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  - Associated variational problem
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  - System of algebraic equations

Outline (2)

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  - Lagrange interpolation functions
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    - Matix of the system
    - Right-hand-side vector
    - Imposition of the essential boundary conditions
  - Results of analytical and FE solutions

Finite Element Method (FEM)

- [generally speaking]
  - a powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied sciences
- [mathematically]
  - a generalization of the classical variational (i.e., Ritz) and weighted-residual (e.g., Galerkin, least-squares, etc.) methods.

Motivation

- Most of the real problems
  - are defined on domains that are geometrically complex
  - may have different boundary conditions on different portions of the boundary
- Therefore, it is usually impossible (or difficult)
  - to find a solution analytically (so one must resort to approximate methods),
  - to generate approximation functions required in the traditional variational methods
- An answer to these problems is a finite-element approach

Main Concept of FEM

- A given domain can be viewed as an assemblage of simple geometric shapes, called finite elements, for which it is possible to systematically generate the approximation functions.
- Remarks:
  - The approximation functions are also called shape functions or interpolation functions since they are often constructed using ideas from interpolation theory.
  - The finite element method is a piecewise (or element-wise) application of the variational and weighted-residual methods.
  - For a given BVP, it is possible to develop different finite element approximations (or finite element models), depending on the choice of a particular variational and weighted-residual formulation.

Major Steps of Finite Element Analysis

- Discretization of the domain into a set of finite elements (mesh generation)
- Weighted-integral or weak formulation of the differential equation over a typical finite element (subdomain)
- Development of the finite element model of the problem using its weighted-integral or weak form. The finite element model consists of a set of algebraic equations among the unknown parameters (degrees of freedom) of the element
- Assembly of finite elements to obtain the global system (i.e., for the total problem) of algebraic equations – for the unknown global degrees of freedom
- Imposition of essential boundary conditions
- Solution of the system of algebraic equations to find (approximate) values in the global degrees of freedom
- Post-computation of solution and quantities of interest

Model Problem (1)

$$\text{(O)DE:} \quad -\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right)+\gamma(x)u(x)=f(x) \quad \text{for } x\in(a,b)$$

- α(x), γ(x), f(x) are the known data of the problem: the first two quantities result from the material properties and geometry of the problem whereas the third one depends on source or loads
- u(x) is the solution to be determined; it is also called dependent variable of the problem (with *x* being the independent variable)

- The domain of this 1D problem is an interval (a, b), and the points *x* = a and *x* = b are the boundary points where boundary conditions are imposed, e.g.,

Model Problem (2)

$$\text{BCs:} \quad \begin{cases} \left(q(\alpha)n_x(\alpha)=\right) -\alpha(x)\dfrac{du}{dx}(\alpha)=\hat{q} & (\text{Neumann b.c.}) \\ u(b)=\hat{u} & (\text{Dirichlet b.c.}) \end{cases}$$

- ˆq and û are the given boundary values
- n_x is the component of the outward unit vector normal to the boundary. In the 1D case there is only one component and: $n_x(a) = -1$, $n_x(b) = +1$
- $q(x) \equiv \alpha(x) du(x)/dx$ is the so-called secondary variable specified on the boundary by the Neumann boundary condition also known as the second kind or natural boundary condition
- u(x) is the primary variable specified on the boundary by the Dirichlet boundary condition also known as the first kind or essential boundary condition

Examples of different physical problems (1)

u (primary var.)	α (material data)	f (source, load)	q (secondary var.)
Heat transfer			
temperature	thermal conductance	heat generation	heat
Flow through porous medium			
fluid-head	permeability	infiltration	source
Flow through pipes			
pressure	pipe resistance	0	source
Flow of viscous fluids			
velocity	viscosity	pressure gradient	shear stress

$$-\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right)+\gamma(x)u(x)=f(x) \quad \text{for } x\in(a,b)$$

Examples of different physical problems (2)

u (primary var.)	α (material data)	f (source, load)	q (secondary var.)
Elastic cables			
displacement	tension	transversal force	point force
Elastic bars			
displacement	axial stiffness	axial force	point force
Torsion of bars			
angle of twist	shear stiffness	0	torque
Electrostatics			
electric potential	Dielectric constant	charge density	electric flux

$$-\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right)+\gamma(x)u(x)=f(x) \quad \text{for } x\in(a,b)$$

Boundary-Value Problem (BVP)

$\Omega = (a,b)$ be an open set (an open interval in case of 1D problems)

Γ be the boundary of Ω, that is, $\Gamma = \{a,b\}$

$\Gamma = \Gamma_q \cup \Gamma_u$ where $\Gamma_q = \{a\}$ and $\Gamma_u = \{b\}$ are disjoint parts of the boundary relating to the Neumann and Dirichlet bounday conditions, respectively

(the data of the problem) $f : \Omega \to \Re,\ \alpha : \Omega \to \Re,\ \gamma : \Omega \to \Re,$

(the values prescribed on the boundary) $\hat{q} : \Gamma_q \to \Re,\ \hat{u} : \Gamma_u \to \Re$

Find $u = ?$ satisfying

differential eq.: $-(\alpha u')' + \gamma u = f$ in $\Omega = (a,b)$

Neumann b.c.: $\alpha u' n_x = \hat{q}$ on $\Gamma_q = \{a\}$

Dirichlet b.c.: $u = \hat{u}$ on $\Gamma_u = \{b\}$

Strong Form

- The classical strong form of a boundary-value problem consists of:
  - the differential equation of the problem,
  - the Neumann boundary conditions, i.e., the natural conditions imposed on the secondary dependent variable (which involves the first derivative of the dependent variable)
  - The Dirichlet (essential) boundary conditions must be satisfied a priori.

$$\text{strong form: } -\left(\alpha u'\right)' + \gamma u = f$$

Weak Form (1)

- (1) Write the weighted-residual statement for the domain equation
- (2) Trade differentiation from u to δu using integration by parts
  - The integration by parts weakens the differentiability requirement for the trial functions u (i.e., for the solution)
- (3) Use the Neumann boundary condition ($\alpha u' n_x = \hat{q}$ on Γ_q) and the property of test function ($\delta u = 0$ on Γ_q) for the boundary term

$$\text{weak form: } \left[-\hat{q}\delta u\right]_{x=a} + \int_a^b \left[\alpha u'\delta u' + \gamma u \delta u - f\,\delta u\right] dx = 0$$

- The weak form is mathematically equivalent to the strong one, that is, if u is a solution to the strong (local, differential) formulation of a BVP, it also satisfies the corresponding weak (global, integral) formulation for any δu (admissible, i.e., sufficiently smooth and $\delta u = 0$ on Γ_u)

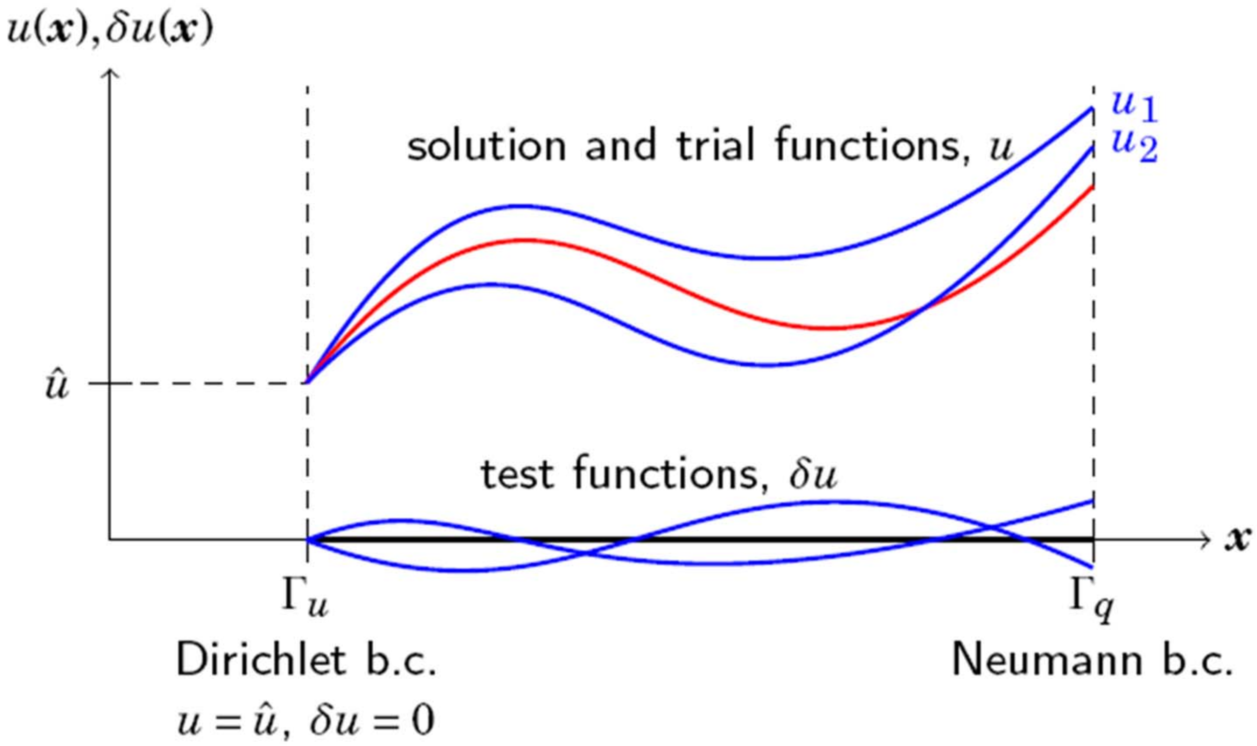
Weak Form (2)

- The essential boundary conditions must be explicitly satisfied by the trial functions: $u = \hat{u}$ on Γ_u . (In case of displacement formulations of many mechanical and structural engineering problems this is called kinematic admissibility requirement.)
- Consequently, the test functions must satisfy the adequate homogeneous essential boundary conditions: $\delta u = 0$ on Γ_u
- The trial functions u (and test functions, δu) need only to be continuous. (Remember that in the case of strong form the continuity of the first derivative of solution u was required.)

Weak Form: Remarks

- The strong form can be derived from the corresponding weak formulation if we take more demanding assumptions for the smoothness of trial functions (i.e., one-order higher differentiability).
- In variational methods, any test function is a variation defined as the difference between any two trial functions. Since any trial function satisfy the essential boundary conditions, the requirement that $\delta u = 0$ on Γ_u follows immediately.

Test and Trial Functions



u_1, u_2: arbitrary trial functions

$$\delta u = u_1 - u_2 \text{ and } \left.\begin{matrix} u_1 = \hat{u} \text{ on } \Gamma_u \\ u_2 = \hat{u} \text{ on } \Gamma_u \end{matrix}\right\} \rightarrow \delta u = 0 \text{ on } \Gamma_u$$

Associated Variational Problem

- U: space of solution (or trial functions)
- W: space of test functions (or weighting functions)
- A: bilinear form defined on U x W
- F: linear form defined on W

- Variational problem:

$$\text{Find } u \in U \text{ so that } A(u, \delta u) = F(\delta u) \quad \forall \delta u \in W$$

Principle of the minimum total potential energy

- The weak form or the variational problem is the statement of the principle of the minimum total potential energy

Find $u \in U$ so that $A(u, \delta u) = F(\delta u) \quad \forall \delta u \in W$

$$\delta P(u) = A(u, \delta u) - F(\delta u) = 0$$

$P(u) = \frac{1}{2} A(u, u) - F(u)$: potential defined by the quadratic functional

$$\frac{1}{2} \delta A(u,u) = \frac{1}{2} \left[\underbrace{A(\delta u, u)}_{A(u,\delta u)} + A(u, \delta u) \right] = A(u, \delta u), \quad \delta F(u) = F(\delta u)$$

since the bilinear form is symmetric

Galerkin Method

- If the problem is well-posed one can try to find an approximated solution u_h by solving the so-called discrete problem which is an approximation of the corresponding variational problem.
- Discrete (approximated) problem:

$$\text{Find } u_h \in U_h \text{ so that } A_h\left(u_h, \delta u_h\right) = F_h\left(\delta u_h\right) \quad \forall \delta u_h \in W_h$$

  - U_h is a finite-dimension space of functions called approximation space whereas u_h is the approximate solution
  - δu_h are discrete test functions from the discrete test space W_h isIn the Galerkin method $W_h = V_h$ (in general, $W_h \neq V_h$)
  - A_h is an approximation of the biliear form A
  - F_h is an approximation of the linear form F

Interpolation

- In the Galerkin method (W=U) the same shape functions, $\phi_i(x)$, are used to interpolate the approximate solution as well as the (discrete) test functions:

$$u_h(x) = \sum_{j=1}^{N} \theta_j \phi_j(x), \quad \delta u_h(x) = \sum_{i=1}^{N} \delta\theta_i \phi_i(x), \quad \theta_i : \text{ degrees of freedom}$$

- Using this interpolation for the approximated problem leads to a system of algebraic equations
  - The left-hand and right-hand sides of the problem equation yield

$$A_h(u_h, \delta u_h) = \sum_{i=1}^{N} \sum_{j=1}^{N} A_h(\phi_j, \phi_i) \theta_j \delta\theta_i = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \theta_j \delta\theta_i$$

$$F_h(\delta u_h) = \sum_{i=1}^{N} F_h(\phi_i) \delta\theta_i = \sum_{i=1}^{N} F_i \delta\theta_i$$

System of Algebraic Equations

- Using this interpolation for the approximated problem leads to a system of algebraic equations
  - The coefficient matrix (stiffness matrix) and the right-hand-side vector are defined as follow

$$A_{ij} = A_h\left(\phi_j, \phi_i\right), \quad F_i = F_h\left(\phi_i\right)$$

  - Now, the approximated problem may be written as

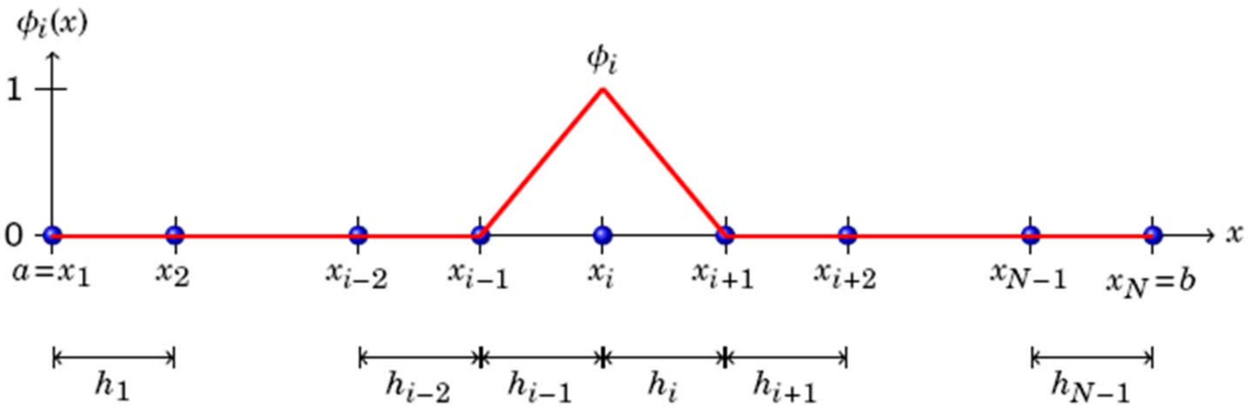
$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left| A_{ij}\theta_j - F_i \right| \delta\theta_i = 0 \quad \forall \delta\theta_i \in V_h$$

  - It is (always) true if the expression in brackets equals zero which gives the system of algebraic equations

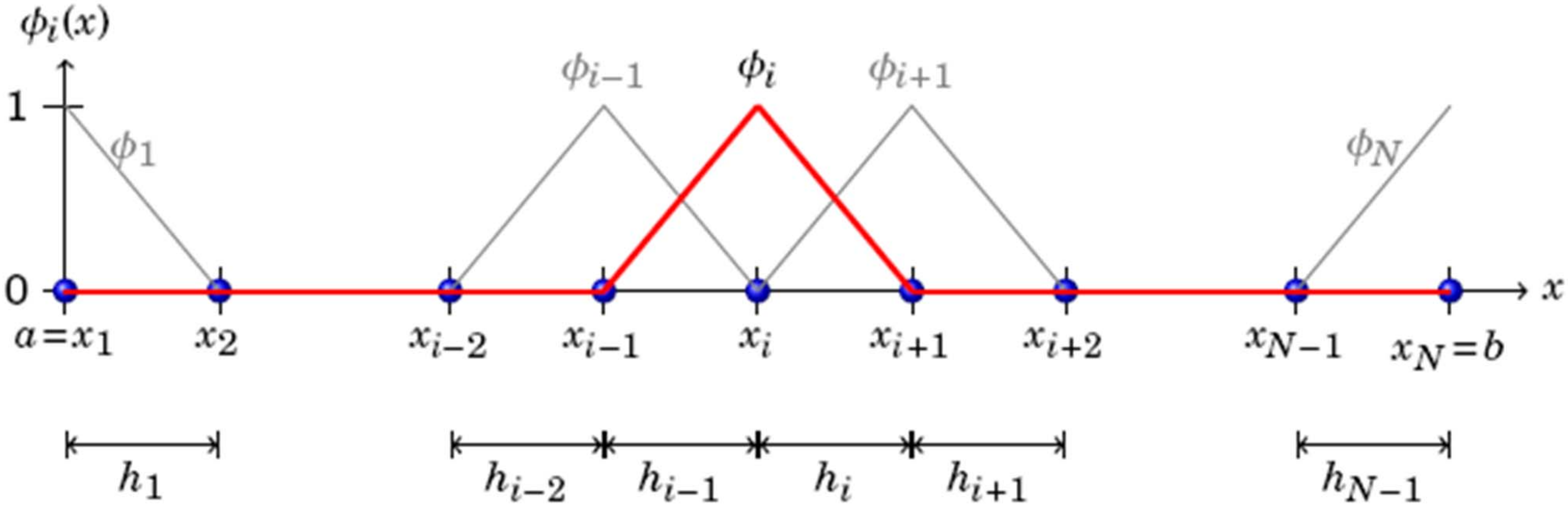
$$\sum_{i=1}^{N} A_{ij}\theta_j = F_i$$

Discretization

- The domain internal is divided into N-1 finite elements (subdomains).
- There are N nodes, each with only 1 degree of freedom.
- Local (or element) shape function is (most often) defined on an element in this way that it is equal 1 in a particular node and 0 in the other(s).
  - Only two linear interpolation functions in 1D finite element
  - Higher-order interpolation functions involve additional nodes inside element
- Global shape function ϕ_i is defined on the whole domain as
  - Local shape functions on (neighboring) elements sharing the node i
  - Identically equal zero on all other elements



Shape Functions

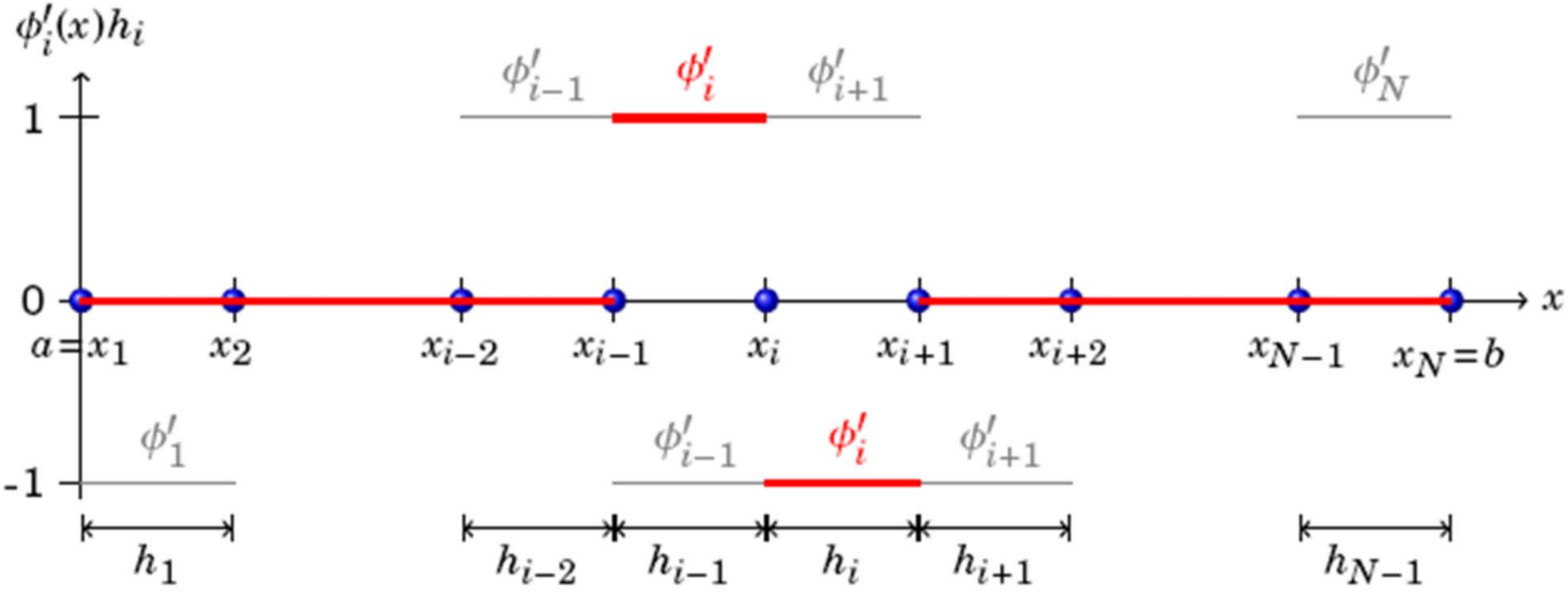


$$\phi_1 = \begin{cases} \dfrac{x_2 - x}{h_1} & \text{for } x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{for } x \in \Omega_{i-1} \\ \dfrac{x_{i+1} - x}{h_i} & \text{for } x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_N = \begin{cases} \dfrac{x - x_{N-1}}{h_{N-1}} & \text{for } x \in \Omega_{N-1} \\ 0 & \text{otherwise} \end{cases}$$

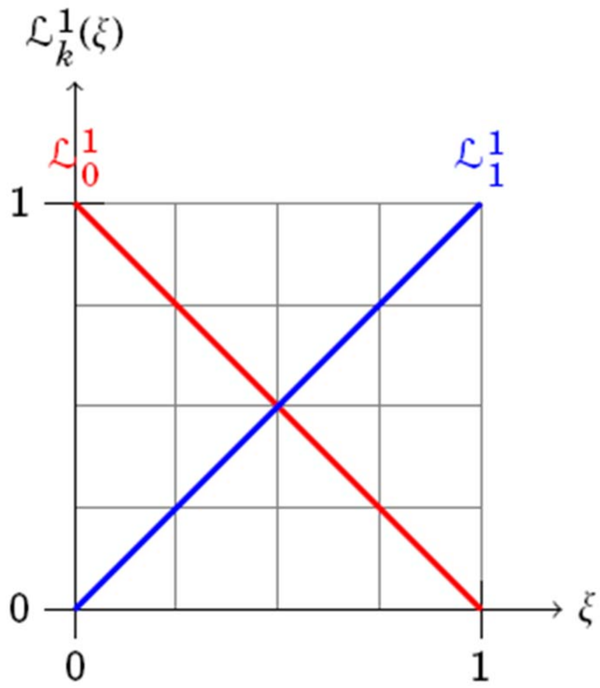
Shape Functions: First Derivatives



$$\phi_1 = \begin{cases} -\dfrac{1}{h_1} & \text{for } x \in \Omega_i \\ 0 & \text{otherwise} \end{cases} \qquad \phi_i = \begin{cases} 1/h_{i-1} & \text{for } x \in \Omega_{i-1} \\ -1/h_i & \text{for } x \in \Omega_i \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N = \begin{cases} \dfrac{1}{h_{N-1}} & \text{for } x \in \Omega_{N-1} \\ 0 & \text{otherwise} \end{cases}$$

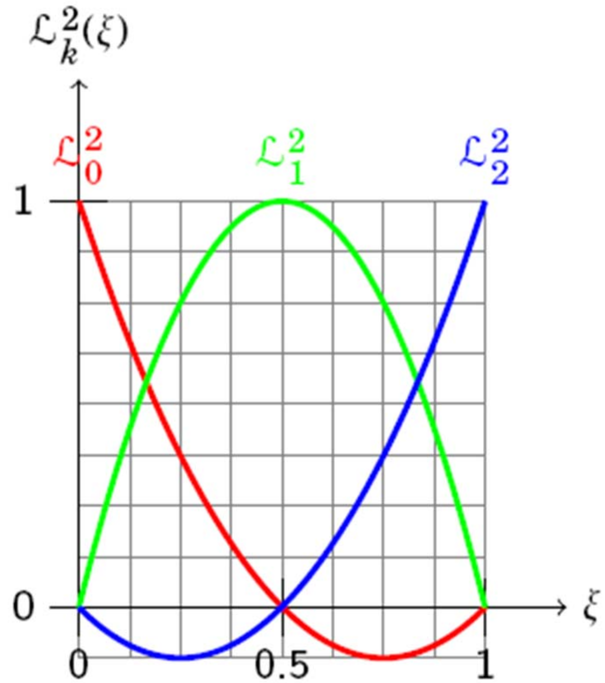
– Discontinuous at interfaces (points) between elements (in the case of linear interpolation they are element-wise constant)

Lagrange interpolation functions



1st order (linear)

$$\mathcal{L}_0^1(\xi) = 1 - \xi\,,$$
$$\mathcal{L}_1^1(\xi) = \xi\,,$$



2nd order (quadratic)

$$\mathcal{L}_0^2(\xi) = (2\xi - 1)(\xi - 1)\,,$$
$$\mathcal{L}_1^2(\xi) = 4\xi(1 - \xi)\,,$$
$$\mathcal{L}_2^2(\xi) = \xi(2\xi - 1)\,.$$

Finite Element System of Algebraic Equations

- The symmetry of the bilinear form A involves the symmetry of the matrix of the FE system of algebraic equations, i.e., $A_{ij} = A_{ji}$.
- A component A_{ij} is defined as an integral (over the problem domain) of a sum of a product of shape functions, ϕ_i and ϕ_j, and a product of their derivatives, ϕ'_i and ϕ'_j.
- The product of two shape functions (or their derivatives) is nonzero only on the elements that contain the both corresponding degrees of freedom (since a shape function corresponding to a particular degree of freedom is nonzero only on the elements sharing it).
- Therefore, the integral can be computed as a sum of the integrals defined only over these finite elements that share the both degrees of freedom (since the contribution from all other elements is null):

$$A_{ij} = \sum_{e \in E} A_{ij}^{(e)} = \sum_{e \in E(i,j)} A_{ij}^{(e)} \qquad \begin{cases} E: \text{ set of all finite elements} \\ E(i,j): \text{ set of finite elements that contain the DOF } i \text{ and } j \end{cases}$$

1D Problem

$$A_{ij} = \begin{cases} A_{11}^{(1)} & \text{for } i = j = 1 \\ A_{ii}^{(i-1)} + A_{ii}^{(i)} & \text{for } i = j = 2,\ldots,(N-1) \\ A_{NN}^{(N-1)} & \text{for } i = j = N \\ A_{i,j+1}^{(i)} & \text{for } \left|i-j\right| = 1 \\ 0 & \text{for } \left|i-j\right| > 1 \end{cases} \xrightarrow[\gamma(x)=\text{const}=\gamma]{\alpha(x)=\text{const}=\alpha} A_{ij} = \begin{cases} \dfrac{\alpha}{h_1} + \dfrac{\gamma h_1}{3} \\ \dfrac{\alpha}{h_{i-1}} + \dfrac{\gamma h_{i-1}}{3} + \dfrac{\alpha}{h_i} + \dfrac{\gamma h_i}{3} \\ \dfrac{\alpha}{h_{N-1}} + \dfrac{\gamma h_{N-1}}{3} \\ -\dfrac{\alpha}{h_i} + \dfrac{\gamma h_i}{6} \\ 0 \end{cases}$$

$$\xrightarrow{f(x)=\text{const}=f} F_i = \begin{cases} \dfrac{fh_1}{2} + \hat{q} & \text{for } i = 1 \\ \dfrac{f\left(h_{i-1} + h_i\right)}{2} & \text{for } i = 2,\ldots,(N-1) \\ F_N = ? & \text{for } i = N \ \left(\text{a reaction to the essential BC}\right) \end{cases}$$

Imposition of Essential BCs (1)

- In general, the assembled matrix A_{ij} is singular and the system of algebraic equations is undetermined. We need to impose the essential boundary conditions to make it solvable.
- Let B be the set of all degrees of freedom where the essential boundary conditions are applied, that is,

  $$\text{for } n \in B : \theta_n = \hat{\theta}_n \text{ where } \hat{\theta}_n \text{ is a given value}$$

- In practice, the essential BCs are imposed as described below.

- Compute a new r.h.s. vector: $\tilde{F}_i = F_i - \sum_{n \in B} A_{in} \hat{\theta}_n \quad \text{for } i = 1, \ldots, N$
- Set: $\tilde{F}_n = \hat{\theta}_n$
- Set: $\tilde{A}_{nn} = 1$ and all other components in the n-th row and n-th column to zero, i.e., $\tilde{A}_{ni} = \tilde{A}_{in} = \delta_{in} \quad \text{for } i = 1, \ldots, N$

Imposition of Essential BCs (2)

Now, the new (slightly modified) system of equations, $\tilde{A}_{ij}\theta_i = \tilde{F}_j$,
is solved for θ_i.

Eventually, a reaction (force, source) is computed $F_n = \sum_{i=1}^{N} A_{ni}\theta_i$

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & \text{for } i = 1,\ldots,(N-1) \\ \delta_{Nj} & \text{for } i = N,\ j = 1,\ldots,N \\ \delta_{iN} & \text{for } i = 1,\ldots,N,\ j = N \end{cases}$$

$$\tilde{F}_i = \begin{cases} F_i - A_{iN}\hat{\theta}_N & \text{for } i = 1,\ldots,(N-1) \\ \hat{\theta}_N & \text{for } i = N \end{cases}$$

$$F_n = \sum_{i=1}^{N} A_{Ni}\theta_i = A_{N,(N-1)}\theta_{N-1} + A_{NN}\hat{\theta}_N$$

Results of Analytical and FE Solutions

$$\alpha(x) = 1,\ \gamma = 3,\ f(x) = 1$$

$$a = 0,\ q(0) = \hat{q} = 1,\ b = 2,\ u(2) = \hat{u} = 0$$