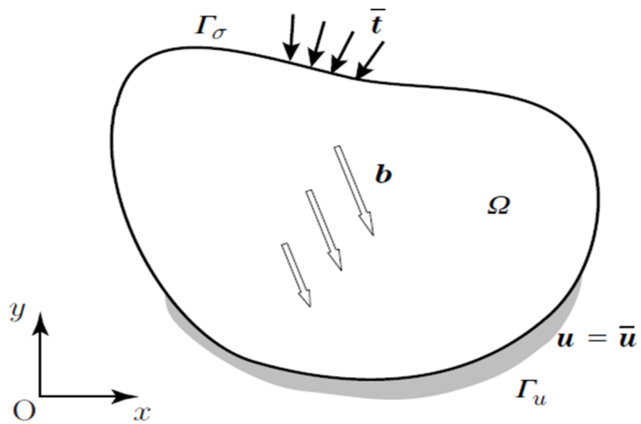
FEM for Steady Problems

  - Governing equation, weak form
  - Discretization, shape function, interpolation
  - Finite element equation, assembly
  - B.C., solution
  - Post-processing
- Potential Flow Problem
- Elasticity Problem
- Characteristics
  - Singularity of coefficient matrix
  - Compatibility and continuity
  - Convergence

2D Elasticity Problem

- Governing equation: strong form
  - Equilibrium equation
  - Strain-displacement relation
  - Stress-strain relation (constitutive equation)
    - Plane stress / Plane strain
  - Boundary conditions
    - Geometrical or Kinematic B.C.: Displacement B.C., support condition
    - Kinetic B.C.: Load B.C., load condition



Governing Equation: Matrix Form (1)

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \partial = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

– Equilibrium equation

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases} \Rightarrow \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = 0 \quad \left(\text{or } \ \partial^T \boldsymbol{\sigma} + \boldsymbol{b} = 0\right)$$

– Strain-displacement relation

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{cases} \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \left(\text{or } \ \boldsymbol{\varepsilon} = \partial \boldsymbol{u}\right)$$

Governing Equation: Matrix Form (2)

– Stress-strain relation (constitutive equation)

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases} \rightarrow \begin{cases} \sigma_x = \dfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right] \\ \sigma_y = \dfrac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu^2\varepsilon_z\right] \\ \tau_{xy} = \dfrac{E}{2(1+\nu)}\gamma_{xy} \end{cases}$$

$$\Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ sym & & D_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (\text{or } \boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon})$$

$$\text{plane stress}: \boldsymbol{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ & 1 & 0 \\ sym & & \dfrac{1-\nu}{2} \end{bmatrix}, \quad \text{plane strain}: \boldsymbol{D} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ & 1 & 0 \\ sym & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Governing Equation: Matrix Form (3)

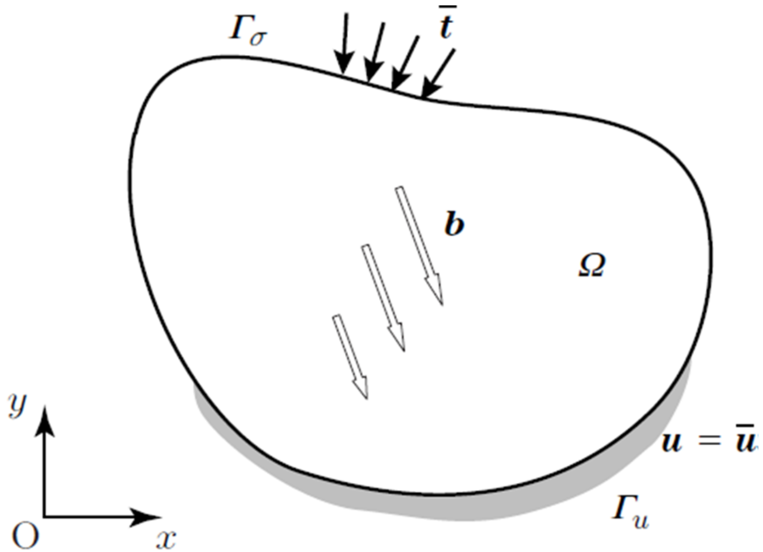
– Boundary Conditions

$$\left.\begin{aligned} u &= \bar{u} \\ v &= \bar{v} \end{aligned}\right\} \text{ on } \Gamma_u \Rightarrow \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} \quad (\text{or } \boldsymbol{u} = \bar{\boldsymbol{u}}) \text{ on } \Gamma_u$$

$$\boldsymbol{t} = \{t_x \quad t_y\}^T (\text{surface force}), \boldsymbol{n} = \{n_x \quad n_y\}^T (\text{normal vector})$$

$$\begin{cases} t_x = \sigma_x n_x + \tau_{xy} n_y \\ t_y = \tau_{xy} n_x + \sigma_y n_y \end{cases} \Rightarrow \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} \quad (\boldsymbol{m\sigma} = \boldsymbol{t} = \bar{\boldsymbol{t}}) \text{ on } \Gamma_\sigma$$

$$\text{where } \bar{\boldsymbol{u}} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix}, \bar{\boldsymbol{t}} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix}, \boldsymbol{m} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$$



Weak Form

Using test function $\begin{Bmatrix} u^* \\ v^* \end{Bmatrix}$ satisfying $\begin{Bmatrix} u^* \\ v^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ $\left(\text{or } \boldsymbol{u}^* = 0\right)$ on Γ_u

$$\int_\Omega \left[u^* \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x \right) + v^* \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y \right) \right] dV = 0$$

$$\int_\Omega \begin{Bmatrix} u^* & v^* \end{Bmatrix} \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \right) dV = 0$$

$$\int_\Omega \boldsymbol{u}^{*T} \left(\partial^T \boldsymbol{\sigma} + \boldsymbol{b} \right) dV = 0 \xrightarrow{\text{Gauss-Green Theorem}} \underbrace{\int_\Gamma \boldsymbol{u}^{*T} \left(\boldsymbol{m\sigma} \right) dS - \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{\sigma} dV}_{} + \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV = 0$$

$$\xrightarrow{\text{kinetic B.C.}} \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{\sigma} dV = \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV + \int_{\Gamma_\sigma} \boldsymbol{u}^{*T} \bar{\boldsymbol{t}} dS \xrightarrow{\boldsymbol{\sigma} = \boldsymbol{D\varepsilon}} \int_\Omega \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{D} \left(\partial \boldsymbol{u} \right) dV = \int_\Omega \boldsymbol{u}^{*T} \boldsymbol{b} dV + \int_{\Gamma_\sigma} \boldsymbol{u}^{*T} \bar{\boldsymbol{t}} dS$$

$$\underbrace{\text{Weak form}\left(u^*\right)}_{\text{based on Galerkin's method}} = \underbrace{\text{Equation of Virtual Work}\left(\delta u\right)}_{\text{based on the displacement method}}$$

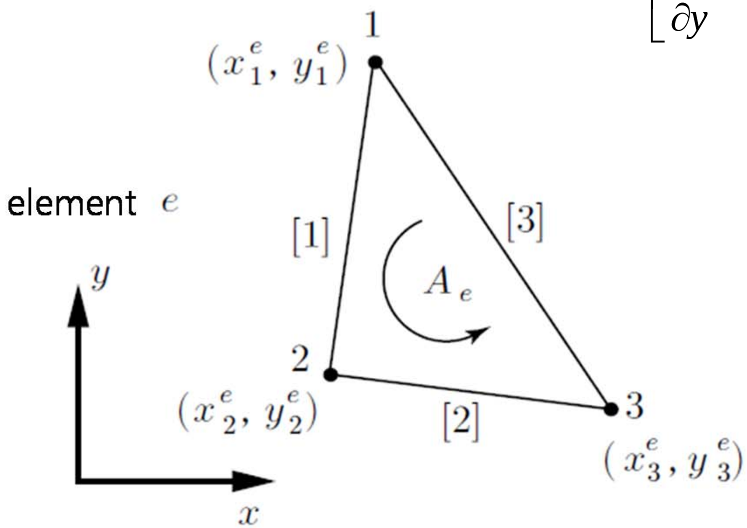
Discretization, Assumed Displacement

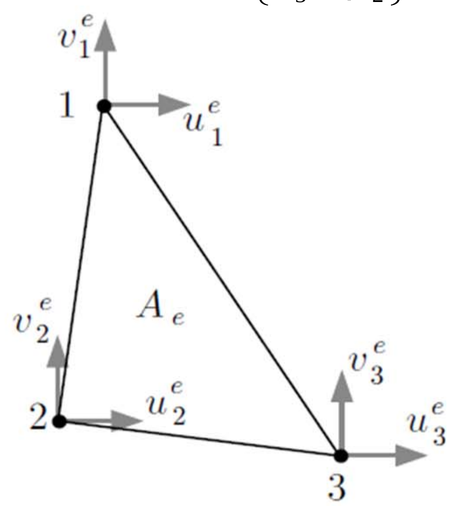
$$\int_{\Omega}\left(\partial\boldsymbol{u}^*\right)^T \boldsymbol{D}\left(\partial\boldsymbol{u}\right)dV=\int_{\Omega}\boldsymbol{u}^{*T}\boldsymbol{b}dV+\int_{\Gamma_\sigma}\boldsymbol{u}^{*T}\bar{\boldsymbol{t}}dS \xrightarrow[\substack{dV=h_e dA\\ dS=h_e ds}]{\Omega\approx\bigcup_{e=1}^{M}\Omega_e} \int_{\Omega_e}\left(\partial\boldsymbol{u}^*\right)^T \boldsymbol{D}_e\left(\partial\boldsymbol{u}\right)h_e dA=\int_{\Omega_e}\boldsymbol{u}^{*T}\boldsymbol{b}h_e dA+\int_{\Gamma_e}\boldsymbol{u}^{*T}\bar{\boldsymbol{t}}h_e ds$$

linear triangle element

CST(Constant Stain Triangle): edges remains straight after the deformation

$$\boldsymbol{u}^T=\{u \quad v\} \text{ where } \begin{cases} u\approx\alpha_1+\alpha_2 x+\alpha_3 y\\ v\approx\beta_1+\beta_2 x+\beta_3 y\end{cases} \rightarrow \boldsymbol{\varepsilon}=\partial\boldsymbol{u}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}\alpha_1+\alpha_2 x+\alpha_3 y\\ \beta_1+\beta_2 x+\beta_3 y\end{Bmatrix}=\begin{Bmatrix}\alpha_2\\ \beta_3\\ \alpha_3+\beta_2\end{Bmatrix}$$





Shape Function

$$u \approx \{1 \quad x \quad y\}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix},\ v \approx \{1 \quad x \quad y\}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} \rightarrow \begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix},\ \begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix}$$

$$u \approx \{1 \quad x \quad y\}\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}^{-1}\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \{1 \quad x \quad y\}\begin{bmatrix}a_1^e & a_2^e & a_3^e\\ b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix} = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix}$$

$$v \approx \{1 \quad x \quad y\}\begin{bmatrix}1 & x_1^e & y_1^e\\ 1 & x_2^e & y_2^e\\ 1 & x_3^e & y_3^e\end{bmatrix}^{-1}\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \{1 \quad x \quad y\}\begin{bmatrix}a_1^e & a_2^e & a_3^e\\ b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix} = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]\begin{Bmatrix}v_1^e\\ v_2^e\\ v_3^e\end{Bmatrix}$$

$$N_\alpha^e(x,y) = a_\alpha^e + b_\alpha^e x + c_\alpha^e y \quad (\alpha = 1,2,3) \rightarrow \begin{cases} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = \begin{cases}1 \text{ if } \alpha = \beta\\ 0 \text{ if } \alpha \neq \beta\end{cases} \left[\text{Kronecker } \delta\right] \text{ continuity} \\ \sum_{\alpha=1}^{3} N_\alpha^e\left(x_\beta^e, y_\beta^e\right) = 1 \left[\text{partition of unity}\right] \text{ convergence}\end{cases}$$

$$\text{where } \begin{cases} a_1^e = \dfrac{1}{2A_e}\left(x_2^e y_3^e - x_3^e y_2^e\right) \\ b_1^e = \dfrac{1}{2A_e}\left(y_2^e - y_3^e\right) \\ c_1^e = \dfrac{1}{2A_e}\left(x_3^e - x_2^e\right)\end{cases} \begin{cases} a_2^e = \dfrac{1}{2A_e}\left(x_3^e y_1^e - x_1^e y_3^e\right) \\ b_2^e = \dfrac{1}{2A_e}\left(y_3^e - y_1^e\right) \\ c_2^e = \dfrac{1}{2A_e}\left(x_1^e - x_3^e\right)\end{cases} \begin{cases} a_3^e = \dfrac{1}{2A_e}\left(x_1^e y_2^e - x_2^e y_1^e\right) \\ b_3^e = \dfrac{1}{2A_e}\left(y_1^e - y_2^e\right) \\ c_3^e = \dfrac{1}{2A_e}\left(x_2^e - x_1^e\right)\end{cases}$$

Interpolation

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} \approx \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{Bmatrix} = \boldsymbol{N}_e \boldsymbol{d}_e \quad \text{where} \quad \begin{cases} \boldsymbol{N}_e : \text{shape function matrix} \\ \boldsymbol{d}_e : \text{element nodal displacement vector} \end{cases}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \approx \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & 0 & \frac{\partial N_3^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_3^e}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \partial \boldsymbol{u} \approx \partial \boldsymbol{N}_e \boldsymbol{d}_e = \boldsymbol{B}_e \boldsymbol{d}_e$$

$$\xrightarrow{\text{CST}} \boldsymbol{B}_e = \begin{bmatrix} b_1^e & 0 & b_2^e & 0 & b_3^e & 0 \\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e \\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e \end{bmatrix} = \frac{1}{2A_e} \begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix}$$

Element Stiffness Equation (1)

FEM based on $\begin{cases} \text{Galerkin method: weak form / test function} \left(\boldsymbol{u}^* \approx \boldsymbol{N}_e \boldsymbol{d}_e^* \right) \text{ / virtual strain} \left(\boldsymbol{\varepsilon}^* \approx \boldsymbol{B}_e \boldsymbol{d}_e^* \right) \\ \text{Displacement method: equation of vitual work / variation of displacement / strain from variation} \end{cases}$

$$\int_{\Omega_e} \left(\partial \boldsymbol{u}^* \right)^T \boldsymbol{D}_e \left(\partial \boldsymbol{u} \right) h_e dA = \int_{\Omega_e} \boldsymbol{u}^{*T} \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{u}^{*T} \boldsymbol{t} h_e ds \rightarrow \boldsymbol{d}_e^{*T} \int_{\Omega_e} \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e h_e dA \boldsymbol{d}_e = \boldsymbol{d}_e^{*T} \int_{\Omega_e} \boldsymbol{N}_e^T \boldsymbol{b} h_e dA + \boldsymbol{d}_e^{*T} \int_{\Gamma_e} \boldsymbol{N}_e^T \boldsymbol{t} h_e ds$$

$$\rightarrow \boldsymbol{K}_e \boldsymbol{d}_e = \boldsymbol{F}_e \quad \text{where} \quad \begin{cases} \boldsymbol{K}_e : \text{element stiffness matrix} \\ \boldsymbol{F}_e : \text{element nodal load vector} \end{cases}$$

$$\boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}_e^T \boldsymbol{D}_e \boldsymbol{B}_e h_e dA = \int_{\Omega_e} \begin{bmatrix} b_1^e & 0 & c_1^e \\ 0 & c_1^e & b_1^e \\ b_2^e & 0 & c_2^e \\ 0 & c_2^e & b_2^e \\ b_3^e & 0 & c_3^e \\ 0 & c_3^e & b_3^e \end{bmatrix} \begin{bmatrix} D_{11}^e & D_{12}^e & D_{13}^e \\ & D_{22}^e & D_{23}^e \\ sym & & D_{33}^e \end{bmatrix} \begin{bmatrix} b_1^e & 0 & b_2^e & 0 & b_3^e & 0 \\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e \\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e \end{bmatrix} h_e dA$$

$$\boldsymbol{F}_e = \boldsymbol{F}_e^b + \boldsymbol{F}_e^t = \int_{\Omega_e} \boldsymbol{N}_e^T \boldsymbol{b} h_e dA + \int_{\Gamma_e} \boldsymbol{N}_e^T \boldsymbol{t} h_e ds = \int_{\Omega_e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} h_e dA + \int_{\Gamma_e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} h_e ds$$

Element Stiffness Equation (2)

TRI3 : node increase; QUAD4 : isoparametric $\Big\} \rightarrow$ numerical integration!

$$\xrightarrow[\text{uniform thickness}]{\text{homogeneous material}} \boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}_e^{\,T} \boldsymbol{D}_e \boldsymbol{B}_e h_e dA = A_e h_e \boldsymbol{B}_e^{\,T} \boldsymbol{D}_e \boldsymbol{B}_e = A_e h_e \begin{bmatrix} b_1^e & 0 & c_1^e \\ 0 & c_1^e & b_1^e \\ b_2^e & 0 & c_2^e \\ 0 & c_2^e & b_2^e \\ b_3^e & 0 & c_3^e \\ 0 & c_3^e & b_3^e \end{bmatrix} \begin{bmatrix} D_{11}^e & D_{12}^e & D_{13}^e \\ & D_{22}^e & D_{23}^e \\ sym & & D_{33}^e \end{bmatrix} \begin{bmatrix} b_1^e & 0 & b_2^e & 0 & b_3^e & 0 \\ 0 & c_1^e & 0 & c_2^e & 0 & c_3^e \\ c_1^e & b_1^e & c_2^e & b_2^e & c_3^e & b_3^e \end{bmatrix}$$

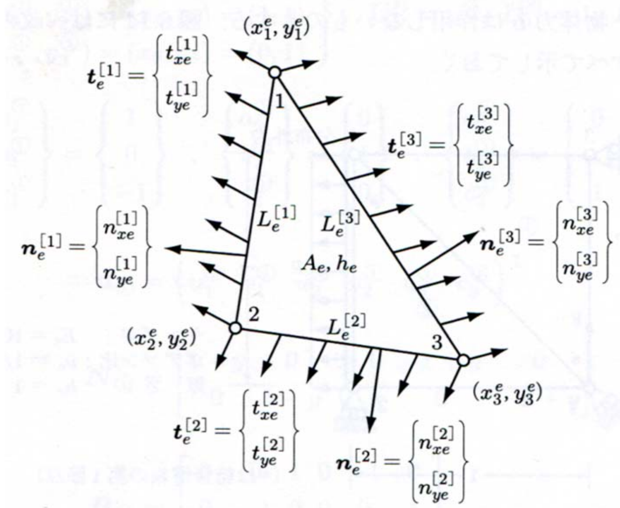
$$\xrightarrow[\text{in the element}]{\text{constant body force}} \boldsymbol{F}_e^b = \frac{A_e h_e}{3} \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix}$$

surface force along the edge neighboring elements: internal force (unknown)

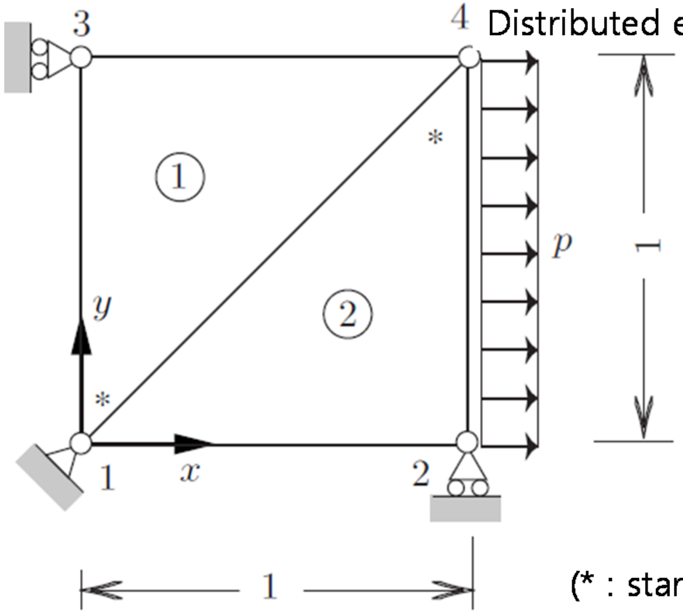
$\begin{cases} \text{natural boundary: surface force (known)} \\ \text{essential boundary: reaction force (unknown)} \end{cases}$

physical quantities distributed in the element $\xrightarrow[\text{based on weak form}]{\text{FEM}}$ equivalent nodal force

$$\xrightarrow[\text{along the edge}]{\text{constant surface force}} \boldsymbol{F}_e^t = \sum_{i=1}^{3} \left(\int_{\Gamma_e^{[i]}} \boldsymbol{N}_e^{\,T} ds \right) \boldsymbol{t}_e^{[i]} h_e = \frac{L_e^{[1]} h_e}{2} \begin{Bmatrix} t_{xe}^{[1]} \\ t_{ye}^{[1]} \\ t_{xe}^{[1]} \\ t_{ye}^{[1]} \\ 0 \\ 0 \end{Bmatrix} + \frac{L_e^{[2]} h_e}{2} \begin{Bmatrix} 0 \\ 0 \\ t_{xe}^{[2]} \\ t_{ye}^{[2]} \\ t_{xe}^{[2]} \\ t_{ye}^{[2]} \end{Bmatrix} + \frac{L_e^{[3]} h_e}{2} \begin{Bmatrix} t_{xe}^{[3]} \\ t_{ye}^{[3]} \\ 0 \\ 0 \\ t_{xe}^{[3]} \\ t_{ye}^{[3]} \end{Bmatrix}$$



Example



Young's modulus: E_e = 100
Poisson ratio: ν_e = 1/3
thickness: h_e = 1

(* : start node at each element)

$$\boldsymbol{D}_e = \frac{E_e}{1-\nu_e^2}\begin{bmatrix} 1 & \nu_e & 0 \\ & 1 & 0 \\ sym & & \frac{1-\nu_e}{2} \end{bmatrix} \rightarrow \boldsymbol{D}_{①} = \boldsymbol{D}_{②} = \frac{100}{1-(1/3)^2}\begin{bmatrix} 1 & 1/3 & 0 \\ & 1 & 0 \\ sym & & \frac{1-1/3}{2} \end{bmatrix} = \frac{75}{2}\begin{bmatrix} 3 & 1 & 0 \\ & 3 & 0 \\ sym & & 1 \end{bmatrix}$$

element	node			area	thickness
	1	2	3		
①	1	4	3	1/2	1
②	4	1	2	1/2	1

node	coordinates	
	x	y
1	0.0	0.0
2	1.0	0.0
3	0.0	1.0
4	1.0	1.0

node	displacement	
	u	v
1	0.0	0.0
2	-	0.0
3	0.0	-

element	edge	force	
		tx	ty
①	[2]	0	0
①	[3]	-	0
②	[2]	0	-
②	[3]	p	0

Example: Element ①

$\left(\text{node: } 1-4-3\right)\quad \boldsymbol{u} = \boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{①}\boldsymbol{d}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①}$

$\boldsymbol{d}_{①} = \left\{ u_1^{①} \quad v_1^{①} \quad u_2^{①} \quad v_2^{①} \quad u_3^{①} \quad v_3^{①} \right\}^T,$

$\boldsymbol{N}_{①} =$

$\boldsymbol{B}_{①} =$

$\boldsymbol{K}_{①} =$

$\boldsymbol{F}_{①} =$

Example: Element ②

$\left(\text{node: } 4-1-2\right)\quad \boldsymbol{u} = \boldsymbol{N}_{②}\boldsymbol{d}_{②},\ \boldsymbol{\varepsilon} = \partial \boldsymbol{N}_{②}\boldsymbol{d}_{②} = \boldsymbol{B}_{②}\boldsymbol{d}_{②}$

$\boldsymbol{d}_{②} = \left\{ u_1^{②} \quad v_1^{②} \quad u_2^{②} \quad v_2^{②} \quad u_3^{②} \quad v_3^{②} \right\}^T$

$\boldsymbol{N}_{②} =$

$\boldsymbol{B}_{②} =$

$\boldsymbol{K}_{②} =$

$\boldsymbol{F}_{②} =$

Example: Assembly

$$\boldsymbol{d} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T$$

$$\boldsymbol{K} \Leftarrow \boldsymbol{K}_{①} \quad \boldsymbol{K} \Leftarrow \boldsymbol{K}_{②}$$

$$\Rightarrow \boldsymbol{K} =, \quad \boldsymbol{F} =$$

Example: Solution

$\boldsymbol{Kd} = \boldsymbol{F}$ with B.C. $u_1 = 0,\ v_1 = 0,\ v_2 = 0,\ u_3 = 0$

$$\boldsymbol{d} = \left\{ 0 \quad 0 \quad \frac{p}{100} \quad 0 \quad 0 \quad -\frac{p}{300} \quad \frac{p}{100} \quad -\frac{p}{300} \right\}^T$$

$\boldsymbol{F} =$

Example: Post-process (stress/strain)

element ① $\left(\text{node: } 1-4-3\right)$ $\boldsymbol{u} = \boldsymbol{N}_{①}\boldsymbol{d}_{①},\ \boldsymbol{\varepsilon} = \partial\boldsymbol{N}_{①}\boldsymbol{d}_{①} = \boldsymbol{B}_{①}\boldsymbol{d}_{①}$

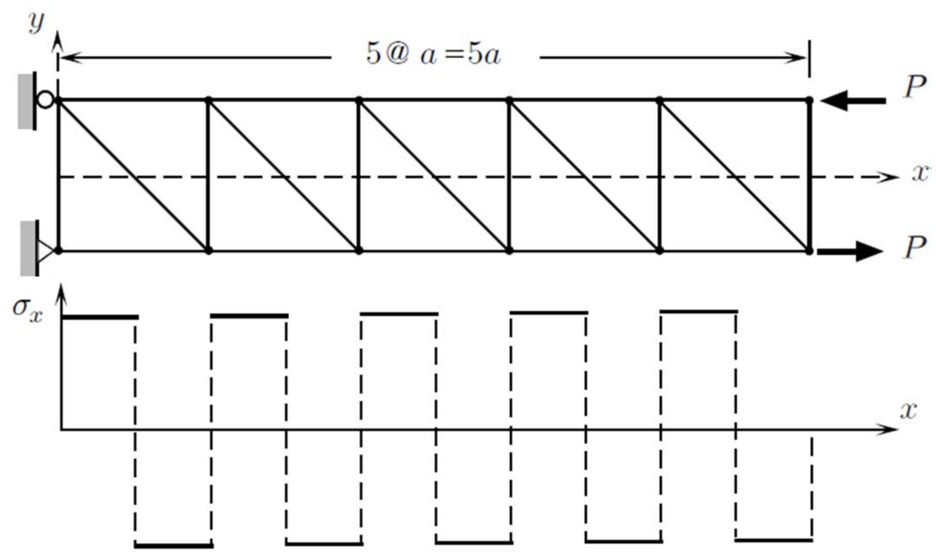
$\boldsymbol{\sigma}_{①} = \boldsymbol{D}_{①}\boldsymbol{\varepsilon}_{①}$

element ② $\left(\text{node: } 4-1-2\right)$ $\boldsymbol{u} = \boldsymbol{N}_{②}\boldsymbol{d}_{②},\ \boldsymbol{\varepsilon} = \partial\boldsymbol{N}_{②}\boldsymbol{d}_{②} = \boldsymbol{B}_{②}\boldsymbol{d}_{②}$

$\boldsymbol{\sigma}_{②} = \boldsymbol{D}_{②}\boldsymbol{\varepsilon}_{②}$

Characteristics (1)

- Singularity of stiffness matrix
  - Check with eigenvalues
  - No energy
  - Need physical constraints to prevent rigid body motion (displacement B.C., support condition)
  - Assembled matrix (singular) → apply B.C. → reduced matrix (regular)
- Approximate solution
  - Equilibrium only at nodes
  - at arbitrary point in elements?

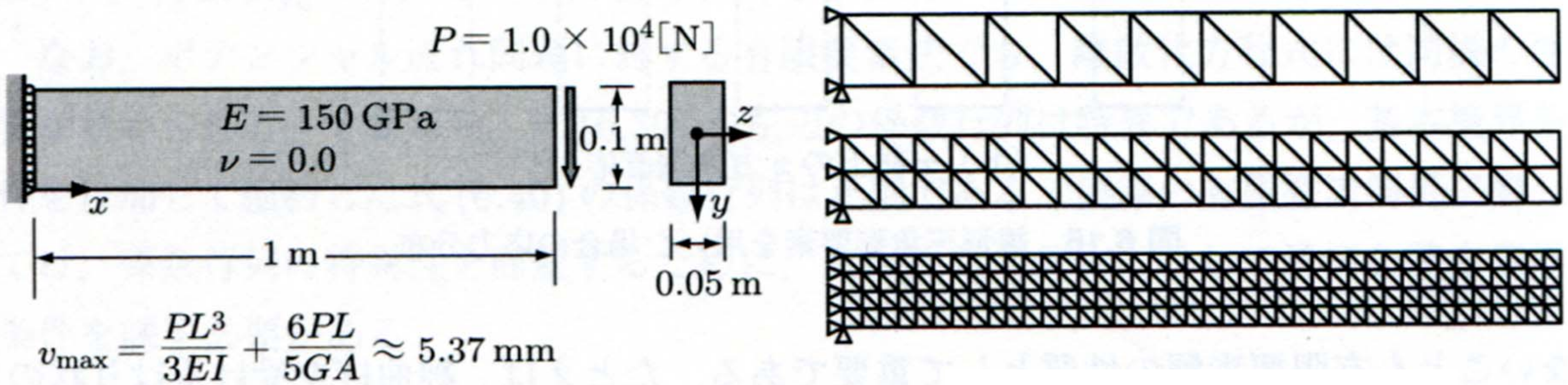


Characteristics (2)

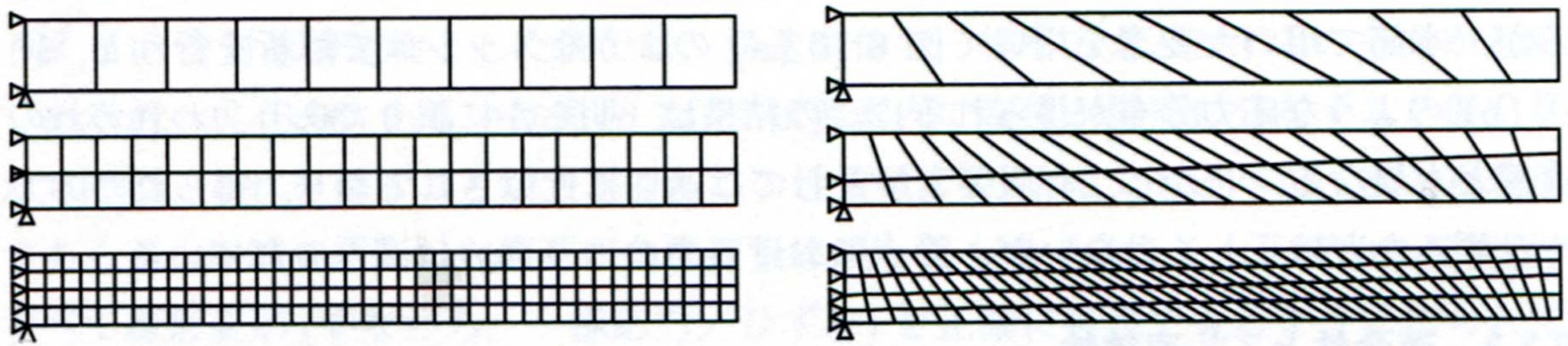
- Compatibility and Continuity
  - Node values including edges are continuous among elements → conforming
  - Non-conforming element?
  - Linear triangular element: (displacement) linear (stress/strain) constant → C^0 continuous
  - Multi-material elastic body: strain?
- Convergence
  - The accuracy of an approximate solution improves as long as the number of nodes or elements is increased.
  - Element-type, mesh pattern

Convergence: Example (1)

- 2D elasticity (plane stress)



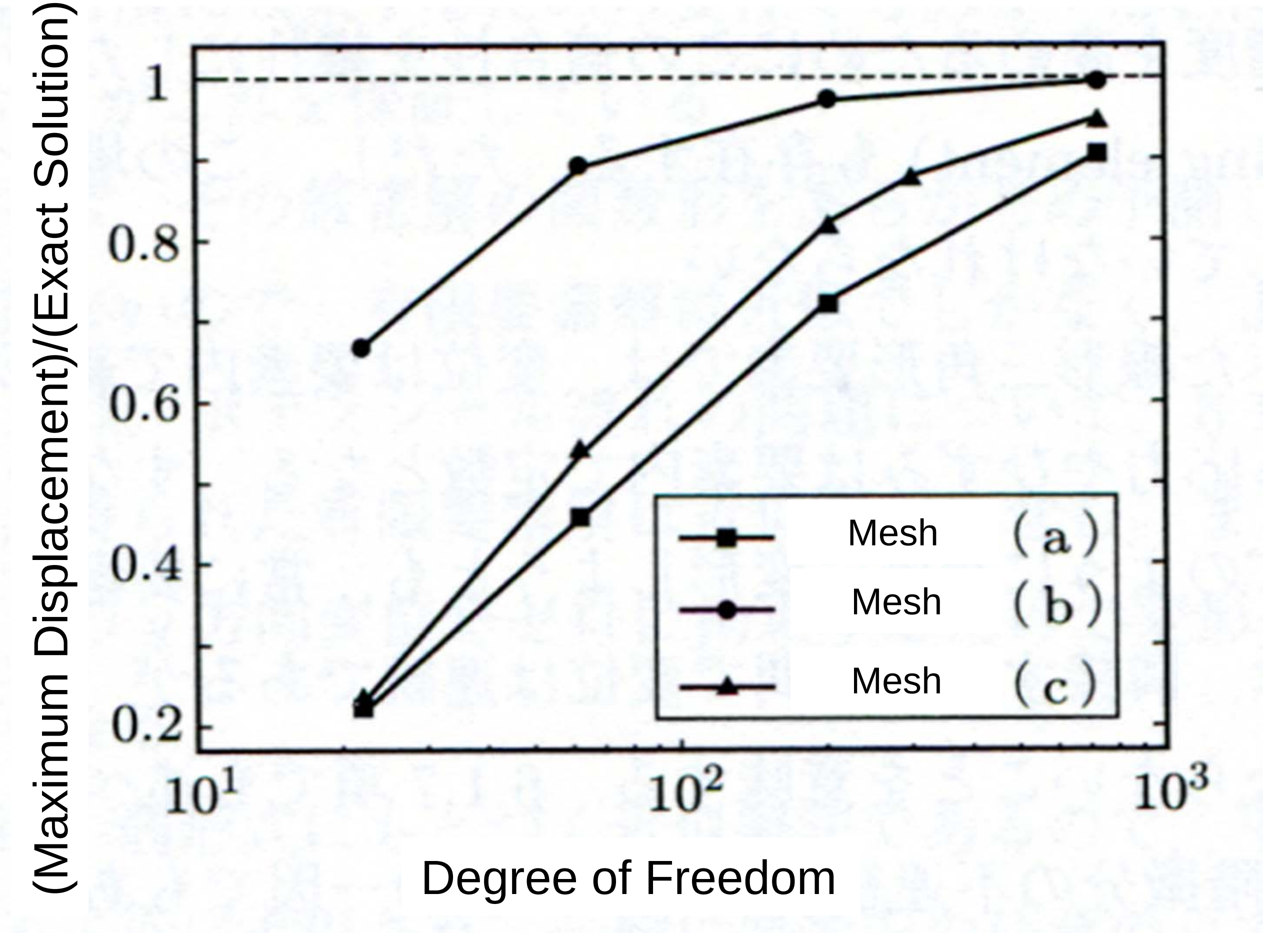
(a) Linear Triangular Element



(b) Linear Quadrilateral Element

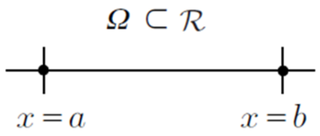
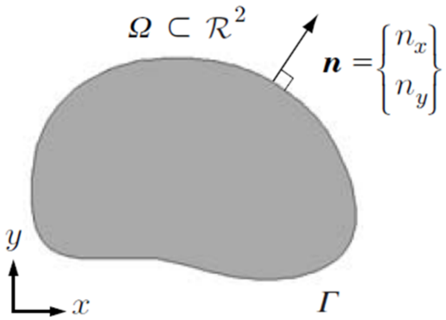
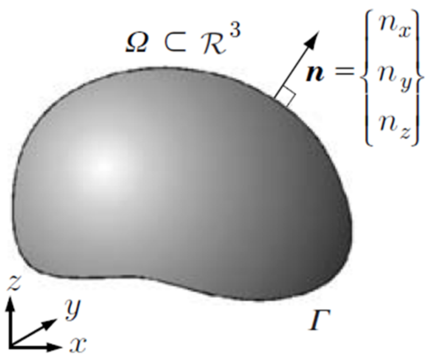
(c) Linear Quadrilateral Element (distorted)

Convergence: Example (2)



Appendix

Integral

domain	infinitesimal domain	domain integral	boundary integral
	dx	$\int_\Omega \square\, dx = \int_a^b \square\, dx$	$-f(a)$ and $f(b)$
	dA 	$\int_\Omega \square\, dA$ (multiple integral $\int_c^d \int_a^b \square\, dxdy$)	$\int_\Gamma \square \boldsymbol{n}\, ds$
	dV 	$\int_\Omega \square\, dV$ (multiple integral $\int_e^f \int_c^d \int_a^b \square\, dxdydz$)	$\int_\Gamma \square \boldsymbol{n}\, dS$

Integral Theorem

- Gauss theorem: scalar f

$$\int_\Omega \nabla f dV = \int_\Gamma f\mathbf{n}dS \quad \text{where} \quad \mathbf{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} \cos(x,\mathbf{n}) \\ \cos(y,\mathbf{n}) \\ \cos(z,\mathbf{n}) \end{Bmatrix}$$

- Gauss's divergence theorem: vector w

$$\int_\Omega \nabla^T \mathbf{w} dV = \int_\Gamma \mathbf{w}^T \mathbf{n} dS$$

- Green-Gauss's theorem

$$\mathbf{w} = v\nabla u \rightarrow \nabla^T \mathbf{w} = \nabla^T (v\nabla u) = \left(\nabla^T v\right)(\nabla u) + v\left(\nabla^T \nabla u\right) = \left(\nabla^T v\right)(\nabla u) + v\Delta u$$

$$\Rightarrow v\Delta u = \nabla^T (v\nabla u) - \left(\nabla^T v\right)(\nabla u)$$

$$\int_\Omega v\Delta u dV = \int_\Gamma (v\nabla u)^T \mathbf{n} dS - \int_\Omega \left(\nabla^T v\right)(\nabla u) dV$$

$$\text{1D}: \int_a^b v\frac{d^2u}{dx^2}dx = \left(v\frac{du}{dx}\bigg|_{x=b} - v\frac{du}{dx}\bigg|_{x=a} \right) - \int_a^b \frac{dv}{dx}\frac{du}{dx}dx \quad (\text{integration by parts})$$

$$\text{2D}: \int_\Omega v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right)dA = \int_\Gamma \left(v\frac{\partial u}{\partial x}n_x + v\frac{\partial u}{\partial x}n_y \right)ds - \int_\Omega \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right)dA$$