boldsymbol{d}^{n}\right)-2\dot{\boldsymbol{d}}^{n}-\frac{\theta\Delta t}{2}\ddot{\boldsymbol{d}}^{n}\end{cases}$$

$$\boldsymbol{M}\ddot{\boldsymbol{d}}^{n+\theta}+\boldsymbol{C}\dot{\boldsymbol{d}}^{n+\theta}+\boldsymbol{K}\boldsymbol{d}^{n+\theta}=\boldsymbol{F}^{n+\theta}\rightarrow$$

$$\left(\frac{6}{(\theta\Delta t)^2}\boldsymbol{M}+\frac{3}{\theta\Delta t}\boldsymbol{C}+\boldsymbol{K}\right)\boldsymbol{d}^{n+\theta}=\boldsymbol{F}^{n+\theta}+\boldsymbol{M}\left\{2\ddot{\boldsymbol{d}}^{n}+\frac{6}{\theta\Delta t}\dot{\boldsymbol{d}}^{n}+\frac{6}{(\theta\Delta t)^2}\boldsymbol{d}^{n}\right\}+\boldsymbol{C}\left(\frac{\theta\Delta t}{2}\ddot{\boldsymbol{d}}^{n}+2\dot{\boldsymbol{d}}^{n}+\frac{3}{\theta\Delta t}\boldsymbol{d}^{n}\right)$$

$$\boldsymbol{d}^{n+\theta}\rightarrow\ddot{\boldsymbol{d}}^{n+\theta}\rightarrow\boldsymbol{d}^{n+1}\rightarrow\dot{\boldsymbol{d}}^{n+1},\ddot{\boldsymbol{d}}^{n+1}$$