# Isoparametric Element

$$\text{physical coordinate}(x, y) \xleftarrow{\text{parametric element}} \underbrace{\text{natural coordinate}(\xi, \eta)}_{\text{master element}}$$

- Numerical integration
  - Trapezoidal rule
  - Newton-Cotes formula
  - Legendre-Gauss formula
- Bilinear isoparametric quadrilateral element
  - Transformation of element shape
  - Interpolation of displacement
  - Element stiffness matrix and load vector

# (Legendre-)Gauss Formula: 1D (1)

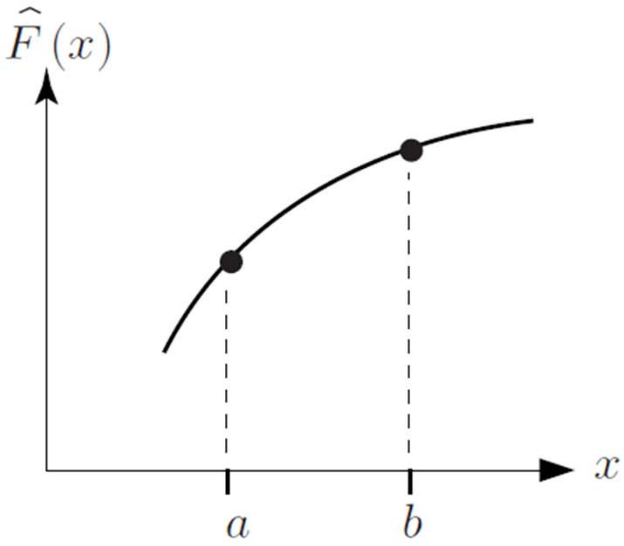
- Numerical integration
  - Sampling point, weighting coefficient

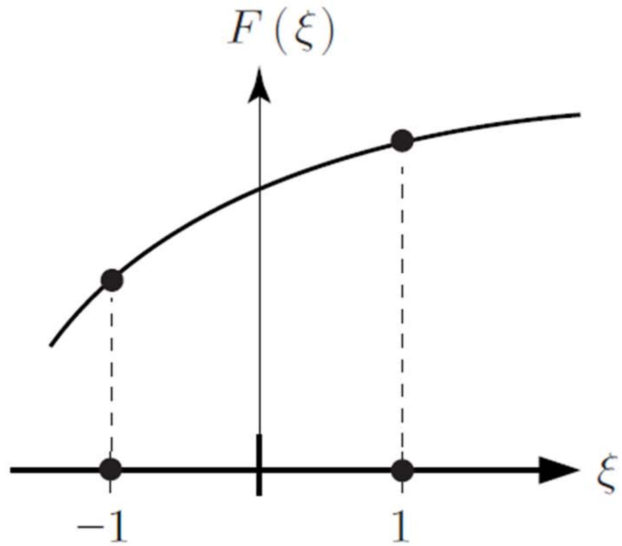
$$(b-a):2=(x-a):(\xi+1)$$

$$\rightarrow \xi=-\frac{b+a}{b-a}+\frac{2}{b-a}x \quad \left(\text{or } x=\frac{a+b}{2}+\frac{b-a}{2}\xi\right)$$

$$\int_a^b \hat{F}(x)\,dx=\frac{b-a}{2}\int_{-1}^{1}F(\xi)\,d\xi$$

$$\int_{-1}^{1}F(\xi)\,d\xi=\begin{cases} w_1F(\xi_1)+w_2F(\xi_2) \leftarrow \text{two points} \\ w_1F(\xi_1)+w_2F(\xi_2)+w_3F(\xi_3) \leftarrow \text{three points}\end{cases}$$



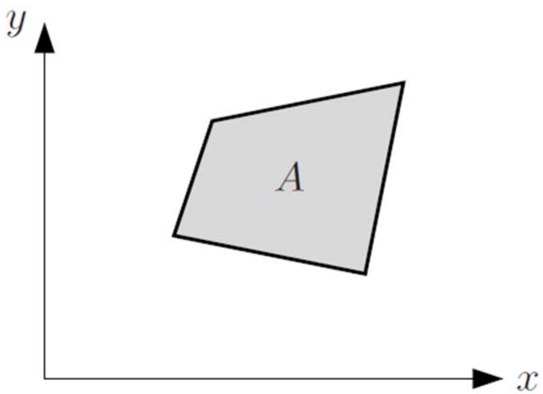
# (Legendre-)Gauss Formula: 1D (2)

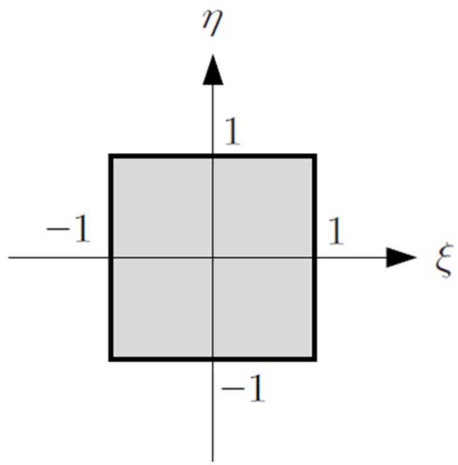
$$\int_{-1}^{1} F(\xi)\,d\xi = w_1 F(\xi_1) + w_2 F(\xi_2) = 1\times F\left(-\frac{1}{\sqrt{3}}\right) + 1\times F\left(\frac{1}{\sqrt{3}}\right)$$

$$\begin{cases} F(\xi)=1 \Rightarrow w_1 + w_2 = 2 \\ F(\xi)=\xi \Rightarrow w_1\xi_1 + w_2\xi_2 = 0 \\ F(\xi)=\xi^2 \Rightarrow w_1\xi_1^{\,2} + w_2\xi_2^{\,2} = \dfrac{2}{3} \\ F(\xi)=\xi^3 \Rightarrow w_1\xi_1^{\,3} + w_2\xi_2^{\,3} = 0 \end{cases} \rightarrow \begin{cases} \xi_1 = -\dfrac{1}{\sqrt{3}},\ \xi_2 = \dfrac{1}{\sqrt{3}} \\ w_1 = w_2 = 1 \end{cases}$$

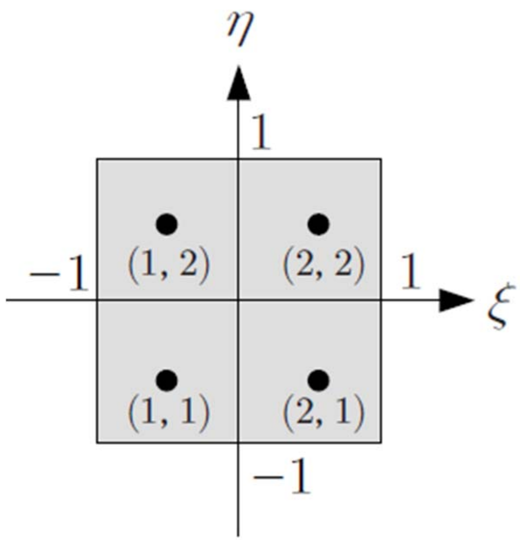
$$\int_{-1}^{1} F(\xi)\,d\xi = w_1 F(\xi_1) + w_2 F(\xi_2) + w_3 F(\xi_3) = \frac{5}{9} F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} F(0) + \frac{5}{9} F\left(\sqrt{\frac{3}{5}}\right)$$

# (Legendre-)Gauss Formula: 2D (1)



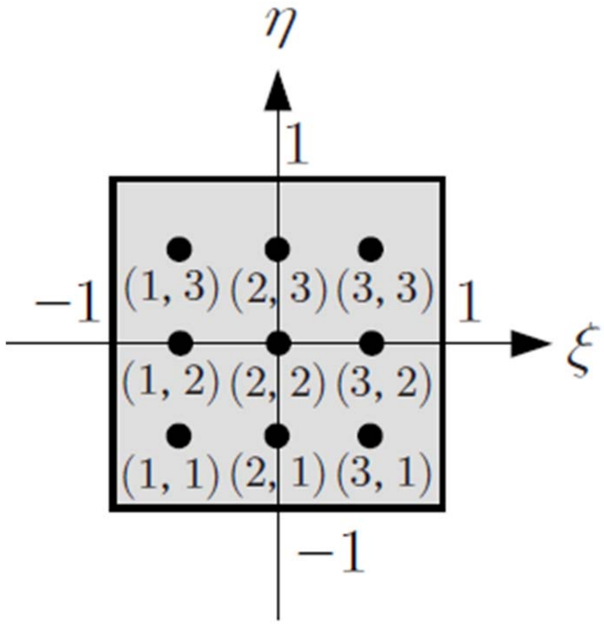




$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \sum_{i=1}^{2}\sum_{j=1}^{2} F(\xi_i,\eta_j)\,w_i w_j$$




| $(i,j)$ | $\xi_i$ | $\eta_j$ | $w_i$ | $w_j$ |
|---|---|---|---|---|
| $(1,1)$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ | 1 | 1 |
| $(1,2)$ | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ | 1 | 1 |
| $(2,1)$ | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ | 1 | 1 |
| $(2,2)$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ | 1 | 1 |

# (Legendre-)Gauss Formula: 2D (2)

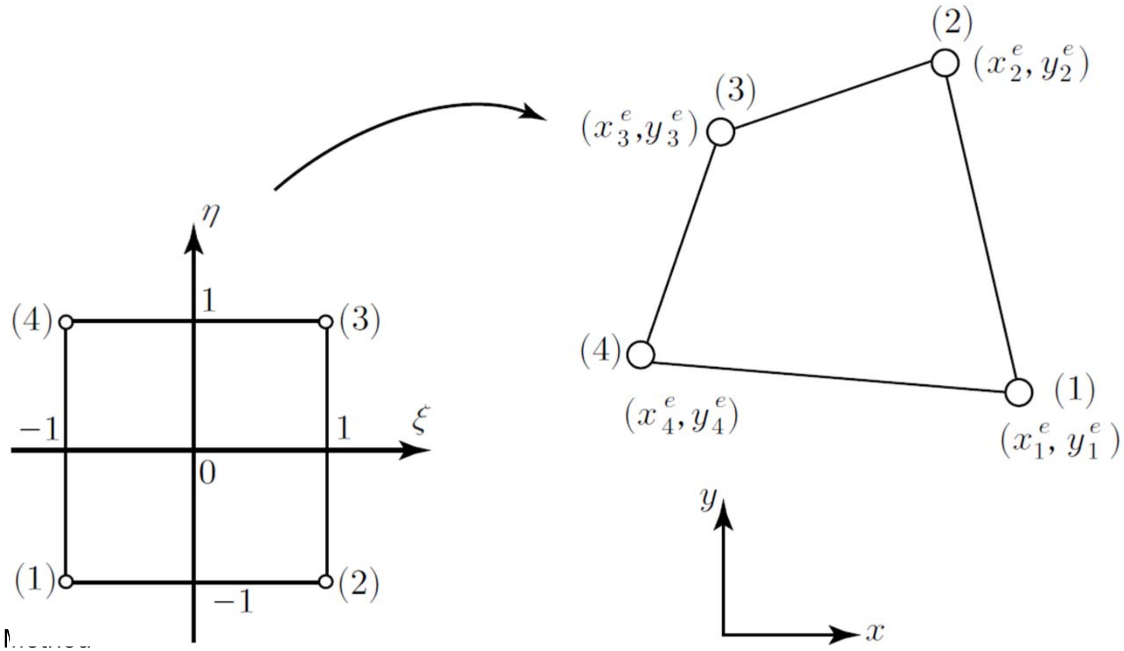



| $(i,j)$ | $\xi_i$ | $\eta_j$ | $w_i$ | $w_j$ |
|---|---|---|---|---|
| $(1,1)$ | $-\sqrt{3/5}$ | $-\sqrt{3/5}$ | $5/9$ | $5/9$ |
| $(1,2)$ | $-\sqrt{3/5}$ | $0$ | $5/9$ | $8/9$ |
| $(1,3)$ | $-\sqrt{3/5}$ | $\sqrt{3/5}$ | $5/9$ | $5/9$ |
| $(2,1)$ | $0$ | $-\sqrt{3/5}$ | $8/9$ | $5/9$ |
| $(2,2)$ | $0$ | $0$ | $8/9$ | $5/9$ |
| $(2,3)$ | $0$ | $\sqrt{3/5}$ | $8/9$ | $5/9$ |
| $(3,1)$ | $\sqrt{3/5}$ | $-\sqrt{3/5}$ | $5/9$ | $5/9$ |
| $(3,2)$ | $\sqrt{3/5}$ | $0$ | $5/9$ | $8/9$ |
| $(3,3)$ | $\sqrt{3/5}$ | $\sqrt{3/5}$ | $5/9$ | $5/9$ |

# Master Element → Physical Element (1)

$$x = c_1 + c_2\xi + c_3\eta + c_4\xi\eta = \{1 \quad \xi \quad \eta \quad \xi\eta\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$

$$x_1^e = \{1 \quad -1 \quad -1 \quad 1\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix},\; x_2^e = \{1 \quad 1 \quad -1 \quad -1\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix},\; x_3^e = \{1 \quad 1 \quad 1 \quad 1\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix},\; x_4^e = \{1 \quad -1 \quad 1 \quad -1\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$



# Master Element → Physical Element (2)

$$\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \Rightarrow x = \{1 \quad \xi \quad \eta \quad \xi\eta\} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} = \{1 \quad \xi \quad \eta \quad \xi\eta\} \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}$$

$$\left\{ N_1^e(\xi,\eta) \quad N_2^e(\xi,\eta) \quad N_3^e(\xi,\eta) \quad N_4^e(\xi,\eta) \right\} = \{1 \quad \xi \quad \eta \quad \xi\eta\} \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$x = \left\{ N_1^e(\xi,\eta) \quad N_2^e(\xi,\eta) \quad N_3^e(\xi,\eta) \quad N_4^e(\xi,\eta) \right\} \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix} \quad \text{where} \quad \begin{cases} N_1^e(\xi,\eta) = \frac{1}{4}(1-\xi-\eta+\xi\eta) = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2^e(\xi,\eta) = \frac{1}{4}(1+\xi-\eta-\xi\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3^e(\xi,\eta) = \frac{1}{4}(1+\xi+\eta+\xi\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4^e(\xi,\eta) = \frac{1}{4}(1-\xi+\eta-\xi\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{cases}$$

# Transformation (1)

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e \end{bmatrix} \begin{Bmatrix} x_1^e \\ y_1^e \\ x_2^e \\ y_2^e \\ x_3^e \\ y_3^e \\ x_4^e \\ y_4^e \end{Bmatrix} = \boldsymbol{N}_e(\xi,\eta)\,\boldsymbol{x}_e$$

$$\left\{ \begin{aligned} &\boldsymbol{x} = \{x \quad y\}^T \\ &\boldsymbol{x}_\alpha^e = \{x_\alpha^e \quad y_\alpha^e\} \ \ (\alpha = 1,\ldots,4) \end{aligned} \right\} \rightarrow \boldsymbol{x} = \sum_{\alpha=1}^{4} N_\alpha^e(\xi,\eta)\,\boldsymbol{x}_\alpha^e$$

$$\left\{ \begin{aligned} dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \\ dy &= \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \end{aligned} \right. \Rightarrow \begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \boldsymbol{J}^T \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \quad \text{where} \quad \underbrace{\boldsymbol{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}}_{\text{Jacobian}}$$

# Transformation (2)

$$\hat{F}(x,y) = F\left(x(\xi,\eta), y(\xi,\eta)\right)$$

$$\begin{cases} \dfrac{\partial \hat{F}}{\partial x} = \dfrac{\partial F}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial F}{\partial \eta}\dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \hat{F}}{\partial y} = \dfrac{\partial F}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial F}{\partial \eta}\dfrac{\partial \eta}{\partial y} \end{cases} \Rightarrow \begin{Bmatrix} \dfrac{\partial \hat{F}}{\partial x} \\ \dfrac{\partial \hat{F}}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial F}{\partial \xi} \\ \dfrac{\partial F}{\partial \eta} \end{Bmatrix} = \boldsymbol{J}^{-1} \begin{Bmatrix} \dfrac{\partial F}{\partial \xi} \\ \dfrac{\partial F}{\partial \eta} \end{Bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \underbrace{\dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \boldsymbol{J}^{-1}}_{\text{inverse Jacobian}} = \begin{bmatrix} J_{11}^{-1} & J_{12}^{-1} \\ J_{21}^{-1} & J_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{j} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

$$j(\xi,\ \eta) = |\boldsymbol{J}| = \det \boldsymbol{J} = J_{11}J_{22} - J_{12}J_{21} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

existence of the inverse of $\boldsymbol{J}$ can be guaranteed by $j > 0$

# Transformation (3)

$$\frac{\partial x}{\partial \xi}=\left\{\frac{\partial N_1^e}{\partial \xi} \quad \frac{\partial N_2^e}{\partial \xi} \quad \frac{\partial N_3^e}{\partial \xi} \quad \frac{\partial N_4^e}{\partial \xi}\right\}\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}=\frac{1}{4}\{-1+\eta \quad 1-\eta \quad 1+\eta \quad -1-\eta\}\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}$$

$$\frac{\partial x}{\partial \eta}=\left\{\frac{\partial N_1^e}{\partial \eta} \quad \frac{\partial N_2^e}{\partial \eta} \quad \frac{\partial N_3^e}{\partial \eta} \quad \frac{\partial N_4^e}{\partial \eta}\right\}\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}=\frac{1}{4}\{-1+\xi \quad -1-\xi \quad 1+\xi \quad 1-\xi\}\begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}$$

$$\frac{\partial y}{\partial \xi}=\left\{\frac{\partial N_1^e}{\partial \xi} \quad \frac{\partial N_2^e}{\partial \xi} \quad \frac{\partial N_3^e}{\partial \xi} \quad \frac{\partial N_4^e}{\partial \xi}\right\}\begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix}=\frac{1}{4}\{-1+\eta \quad 1-\eta \quad 1+\eta \quad -1-\eta\}\begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix}$$

$$\frac{\partial y}{\partial \eta}=\left\{\frac{\partial N_1^e}{\partial \eta} \quad \frac{\partial N_2^e}{\partial \eta} \quad \frac{\partial N_3^e}{\partial \eta} \quad \frac{\partial N_4^e}{\partial \eta}\right\}\begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix}=\frac{1}{4}\{-1+\xi \quad -1-\xi \quad 1+\xi \quad 1-\xi\}\begin{Bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{Bmatrix}$$

# Interpolation of Displacement (1)

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \\ u_4^e \\ v_4^e \end{Bmatrix}$$

$$\boldsymbol{u} \approx \boldsymbol{N}_e \boldsymbol{d}_e = \sum_{\alpha=1}^{4} N_\alpha^e \left(\xi, \eta\right) \boldsymbol{d}_\alpha^e \quad \text{where} \quad \boldsymbol{d}_e{}^T = \left\{ u_1^e \quad v_1^e \quad u_2^e \quad v_2^e \quad u_3^e \quad v_3^e \quad u_4^e \quad v_4^e \right\}$$

$$\boldsymbol{\varepsilon} \approx \partial \boldsymbol{u}^h \approx \partial \boldsymbol{N}_e \boldsymbol{d}_e = \boldsymbol{B}_e \boldsymbol{d}_e = \sum_{\alpha=1}^{4} B_\alpha^e \left(\xi, \eta\right) \boldsymbol{d}_\alpha^e$$

# Interpolation of Displacement (2)

$$
\boldsymbol{B}_e = \partial \boldsymbol{N}_e = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e & 0 & N_4^e \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} & 0 & \frac{\partial N_4^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & 0 & \frac{\partial N_3^e}{\partial y} & 0 & \frac{\partial N_4^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_4^e}{\partial y} & \frac{\partial N_4^e}{\partial x} \end{bmatrix}
$$

$$
= \begin{bmatrix} \boldsymbol{B}_1^e & \boldsymbol{B}_2^e & \boldsymbol{B}_3^e & \boldsymbol{B}_4^e \end{bmatrix}
$$

# Interpolation of Displacement (2)

$$\boldsymbol{B}_\alpha^e = \begin{bmatrix} \dfrac{\partial N_\alpha^e(\xi,\eta)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_\alpha^e(\xi,\eta)}{\partial y} \\ \dfrac{\partial N_\alpha^e(\xi,\eta)}{\partial y} & \dfrac{\partial N_\alpha^e(\xi,\eta)}{\partial x} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_\alpha^e}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial N_\alpha^e}{\partial \eta}\dfrac{\partial \eta}{\partial x} & 0 \\ 0 & \dfrac{\partial N_\alpha^e}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial N_\alpha^e}{\partial \eta}\dfrac{\partial \eta}{\partial y} \\ \dfrac{\partial N_\alpha^e}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial N_\alpha^e}{\partial \eta}\dfrac{\partial \eta}{\partial y} & \dfrac{\partial N_\alpha^e}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial N_\alpha^e}{\partial \eta}\dfrac{\partial \eta}{\partial x} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial N_\alpha^e}{\partial \xi} J_{11}{}^{-1}(\xi,\eta) + \dfrac{\partial N_\alpha^e}{\partial \eta} J_{12}{}^{-1}(\xi,\eta) & 0 \\ 0 & \dfrac{\partial N_\alpha^e}{\partial \xi} J_{21}{}^{-1}(\xi,\eta) + \dfrac{\partial N_\alpha^e}{\partial \eta} J_{22}{}^{-1}(\xi,\eta) \\ \dfrac{\partial N_\alpha^e}{\partial \xi} J_{21}{}^{-1}(\xi,\eta) + \dfrac{\partial N_\alpha^e}{\partial \eta} J_{22}{}^{-1}(\xi,\eta) & \dfrac{\partial N_\alpha^e}{\partial \xi} J_{11}{}^{-1}(\xi,\eta) + \dfrac{\partial N_\alpha^e}{\partial \eta} J_{12}{}^{-1}(\xi,\eta) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial N_\alpha^e}{\partial \xi}\left(\dfrac{1}{j}\dfrac{\partial y}{\partial \eta}\right) + \dfrac{\partial N_\alpha^e}{\partial \eta}\left(-\dfrac{1}{j}\dfrac{\partial y}{\partial \xi}\right) & 0 \\ 0 & \dfrac{\partial N_\alpha^e}{\partial \xi}\left(-\dfrac{1}{j}\dfrac{\partial x}{\partial \eta}\right) + \dfrac{\partial N_\alpha^e}{\partial \eta}\left(\dfrac{1}{j}\dfrac{\partial x}{\partial \xi}\right) \\ \dfrac{\partial N_\alpha^e}{\partial \xi}\left(-\dfrac{1}{j}\dfrac{\partial x}{\partial \eta}\right) + \dfrac{\partial N_\alpha^e}{\partial \eta}\left(\dfrac{1}{j}\dfrac{\partial x}{\partial \xi}\right) & \dfrac{\partial N_\alpha^e}{\partial \xi}\left(\dfrac{1}{j}\dfrac{\partial y}{\partial \eta}\right) + \dfrac{\partial N_\alpha^e}{\partial \eta}\left(-\dfrac{1}{j}\dfrac{\partial y}{\partial \xi}\right) \end{bmatrix}$$

# Element Equation (1)

$$\int_{\Omega_e} \left(\partial \boldsymbol{u}^*\right)^T \boldsymbol{D}_e \left(\partial \boldsymbol{u}\right) h_e dA = \int_{\Omega_e} \boldsymbol{u}^{*T} \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{u}^{*T} \overline{\boldsymbol{t}} h_e ds$$

$$\xrightarrow{\boldsymbol{u}^{*e} = \delta \boldsymbol{u}}$$

$$\int_{\Omega_e} \left(\partial \delta \boldsymbol{u}\right)^T \boldsymbol{D} \left(\partial \boldsymbol{u}\right) h_e dA = \int_{\Omega_e} \delta \boldsymbol{u}^T \boldsymbol{b} h_e dA + \int_{\Gamma_e} \delta \boldsymbol{u}^T \boldsymbol{t} h_e ds$$

$$\delta \boldsymbol{d}_e{}^T \int_{\Omega_e} \boldsymbol{B}_e{}^T \boldsymbol{D} \boldsymbol{B}_e h_e dA \boldsymbol{d}_e = \delta \boldsymbol{d}_e{}^T \int_{\Omega_e} \boldsymbol{N}_e{}^T \boldsymbol{b} h_e dA + \delta \boldsymbol{d}_e{}^T \int_{\Gamma_e} \boldsymbol{N}_e{}^T \boldsymbol{t} h_e ds$$

$$\Leftrightarrow \boldsymbol{K}_e \boldsymbol{d}_e = \boldsymbol{F}_e$$

$$\boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}_e{}^T \left(\xi, \eta\right) \boldsymbol{D} \boldsymbol{B}_e \left(\xi, \eta\right) h_e dA$$

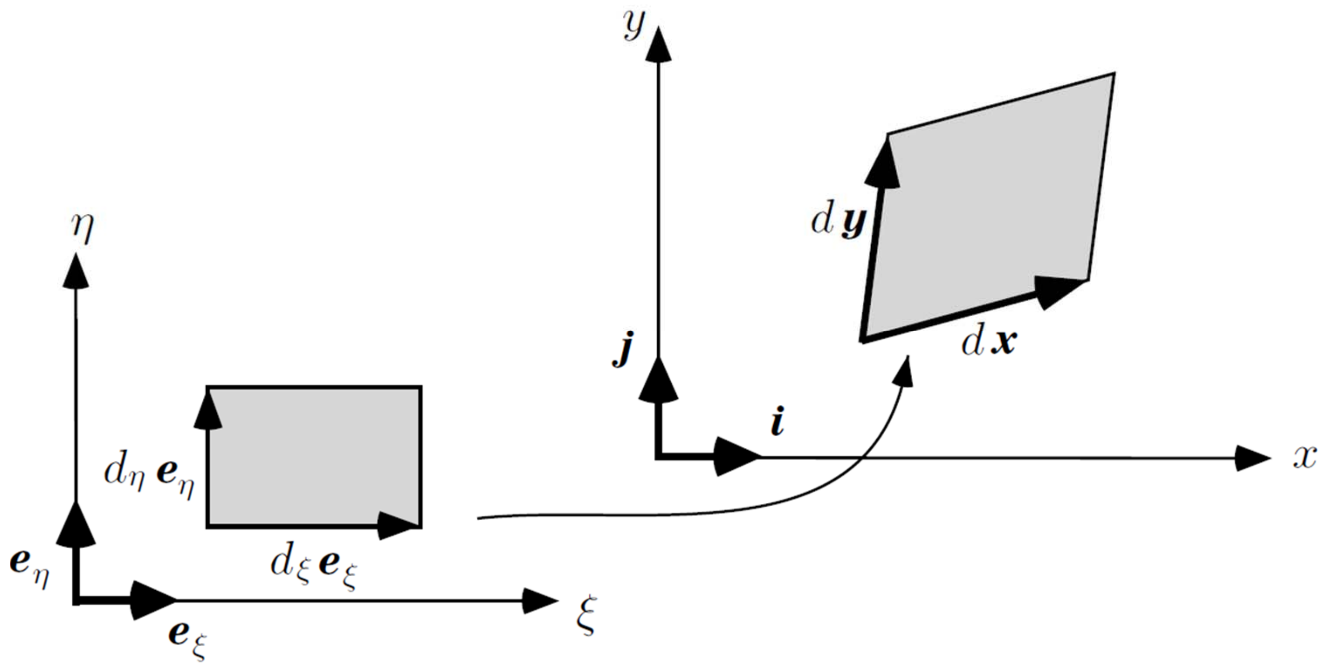
$$\boldsymbol{F}_e = \boldsymbol{F}_e^b + \boldsymbol{F}_e^\sigma = \int_{\Omega_e} \boldsymbol{N}_e{}^T \left(\xi, \eta\right) \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{N}_e{}^T \left(\xi, \eta\right) \boldsymbol{t} h_e ds$$

# Element Equation (2)

$$dV = h_e dA = \left(d\boldsymbol{x} \times d\boldsymbol{y}\right)\cdot\left(h_e \boldsymbol{k}\right) = h_e \left(\frac{\partial x}{\partial \xi} d\xi \boldsymbol{e}_\xi + \frac{\partial x}{\partial \eta} d\eta \boldsymbol{e}_\eta\right) \times \left(\frac{\partial y}{\partial \xi} d\xi \boldsymbol{e}_\xi + \frac{\partial y}{\partial \eta} d\eta \boldsymbol{e}_\eta\right)\cdot \boldsymbol{e}_\varsigma$$

$$= h_e \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) d\xi d\eta = h_e j d\xi d\eta$$

$$\int_{\Omega_e} \hat{F}\left(x, y\right) dV = \int_{\Omega_e} \hat{F}\left(x, y\right) h_e dA = \int_{-1}^{1}\int_{-1}^{1} F\left(\xi, \eta\right) h_e j d\xi d\eta = h_e \sum_{i=1}^{2}\sum_{j=1}^{2} F\left(\xi_i, \eta_j\right) j\left(\xi_i, \eta_j\right) w_i w_j$$

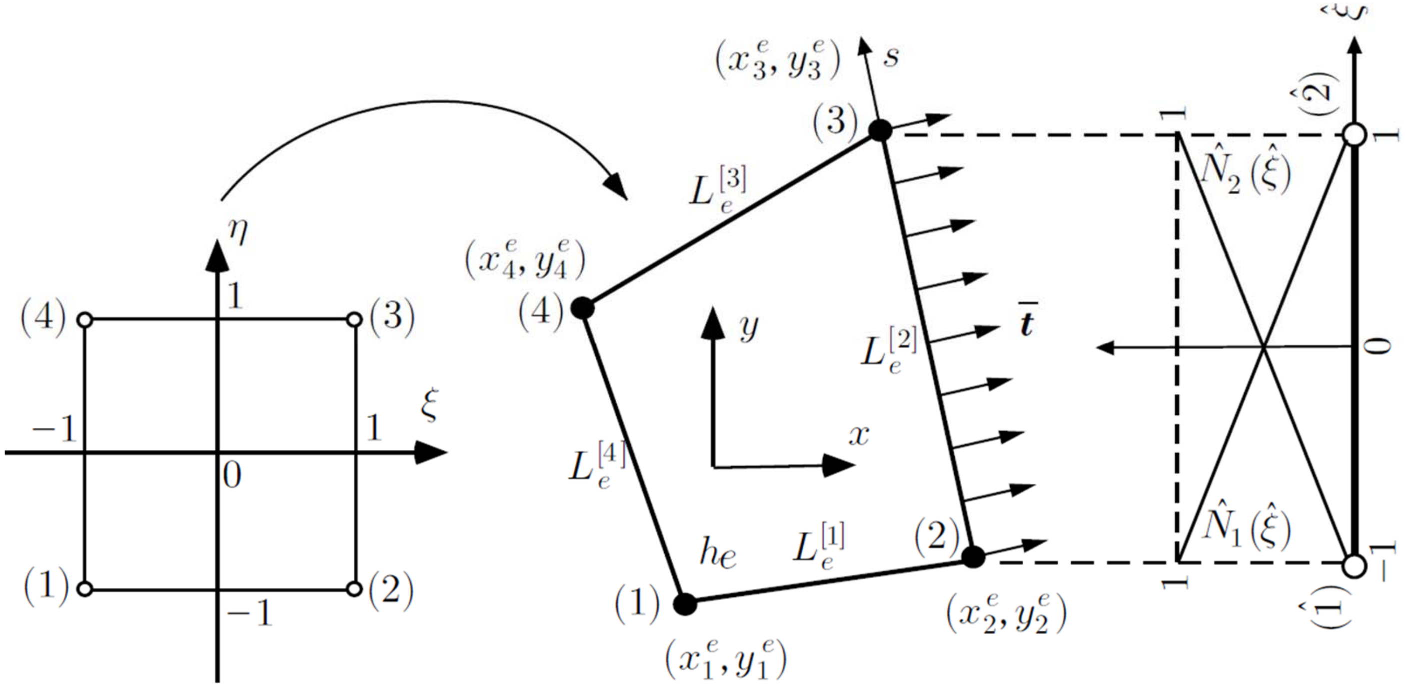


# Element Equation (3)

$$\begin{cases} \boldsymbol{K}_e = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}_e{}^T \boldsymbol{D}\boldsymbol{B}_e h_e \, j \, d\xi \, d\eta \\ \boldsymbol{F}_e = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{N}_e{}^T \boldsymbol{b} h_e \, j \, d\xi \, d\eta \end{cases}$$

$$s = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{\partial x}{\partial \xi} d\xi\right)^2 + \left(\frac{\partial y}{\partial \xi} d\xi\right)^2} = \frac{\partial s}{\partial \xi} d\xi$$

$$\begin{cases} N_1^e(1,\eta) = 0 \\ N_2^e(1,\eta) = \frac{1}{2}(1-\eta) \Rightarrow \hat{N}_{\hat{1}}^e(\hat{\xi}) = \frac{1}{2}(1-\hat{\xi}) \\ N_3^e(1,\eta) = \frac{1}{2}(1+\eta) \Rightarrow \hat{N}_{\hat{2}}^e(\hat{\xi}) = \frac{1}{2}(1+\hat{\xi}) \\ N_4^e(1,\eta) = 0 \end{cases}$$



# Element Equation (4)

$$dx = \frac{\partial x}{\partial \hat{\xi}} d\hat{\xi} = \frac{\partial}{\partial \hat{\xi}} \left( \sum_{\hat{\alpha}=1}^{2} \hat{N}_{\hat{\alpha}}^{e}\left(\hat{\xi}\right) x_{\hat{\alpha}}^{e} \right) d\hat{\xi}, \quad dy = \frac{\partial y}{\partial \hat{\xi}} d\hat{\xi} = \frac{\partial}{\partial \hat{\xi}} \left( \sum_{\hat{\alpha}=1}^{2} \hat{N}_{\hat{\alpha}}^{e}\left(\hat{\xi}\right) y_{\hat{\alpha}}^{e} \right) d\hat{\xi}$$

$$s \approx \sqrt{\left( \sum_{\hat{\alpha}=1}^{2} \frac{\partial \hat{N}_{\hat{\alpha}}^{e}\left(\hat{\xi}\right)}{\partial \xi} x_{\hat{\alpha}}^{e} \right)^2 + \left( \sum_{\hat{\alpha}=1}^{2} \frac{\partial \hat{N}_{\hat{\alpha}}^{e}\left(\hat{\xi}\right)}{\partial \xi} y_{\hat{\alpha}}^{e} \right)^2} d\hat{\xi} \hat{N}_{\hat{\alpha}}^{e}\left(\hat{\xi}\right) = \frac{1}{2}\sqrt{\left(x_{\hat{1}}^{e} - x_{\hat{2}}^{e}\right)^2 + \left(y_{\hat{1}}^{e} - y_{\hat{2}}^{e}\right)^2} d\hat{\xi} = \frac{1}{2} L_e^{[2]} d\hat{\xi}$$

$$\boldsymbol{F}_e^{\sigma} = \int_{\Gamma_e} \boldsymbol{N}_e\left(1,\eta\right)^T \bar{\boldsymbol{t}} h_e ds = \int_{-1}^{1} \hat{\boldsymbol{N}}_e^{\ T}\left(\hat{\xi}\right) \bar{\boldsymbol{t}} h_e \frac{\partial s}{\partial \hat{\xi}} d\hat{\xi} = \int_{-1}^{1} \hat{\boldsymbol{N}}_e^{\ T}\left(\hat{\xi}\right) \bar{\boldsymbol{t}} h_e \frac{L_e^{[2]}}{2} d\hat{\xi}$$

$$\hat{\boldsymbol{N}}_e\left(\hat{\xi}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hat{N}_1^e\left(\hat{\xi}\right) & 0 \\ 0 & \hat{N}_1^e\left(\hat{\xi}\right) \\ \hat{N}_2^e\left(\hat{\xi}\right) & 0 \\ 0 & \hat{N}_2^e\left(\hat{\xi}\right) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2}\left(1-\hat{\xi}\right) & 0 \\ 0 & \frac{1}{2}\left(1-\hat{\xi}\right) \\ \frac{1}{2}\left(\hat{\xi}-1\right) & 0 \\ 0 & \frac{1}{2}\left(\hat{\xi}-1\right) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{F}_e^{\sigma} = \frac{h_e L_e^{[2]}}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} = \frac{h_e L_e^{[2]}}{2} \begin{Bmatrix} 0 \\ 0 \\ \bar{t}_x \\ \bar{t}_y \\ \bar{t}_x \\ \bar{t}_y \\ 0 \\ 0 \end{Bmatrix}$$