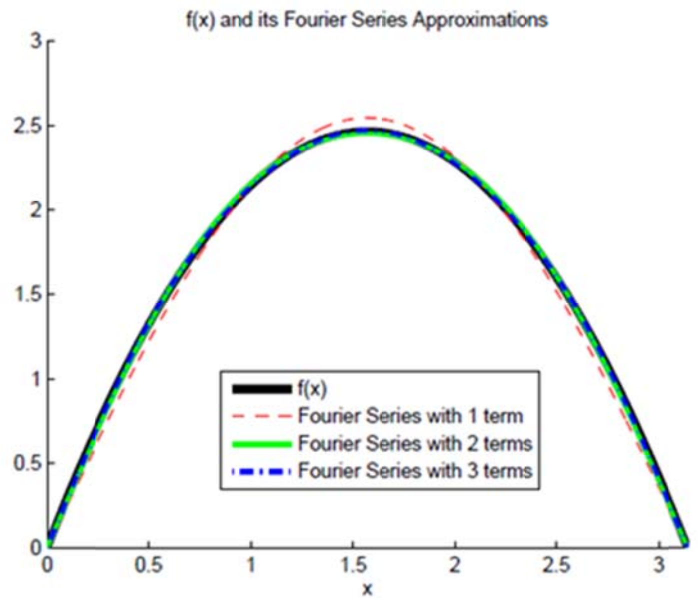
4.1.5

$$f(x)=\begin{cases} x(\pi-x)\text{for } 0<x<\pi \\ -x(\pi-x)\ \text{for } -\pi<x<0 \end{cases} \rightarrow f''(x)=\begin{cases} 2\ \text{for } -\pi<x<0 \\ -2\ \text{for } 0<x<\pi \end{cases}$$

$$\rightarrow f''(x)=\underbrace{2-4S(x)\ \text{for } -\pi<x<\pi}_{\text{step function: decay rate } \frac{1}{k}}$$

Each time a function is integrated the decay rate becomes 1/k times faster.



4.1.7

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} \sin x dx = \frac{1}{\pi}$$

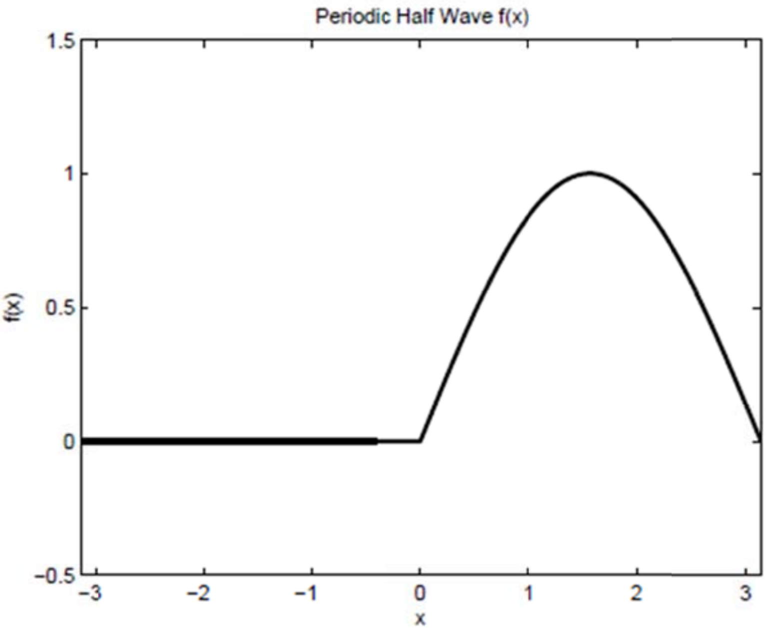
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos kx dx = \begin{cases} 0 \text{ for odd } k \\ -\frac{2}{\pi}\left(\frac{1}{k^2-1}\right) \text{ for even } k \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin kx dx = \begin{cases} 0 \text{ for } k \neq 1 \\ \frac{1}{2} \text{ for } k = 1 \end{cases}$$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j)x)}{(2j)^2-1} + \frac{1}{2}\sin x \xrightarrow{check!} f(x) = \frac{1}{2}\sin x + \frac{1}{2}|\sin x|$$

7)

$$f(x) = \begin{cases} \sin(x), & \text{for } 0 < x < \pi \\ 0, & \text{for } -\pi < x < 0 \end{cases}$$



4.1.13

$$f(x)=\begin{cases}1 \text{ for } |x|<\dfrac{\pi}{2}\\ 0 \text{ for } \dfrac{\pi}{2}<|x|<\pi\end{cases}$$

(a) $\int_{-\pi}^{\pi}\left|f(x)\right|^2 dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx=\pi$

(b) $\begin{cases}c_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx=\dfrac{1}{2}\\ c_k=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{ikx}dx=\dfrac{1}{2\pi}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ikx}dx=\dfrac{1}{2\pi ik}\left[exp\left(\dfrac{ik\pi}{2}\right)-exp\left(\dfrac{-ik\pi}{2}\right)\right]=\dfrac{1}{k\pi}sin\left(\dfrac{k\pi}{2}\right)\end{cases}$

(c) $\int_{-\pi}^{\pi}\left|f(x)\right|^2 dx=2\pi\sum_{k=-\infty}^{\infty}\left|c_k\right|^2=2\pi\left|c_0\right|^2+4\pi\sum_{k=1}^{\infty}\left|c_k\right|^2=\dfrac{\pi}{2}+\dfrac{4}{\pi}\underbrace{\sum_{k=1}^{\infty}\dfrac{sin^2\left(\frac{k\pi}{2}\right)}{k^2}}_{\frac{\pi^2}{8}}=\pi$