# 4.1 Fourier Series for Periodic Functions

- Formula for coefficients
- Decay rate of coefficients
- Rules for the derivative
- Requirements for Fourier to work perfectly

$$S(x) = b_1 \sin x + b_2 \sin 2x + \cdots$$
$$C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$
$$F(x) = C(x) + S(x)$$

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$-u''(x) = \delta(x-a) \xrightarrow{u(x)=\sum_{k=-\infty}^{\infty} c_k e^{ikx}} \sum k^2 c_k e^{ikx} = \sum d_k e^{ikx}$$

# Fourier Sines

$$S(x) = b_1 \sin x + b_2 \sin 2x + \cdots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x) \sin kx dx = \frac{2}{\pi}\int_{0}^{\pi} S(x) \sin kx dx$$

- Orthogonality
  - Breaks the problem down into one-dimensional projections
- Analogy to “weak form”
- Best?
- Example: square wave
  - Gibbs phenomenon

# Fourier Cosines

$$C(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} C(x)dx = \frac{2}{\pi}\int_{-\pi}^{\pi} C(x)dx$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} C(x)\cos kx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} C(x)\cos kx dx$$

- Example: delta function
  - Cannot converge
  - Partial sum up to cos Nx

# Complete Form

- All sines and cosines are mutually orthogonal over every "2π interval"
- Example: square pulse

$$\begin{cases} F(x) = C(x) + S(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \\ a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} C(x)dx \\ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} C(x)\cos kx dx \\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin kx dx \end{cases}$$

$$\rightarrow \begin{cases} F_{even}(x) = \frac{F(x) + F(-x)}{2} = C(x) \\ F_{odd}(x) = \frac{F(x) - F(-x)}{2} = S(x) \end{cases}$$

# Energy

- Energy in x-space = Energy in k-space
  - Square wave
  - Delta function
  - Complex function
- Function space $L^2$ = Hilbert space
  - Space of functions with finite energy, finite length

# Complex Form

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-ikx} dx$$

- Complex exponentials are orthogonal
- Complex inner product
- Orthogonality

- Rules for operating on F(x)

$$\text{derivative}: \frac{dF}{dx} \to ikc_k$$

$$\text{integral}: \int F(x) dx \to \frac{c_k}{ik}$$

$$\text{shift}: F(x-s) \to e^{-iks} c_k$$

$$\text{energy}:$$