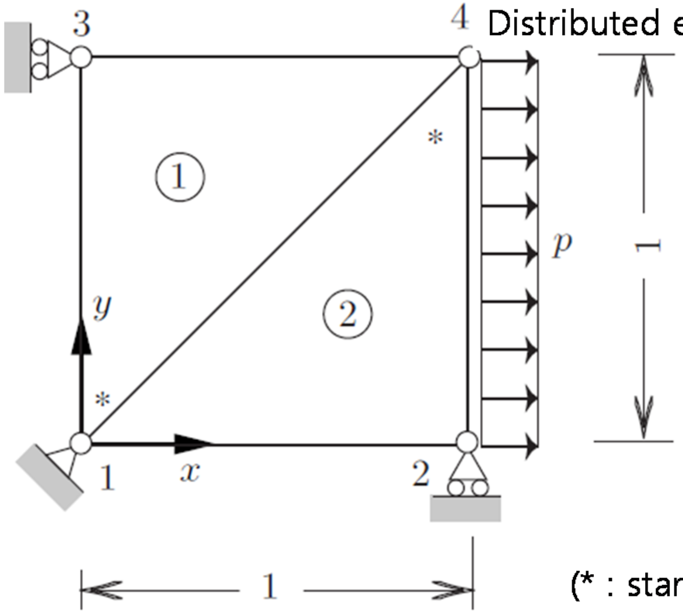
Example



Young's modulus: $E_e = 100$
Poisson ratio: $\nu_e = 1/3$
thickness: $h_e = 1$

(* : start node at each element)

$$\boldsymbol{D}_e = \frac{E_e}{1-\nu_e^2}\begin{bmatrix} 1 & \nu_e & 0 \\ & 1 & 0 \\ sym & & \frac{1-\nu_e}{2} \end{bmatrix} \rightarrow \boldsymbol{D}_{①} = \boldsymbol{D}_{②} = \frac{100}{1-(1/3)^2}\begin{bmatrix} 1 & 1/3 & 0 \\ & 1 & 0 \\ sym & & \frac{1-1/3}{2} \end{bmatrix} = \frac{75}{2}\begin{bmatrix} 3 & 1 & 0 \\ & 3 & 0 \\ sym & & 1 \end{bmatrix}$$

element	node			area	thickness
	1	2	3		
①	1	4	3	1/2	1
②	4	1	2	1/2	1

node	coordinates	
	x	y
1	0.0	0.0
2	1.0	0.0
3	0.0	1.0
4	1.0	1.0

node	displacement	
	u	v
1	0.0	0.0
2	-	0.0
3	0.0	-

element	edge	force	
		tx	ty
①	[2]	0	0
①	[3]	-	0
②	[2]	0	-
②	[3]	p	0

Example: Element ①

$$\left(\text{node: } 1-4-3\right) \quad \boldsymbol{u} = \boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{①}\boldsymbol{d}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①}$$

$$\left.\begin{array}{l} \left(x_1^{①}, y_1^{①}\right) = \left(x_1, y_1\right) = (0,0) \\ \left(x_2^{①}, y_2^{①}\right) = \left(x_4, y_4\right) = (1,1) \\ \left(x_3^{①}, y_3^{①}\right) = \left(x_3, y_3\right) = (0,1) \end{array}\right\} \ L_{①}^{[1]} = \sqrt{2},\ L_{①}^{[2]} = 1,\ L_{①}^{[3]} = 1 \rightarrow \begin{Bmatrix} a_1^{①} \\ b_1^{①} \\ c_1^{①} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix},\ \begin{Bmatrix} a_2^{①} \\ b_2^{①} \\ c_2^{①} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix},\ \begin{Bmatrix} a_3^{①} \\ b_3^{①} \\ c_3^{①} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

$$\boldsymbol{d}_{①} = \left\{u_1^{①} \quad v_1^{①} \quad u_2^{①} \quad v_2^{①} \quad u_3^{①} \quad v_3^{①}\right\}^T, \boldsymbol{N}_{①} = \begin{bmatrix} 1-y & 0 & x & 0 & -x+y & 0 \\ 0 & 1-y & 0 & x & 0 & -x+y \end{bmatrix}, \boldsymbol{B}_{①} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\boldsymbol{K}_{①} = A_{①}h_{①}\boldsymbol{B}_{①}{}^T\boldsymbol{D}_{①}\boldsymbol{B}_{①} = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\frac{75}{2}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} = \frac{75}{4}\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 3 & -1 & 0 & 1 & -3 \\ 0 & -1 & 3 & 0 & -3 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & -3 & 1 & 4 & -2 \\ 1 & -3 & 1 & -1 & -2 & 4 \end{bmatrix}$$

$$\boldsymbol{F}_{①} = \frac{\sqrt{2}}{2}\left\{t_{x①}^{[1]} \quad t_{y①}^{[1]} \quad t_{x①}^{[1]} \quad t_{y①}^{[1]} \quad 0 \quad 0\right\}^T + \frac{1}{2}\left\{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right\}^T + \frac{1}{2}\left\{t_{x①}^{[3]} \quad 0 \quad 0 \quad 0 \quad t_{x①}^{[3]} \quad 0\right\}^T$$

Example: Element ②

$$\left(\text{node: } 4-1-2\right)\quad \boldsymbol{u}=\boldsymbol{N}_{②}\boldsymbol{d}_{②},\ \boldsymbol{\varepsilon}=\partial\boldsymbol{N}_{②}\boldsymbol{d}_{②}=\boldsymbol{B}_{②}\boldsymbol{d}_{②}$$

$$\left.\begin{array}{l}\left(x_1^{②},y_1^{②}\right)=\left(x_4,y_4\right)=\left(1,1\right)\\ \left(x_2^{②},y_2^{②}\right)=\left(x_1,y_1\right)=\left(0,0\right)\\ \left(x_3^{②},y_3^{②}\right)=\left(x_2,y_2\right)=\left(1,0\right)\end{array}\right\}\ L_{②}^{[1]}=\sqrt{2},\ L_{②}^{[2]}=1,\ L_{②}^{[3]}=1\rightarrow\begin{Bmatrix}a_1^{②}\\ b_1^{②}\\ c_1^{②}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix},\ \begin{Bmatrix}a_2^{②}\\ b_2^{②}\\ c_2^{②}\end{Bmatrix}=\begin{Bmatrix}1\\ -1\\ 0\end{Bmatrix},\ \begin{Bmatrix}a_3^{②}\\ b_3^{②}\\ c_3^{②}\end{Bmatrix}=\begin{Bmatrix}0\\ 1\\ -1\end{Bmatrix}$$

$$\boldsymbol{d}_{②}=\left\{u_1^{②}\quad v_1^{②}\quad u_2^{②}\quad v_2^{②}\quad u_3^{②}\quad v_3^{②}\right\}^T,\boldsymbol{N}_{②}=\begin{bmatrix}y&0&1-x&0&x-y&0\\ 0&y&0&1-x&0&x-y\end{bmatrix},\boldsymbol{B}_{②}=\begin{bmatrix}0&0&-1&0&1&0\\ 0&1&0&0&0&-1\\ 1&0&0&-1&-1&1\end{bmatrix}$$

$$\boldsymbol{K}_{②}=A_{②}h_{②}\boldsymbol{B}_{②}^{\ T}\boldsymbol{D}_{②}\boldsymbol{B}_{②}=\frac{1}{2}\begin{bmatrix}0&0&1\\ 0&1&0\\ -1&0&0\\ 0&0&-1\\ 1&0&-1\\ 0&-1&1\end{bmatrix}\frac{75}{2}\begin{bmatrix}3&1&0\\ 1&3&0\\ 0&0&1\end{bmatrix}\begin{bmatrix}0&0&-1&0&1&0\\ 0&1&0&0&0&-1\\ 1&0&0&-1&-1&1\end{bmatrix}=\frac{75}{4}\begin{bmatrix}1&0&0&-1&-1&1\\ 0&3&-1&0&1&-3\\ 0&-1&3&0&-3&1\\ -1&0&0&1&1&-1\\ -1&1&-3&1&4&-2\\ 1&-3&1&-1&-2&4\end{bmatrix}$$

$$\boldsymbol{F}_{②}=\frac{\sqrt{2}}{2}\left\{t_{x②}^{[1]}\quad t_{y②}^{[1]}\quad t_{x②}^{[1]}\quad t_{y②}^{[1]}\quad 0\quad 0\right\}^T+\frac{1}{2}\left\{0\quad 0\quad 0\quad t_{y②}^{[2]}\quad 0\quad t_{y②}^{[2]}\right\}^T+\frac{1}{2}\left\{p\quad 0\quad 0\quad 0\quad p\quad 0\right\}^T$$

Example: Assembly

$$\boldsymbol{d} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T$$

$$\boldsymbol{K} \Leftarrow \boldsymbol{K}_{①}$$

$$\frac{75}{4}\begin{bmatrix} 1 & 0 & (0) & (0) & -1 & 1 & 0 & -1 \\ 0 & 3 & (0) & (0) & 1 & -3 & -1 & 0 \\ (0) & (0) & (0) & (0) & (0) & (0) & (0) & (0) \\ (0) & (0) & (0) & (0) & (0) & (0) & (0) & (0) \\ -1 & 1 & (0) & (0) & 4 & -2 & -3 & 1 \\ 1 & 3 & (0) & (0) & -2 & 4 & 1 & -1 \\ 0 & -1 & (0) & (0) & -3 & 1 & 3 & 0 \\ -1 & 0 & (0) & (0) & 1 & -1 & 0 & 1 \end{bmatrix} \begin{array}{c} u_1 \leftarrow u_1^{①} \\ v_1 \leftarrow v_1^{①} \\ u_2 \\ v_2 \\ u_3 \leftarrow u_3^{①} \\ v_3 \leftarrow v_3^{①} \\ u_4 \leftarrow u_2^{①} \\ v_4 \leftarrow v_2^{①} \end{array}$$

$$\boldsymbol{K} \Leftarrow \boldsymbol{K}_{②}$$

$$\frac{75}{4}\begin{bmatrix} 3 & 0 & -3 & 1 & (0) & (0) & 0 & -1 \\ 0 & 1 & 1 & -1 & (0) & (0) & -1 & 0 \\ -3 & 1 & 4 & -2 & (0) & (0) & -1 & 1 \\ 1 & -1 & -2 & 4 & (0) & (0) & 1 & -3 \\ (0) & (0) & (0) & (0) & (0) & (0) & (0) & (0) \\ (0) & (0) & (0) & (0) & (0) & (0) & (0) & (0) \\ 0 & -1 & -1 & 1 & (0) & (0) & 1 & 0 \\ -1 & 0 & 1 & -3 & (0) & (0) & 0 & 3 \end{bmatrix} \begin{array}{c} u_1 \leftarrow u_2^{②} \\ v_1 \leftarrow v_2^{②} \\ u_2 \leftarrow u_3^{②} \\ v_2 \leftarrow v_3^{②} \\ u_3 \\ v_3 \\ u_4 \leftarrow u_1^{②} \\ v_4 \leftarrow v_1^{②} \end{array}$$

$$\Rightarrow \boldsymbol{K} = \frac{75}{4}\begin{bmatrix} 4 & 0 & -3 & 1 & -1 & 1 & 0 & -2 \\ 0 & 4 & 1 & -1 & 1 & -3 & -2 & 0 \\ -3 & 1 & 4 & -2 & (0) & (0) & -1 & 1 \\ 1 & -1 & -2 & 4 & (0) & (0) & 1 & -3 \\ -1 & 1 & (0) & (0) & 4 & -2 & -3 & 1 \\ 1 & 3 & (0) & (0) & -2 & 4 & 1 & -1 \\ 0 & -2 & -1 & 1 & -3 & 1 & 4 & 0 \\ -2 & 0 & 1 & -3 & 1 & -1 & 0 & 4 \end{bmatrix}, \quad \boldsymbol{F} = \frac{1}{2}\begin{Bmatrix} \sqrt{2}\left(t_{x①}^{[1]} + t_{x②}^{[1]}\right) + t_{x①}^{[3]} \\ \sqrt{2}\left(t_{y①}^{[1]} + t_{y②}^{[1]}\right) + t_{y①}^{[3]} \\ p \\ t_{y①}^{[2]} \\ t_{x①}^{[3]} \\ 0 \\ \sqrt{2}\left(t_{x①}^{[1]} + t_{x②}^{[1]}\right) + p \\ \sqrt{2}\left(t_{y①}^{[1]} + t_{y②}^{[1]}\right) \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} t_{x①}^{[3]} \\ t_{y①}^{[2]} \\ p \\ t_{y①}^{[2]} \\ t_{x①}^{[3]} \\ 0 \\ p \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1^x \\ R_1^y \\ \dfrac{p}{2} \\ R_2^y \\ R_3^x \\ 0 \\ \dfrac{p}{2} \\ 0 \end{Bmatrix}$$

Example: Solution

$\boldsymbol{Kd} = \boldsymbol{F}$ with B.C. $u_1 = 0,\ v_1 = 0,\ v_2 = 0,\ u_3 = 0$

$\boldsymbol{d} = \{0 \quad 0 \quad u_2 \quad 0 \quad 0 \quad v_3 \quad u_4 \quad v_4\}^T$

$$\frac{75}{4}\begin{bmatrix} 4 & 0 & -3 & 1 & -1 & 1 & 0 & -2 \\ 0 & 4 & 1 & -1 & 1 & -3 & -2 & 0 \\ -3 & 1 & 4 & -2 & (0) & (0) & -1 & 1 \\ 1 & -1 & -2 & 4 & (0) & (0) & 1 & -3 \\ -1 & 1 & (0) & (0) & 4 & -2 & -3 & 1 \\ 1 & 3 & (0) & (0) & -2 & 4 & 1 & -1 \\ 0 & -2 & -1 & 1 & -3 & 1 & 4 & 0 \\ -2 & 0 & 1 & -3 & 1 & -1 & 0 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_1^x \\ R_1^y \\ \frac{p}{2} \\ R_2^y \\ R_3^x \\ 0 \\ \frac{p}{2} \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} -3 & 1 & 4 & -2 & 0 & 0 & -1 & 1 \\ 1 & 3 & 0 & 0 & -2 & 4 & 1 & -1 \\ 0 & -2 & -1 & 1 & -3 & 1 & 4 & 0 \\ 2 & 0 & 1 & -3 & 1 & -1 & 0 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \frac{p}{2} \\ 0 \\ \frac{p}{2} \\ 0 \end{Bmatrix}$$

$$\rightarrow \begin{bmatrix} 4 & 0 & -1 & 1 \\ 0 & 4 & 1 & -1 \\ -1 & 1 & 4 & 0 \\ 1 & -1 & 0 & 4 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \frac{p}{2} \\ 0 \\ \frac{p}{2} \\ 0 \end{Bmatrix} - \begin{bmatrix} -3 & 1 & -2 & 0 \\ 1 & 3 & 0 & -2 \\ 0 & -2 & 1 & -3 \\ -2 & 0 & -3 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{p}{2} \\ 0 \\ \frac{p}{2} \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \frac{p}{100}\begin{Bmatrix} 1 \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{Bmatrix}\left(= \frac{p}{E}\begin{Bmatrix} 1 \\ -\nu \\ 1 \\ -\nu \end{Bmatrix} \right)$$

$$\boldsymbol{d} = \left\{0 \quad 0 \quad \frac{p}{100} \quad 0 \quad 0 \quad -\frac{p}{300} \quad \frac{p}{100} \quad -\frac{p}{300}\right\}^T, \boldsymbol{F} = \left\{-\frac{p}{2} \quad 0 \quad \frac{p}{2} \quad 0 \quad -\frac{p}{2} \quad 0 \quad \frac{p}{2} \quad 0\right\}^T$$

Example: Post-process (stress/strain)

element ① $\left(\text{node: } 1-4-3\right)$ $\boldsymbol{u} = \boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{①}\boldsymbol{d}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①}$

$$\boldsymbol{\varepsilon}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ p/100 \\ -p/300 \\ 0 \\ -p/300 \end{Bmatrix} \begin{matrix} u_1^{①} = u_1 \\ v_1^{①} = v_1 \\ u_2^{①} = u_4 \\ v_2^{①} = v_4 \\ u_3^{①} = u_3 \\ v_3^{①} = v_3 \end{matrix} = \frac{p}{100} \begin{Bmatrix} 1 \\ -\dfrac{1}{3} \\ 0 \end{Bmatrix}$$

element ② $\left(\text{node: } 4-1-2\right)$ $\boldsymbol{u} = \boldsymbol{N}_{②}\boldsymbol{d}_{②},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{②}\boldsymbol{d}_{②} = \boldsymbol{B}_{②}\boldsymbol{d}_{②}$

$$\boldsymbol{\varepsilon}_{②} = \boldsymbol{B}_{②}\boldsymbol{d}_{②} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} p/100 \\ -p/300 \\ 0 \\ 0 \\ p/100 \\ 0 \end{Bmatrix} \begin{matrix} u_1^{②} = u_4 \\ v_1^{②} = v_4 \\ u_2^{②} = u_1 \\ v_2^{②} = v_1 \\ u_3^{②} = u_2 \\ v_3^{②} = v_2 \end{matrix} = \frac{p}{100} \begin{Bmatrix} 1 \\ -\dfrac{1}{3} \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\sigma}_{①} = \boldsymbol{D}_{①}\boldsymbol{\varepsilon}_{①} = \frac{75}{2} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{p}{100} \begin{Bmatrix} 1 \\ -1/3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix} = \boldsymbol{\sigma}_{②} = \boldsymbol{D}_{②}\boldsymbol{\varepsilon}_{②}$$