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Boundary-Value Problem

- Let B be a domain with the boundary ∂B, and
  - L(.) be a (second order) differential operator
  - f = f (**x**) be a known source term in B
- Boundary-Value Problem:

Find $u = u(\boldsymbol{x}) = ?$ satisfying PDE $L(u) = f$ in B

and subject to (at least one of) the following boundary conditions

$$u = \hat{u} \text{ on } \partial B_1, \quad \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \hat{\gamma} \text{ on } \partial B_2, \quad \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha} u = \hat{\beta} \text{ on } \partial B_3$$

where $\hat{u} = \hat{u}(\boldsymbol{x})$, $\hat{\gamma} = \hat{\gamma}(\boldsymbol{x})$, $\hat{\alpha} = \hat{\alpha}(\boldsymbol{x})$, $\hat{\beta} = \hat{\beta}(\boldsymbol{x})$ are fields prescribed on adequate parts of the boundary $\partial B = \partial B_1 \cup \partial B_2 \cup \partial B_3$

  - the boundary parts are mutually disjoint
  - for f ≡ 0 the PDE is called homogeneous

Types of Boundary Conditions

- the first kind (or Dirichlet b.c) $u = \hat{u} \ \text{ on } \partial B_1$
- the second kind (or Neumann b.c) $\frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \hat{\gamma} \ \text{ on } \partial B_2$
- the third kind (or Robin b.c.) $\frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha} u = \hat{\beta} \ \text{ on } \partial B_3$
  - sometimes termed also the generalized Neumann b.c., it can be presented as

$$\frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \hat{\gamma} + \hat{\alpha}\left(\hat{u} - u\right) \ \text{ on } \partial B_3$$
$$\hat{\beta} = \hat{\gamma} + \hat{\alpha}\hat{u}$$

General Idea

- Weighted Residual Method (WRM) assumes that a solution can be approximated analytically or piecewise analytically. In general,
  - a solution to a PDE can be expressed as a linear combination of a base set of functions where the coefficients are determined by a chosen method, and
  - the method attempts to minimize the approximation error
- In fact, WRM represents a particular group of methods where an integral error is minimized in a certain way. Depending on this way the WRM can generate:
  - finite volume method
  - finite element methods
  - spectral methods
  - finite difference methods

Approximation

- Assumption: the exact solution, *u*, can be approximated by a linear combination of *N* (linearly-independent) analytical functions

$$u(\boldsymbol{x}) \approx \tilde{u}(\boldsymbol{x}) = \sum_{s=1}^{N} U_s \phi_s(\boldsymbol{x})$$

$\tilde{u}$ is an approximated solution

U_s are unknown coefficients, the so-called degrees of freedom

$\phi_s = \phi_s(\boldsymbol{x})$ form a base set of selected functions (trial functions or shape functions)

This set of functions generates the space of approximated solutions.

$s = 1, \ldots, N$ where N is the number of degrees of freedom

Residua or Error Functions

- In general, an approximated solution, ũ, does not satisfy exactly the PDE and/or some (or all) boundary conditions. The generated errors can be described by the following error functions:
  - the PDE residuum $R_0(\tilde{u}) = L(\tilde{u}) - f$
  - the Dirichlet condition residuum $R_1(\tilde{u}) = \tilde{u} - \hat{u}$
  - the Neumann condition residuum $R_2(\tilde{u}) = \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} - \hat{\gamma}$
  - the Robin condition residuum $R_3(\tilde{u}) = \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha} u - \hat{\beta}$

Minimization of Errors

- Demand: Minimize the errors in a weighted integral sense

$$\int_B R_0(\tilde{u})\overset{0}{\psi}_r + \int_{\partial B_1} R_1(\tilde{u})\overset{1}{\psi}_r + \int_{\partial B_2} R_2(\tilde{u})\overset{2}{\psi}_r + \int_{\partial B_3} R_3(\tilde{u})\overset{3}{\psi}_r = 0$$

$$\left\{\overset{0}{\psi}_r\right\}, \left\{\overset{1}{\psi}_r\right\}, \left\{\overset{2}{\psi}_r\right\} \text{ and } \left\{\overset{3}{\psi}_r\right\} (r = 1, \ldots, M) \text{ are sets of weight functions}$$

  - Note that *M* weight functions yield *M* conditions (or equations) from which to determine the *N* coefficients U_s. To determine these *N* coefficients uniquely we need *N* independent conditions (equations). Now, using the formulae for residua results in

$$\int_B L(\tilde{u})\overset{0}{\psi}_r + \int_{\partial B_1} \tilde{u}\overset{1}{\psi}_r + \int_{\partial B_2} \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n}\overset{2}{\psi}_r + \int_{\partial B_3} \left(\frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha}\tilde{u}\right)\overset{3}{\psi}_r$$

$$= \int_B f\overset{0}{\psi}_r + \int_{\partial B_1} \hat{u}\overset{1}{\psi}_r + \int_{\partial B_2} \hat{\gamma}\overset{2}{\psi}_r + \int_{\partial B_3} \hat{\beta}\overset{3}{\psi}_r$$

System of Algebraic Equations

Applying the approximation $\tilde{u} = \sum_{s=1}^{N} U_s \phi_s$, and using the linearity of operators,

$$L(\tilde{u}) = \sum_{s=1}^{N} U_s L(\phi_s), \quad \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \sum_{s=1}^{N} U_s \frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n}$$

leads to the following system of algebraic equations

$$\sum_{s=1}^{N} A_{rs} U_s = B_r$$

where $A_{rs} = \int_B L(\phi_s) \overset{0}{\psi}_r + \int_{\partial B_1} \phi_s \overset{1}{\psi}_r + \int_{\partial B_2} \frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} \overset{2}{\psi}_r + \int_{\partial B_3} \left(\frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha} \phi_s \right) \overset{3}{\psi}_r$

$$B_r = \int_B f \overset{0}{\psi}_r + \int_{\partial B_1} \hat{u} \overset{1}{\psi}_r + \int_{\partial B_2} \hat{\gamma} \overset{2}{\psi}_r + \int_{\partial B_3} \hat{\beta} \overset{3}{\psi}_r$$

Categories of WRM

- There are four main categories of weight functions which generate the following categories of WRM:
  - Subdomain method
    - Chosen subdomain
  - Collocation method
    - Dirac delta functions
  - Least squares method
    - Derivatives of residual itself
  - Galerkin method
    - Base functions

Subdomain Method

- Here the domain is divided in *M* subdomains ΔBr where

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Delta B_r \\ 0 & \text{outside} \end{cases}$$

- such that this method minimizes the residual error in each of the chosen subdomains. Note that the choice of the subdomains is free. In many cases an equal division of the total domain is likely the best choice. However, if higher resolution (and a corresponding smaller error) in a particular area is desired, a non-uniform choice may be more appropriate.

Collocation Method

- In this method the weight functions are chosen to be Dirac delta functions

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_r)$$

such that the error is zero at the chosen nodes $\mathbf{x}_r$

Least Squares Method

- This method uses derivatives of the residual itself as weight functions in the form

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \frac{\partial R_0\left(\tilde{u}(\boldsymbol{x})\right)}{\partial U_r}$$

- The motivation for this choice is to minimize $\int_B {R_0}^2$ of the computational domain. Note that (if the boundary conditions are satisfied) this choice of the weight function implies

$$\frac{\partial}{\partial U_r}\left(\int_B {R_0}^2\right) = 0$$

for all values of U_r

Galerkin Method

- In this method the weight functions are chosen to be identical to the base functions

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \phi_r(\boldsymbol{x})$$

- In particular, if the base function set is orthogonal

$$\int_B \phi_r \phi_s = 0 \quad \text{if} \quad r \neq s$$

this choice of weight functions implies that the residual R_0 is rendered orthogonal with the minimization condition

$$\int_B R_0 \overset{0}{\psi}_r = 0$$

for all base functions

A simple BVP approached by WRM

- Boundary Value Problem (for an ODE):

Find $u = u(\boldsymbol{x}) = ?$ satisfying

$$\frac{d^2u}{dx^2} - \frac{du}{dx} = 0 \quad \text{in } B = [a,b]$$

subject to boundary conditions on $\partial B = \partial B_1 \cup \partial B_2 = \{a\} \cup \{b\}$

$$u\big|_{x=a} = \hat{u} \quad \text{(Dirichlet)}, \quad \left.\frac{du}{dx}\right|_{x=b} = \hat{\gamma} \quad \text{(Neumann)}$$

WRM approach

Residua for an approximated solution $\tilde{u}$

$$R_0(\tilde{u}) = \frac{d^2\tilde{u}}{dx^2} - \frac{d\tilde{u}}{dx}, \quad R_1(\tilde{u}) = \tilde{u}\big|_{x=a} - \hat{u}, \quad R_2(\tilde{u}) = \left.\frac{d\tilde{u}}{dx}\right|_{x=b} - \hat{\gamma}$$

Minimization of weighted residual error

$$\int_a^b \left(\frac{d^2\tilde{u}}{dx^2} - \frac{d\tilde{u}}{dx} \right) \overset{0}{\psi}_r + \left| (\tilde{u} - \hat{u}) \overset{1}{\psi}_r \right|_{x=a} + \left| \left(\frac{d\tilde{u}}{dx} - \hat{\gamma} \right) \overset{2}{\psi}_r \right|_{x=b} = 0$$

system of algebraic equations $\sum_{s=1}^{N} A_{rs} U_s = B_r$

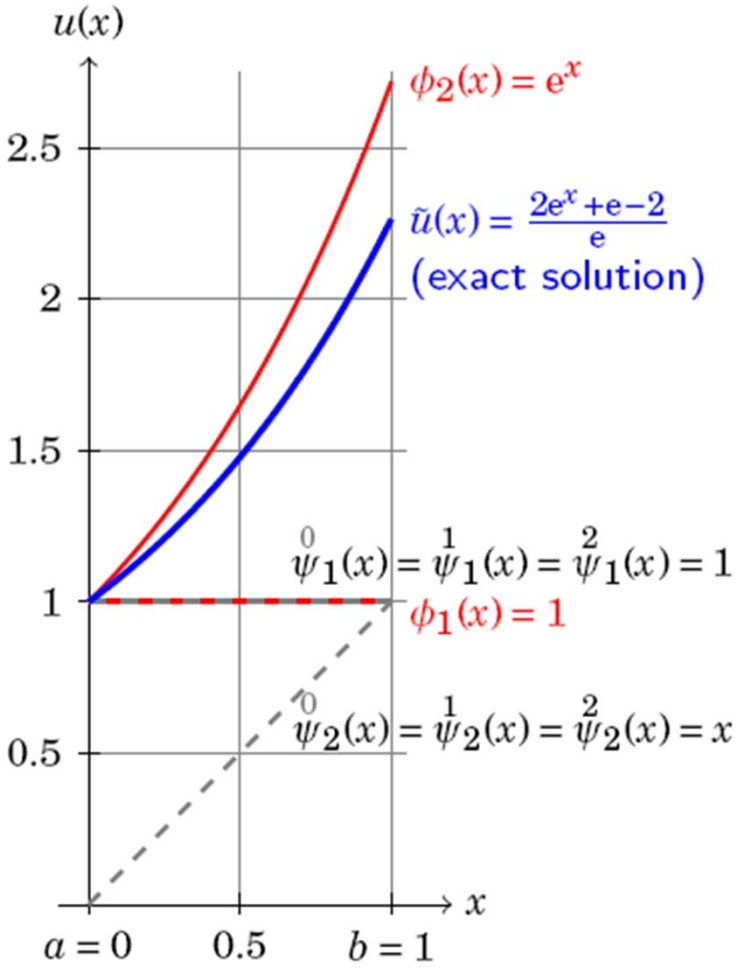
where $A_{rs} = \int_a^b \left(\frac{d^2\phi_s}{dx^2} - \frac{d\phi_s}{dx} \right) \overset{0}{\psi}_r + \left| \phi_s \overset{1}{\psi}_r \right|_{x=a} + \left| \frac{d\phi_s}{dx} \overset{2}{\psi}_r \right|_{x=b}$

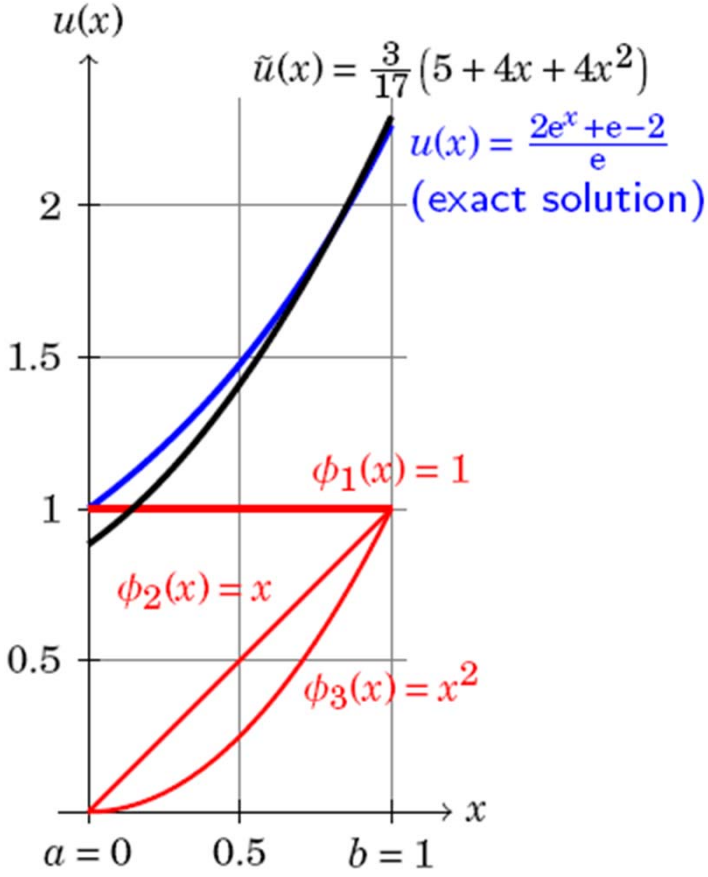
$$B_r = \left| \hat{u} \overset{1}{\psi}_r \right|_{x=a} + \left| \hat{\gamma} \overset{2}{\psi}_r \right|_{x=b}$$

Numerical (and exact) Solution

- Boundary limits and values: $a = 0,\ b = 1,\ \hat{u} = 1,\ \hat{\gamma} = 2$
- Shape functions: $\{\phi_s\} = \{1, e^x\},\quad \left\{{}^{0}_{}\psi_r\right\}, \left\{{}^{1}_{}\psi_r\right\}, \left\{{}^{2}_{}\psi_r\right\} = \{1, x\}$
- Weight functions: $\{\phi_s\} = \left\{{}^{0}_{}\psi_s\right\}, \left\{{}^{1}_{}\psi_s\right\}, \left\{{}^{2}_{}\psi_s\right\} = \{1, x, x^2\}$
- System of equations:
- Coefficients:
- Approximated solution:

Comparison


u(x)
2.5
2
1.5
1
0.5
$\phi_2(x) = e^x$
$\tilde{u}(x) = \frac{2e^x + e - 2}{e}$
(exact solution)
$\overset{0}{\psi}_1(x) = \overset{1}{\psi}_1(x) = \overset{2}{\psi}_1(x) = 1$
$\phi_1(x) = 1$
$\overset{0}{\psi}_2(x) = \overset{1}{\psi}_2(x) = \overset{2}{\psi}_2(x) = x$
x
a = 0
0.5
b = 1


u(x)
$\tilde{u}(x) = \frac{3}{17}(5 + 4x + 4x^2)$
$u(x) = \frac{2e^x + e - 2}{e}$
(exact solution)
2
1.5
1
0.5
$\phi_1(x) = 1$
$\phi_2(x) = x$
$\phi_3(x) = x^2$
x
a = 0
0.5
b = 1

Outline

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  - Problem approximation and solution
- General features

Direct Variational Methods

- Direct methods – the methods which (bypassing the derivation of the Euler equations) go directly from a variational statement of the problem to the solution
  - The assumed solutions in the variational methods are in the form of a finite linear combination of undetermined parameters with appropriately chosen functions.
  - In these methods a continuous function is represented by a finite linear combination of functions. However, in general, the solution of a continuum problem cannot be represented by a finite set of functions an error is introduced into the solution.
  - The solution obtained is an approximation of the true solution for the equations describing a physical problem.
  - As the number of linearly independent terms in the assumed solution is increased, the error in the approximation will be reduced (the assumed solution converges to the desired solution).
- Classical variational methods of approximation are: Ritz, Galerikin, Petrov-Galerkin (weighted residuals).

Mathematical Preliminaries

If A is a strictly positive operator (i.e., $\langle Au,u\rangle_H > 0$ holds for all $0 \neq u \in D_A$, and $\langle Au,u\rangle_H = 0$ if and only if $u = 0$), then $Au = f$ in H has at most one solution $\bar{u} \in D_A$ in H

$A: D_A \to H$ be a positive operator (in D_A), and $f \in H$

$\Pi: D_A \to H$ be a quadratic functional defined as $\Pi(u) = \frac{1}{2}\langle Au,u\rangle_H - \langle f,u\rangle_H$

- If $\bar{u} \in D_A$ is a solution to the operator equation $Au = f$ in H, then the quadratic functional $\Pi(u)$ assumes its minimal value in D_A for the element $\bar{u}$, i.e., $\Pi(u) \geq \Pi(\bar{u})$ and $\Pi(u) = \Pi(\bar{u})$ only for $u = \bar{u}$
- Conversly, if $\Pi(u)$ assumes its minimal value, among all $u \in D_A$, for the element $\bar{u}$, then $\bar{u}$ is the solution of the operator equation, that is, $A\bar{u} = f$

Example

$$\langle Au, v \rangle_H = \int_0^L (Au) v dx = \int_0^L u (Av) dx = \langle u, Av \rangle_H$$

Basic Idea

- The problem must be stated in a variational form, as a minimization problem: find ū minimizing certain functional $\Pi(u)$.
- The solution is approximated by a finite linear combination of the form

$$\bar{u}(\boldsymbol{x}) \approx \tilde{u}^{(N)}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \phi_j(\boldsymbol{x}) + \phi_0(\boldsymbol{x})$$

c_j : undetermined parameters termed Ritz coefficients

ϕ_0, ϕ_j : approximation functions $(j = 1, \ldots, N)$

- The parameters c_j are determined by requiring that the variational statement holds for the approximate solution, that is, $\Pi(\bar{u})$ is minimized with respect to c_j (j = 1,...,N)
- Remark: The approximate solution may be exact if the set of approximation functions is well chosen (i.e., it expands a space which contains the solution).

Ritz equations for the parameters

– By substituting the approximate form of solution into the functional Π one obtains Π as a function of the parameters c_j (after carrying out the indicated integration):

$$\Pi\left(\tilde{u}^{(N)}\right) = \tilde{\Pi}\left(c_1, \ldots, c_N\right)$$

The Ritz parameters are determined (or adjusted) such that $\delta\Pi = 0$

$$\delta\Pi = \frac{\partial\Pi}{\partial c_1}\delta c_1 + \cdots + \frac{\partial\Pi}{\partial c_N}\delta c_N = \sum_{i=1}^{N}\frac{\partial\Pi}{\partial c_i}\delta c_i = 0$$

Since the parameters c_j are independent, it follows that

$$\frac{\partial\Pi}{\partial c_i} = 0 \quad \text{for} \quad j = 1, \ldots, N$$

These are the N Ritz equations to determine the N Ritz parameters c_j.

Quadratic Functional

If the functional $\Pi(u)$ is quadratic in u, then its variation can be expressed as

$$\delta\Pi = B(u,\delta u) - L(\delta u)$$

where $B(.,.)$ and $L(.)$ are certain bilinear and linear forms, respectively.

Applying the Ritz approximation: $\tilde{u}^{(N)} = \sum_{j=1}^{N} \phi_j c_j + \phi_0, \quad \delta\tilde{u}^{(N)} = \sum_{i=1}^{N} \phi_i \delta c_i$

$$\delta\Pi = B(\tilde{u},\delta\tilde{u}) - L(\delta\tilde{u}) = \sum_{i=1}^{N}\left[\sum_{j=1}^{N} A_{ij}c_j - b_i\right]\delta c_i = 0$$

the Ritz equations form a system of linear algebraic equations:

$$\frac{\partial\Pi}{\partial c_i} = \left[\sum_{j=1}^{N} A_{ij}c_j - b_i\right] = 0 \quad \text{or} \quad \sum_{j=1}^{N} A_{ij}c_j = b_i \quad (i = 1,\ldots,N)$$

where A_{ij} is the governing equations and b_i is the right-hand side vector

Properties of approximation functions

- A convergent Ritz approximation requires the following:
  - ϕ_0 must satisfy the specified essential boundary conditions. When these conditions are homogeneous, then $\phi_0(\mathbf{x}) = 0$.
  - ϕ_i must satisfy the following three conditions:
    - be continuous, as required by the variational statement being used
    - satisfy the homogeneous form of the specified essential boundary conditions
    - the set $\{\phi_i\}$ must be linearly independent and complete
- If these requirements are satisfied then:
  - the Ritz approximation has a unique solution $\tilde{u}^{(N)}(x)$
  - this solution converges to the true solution of the problem as the value of *N* is increased

Problem Definition

$$\text{(O)DE: } -\frac{d}{dx}\left(\alpha(\boldsymbol{x})\frac{du(\boldsymbol{x})}{dx}\right) = f(\boldsymbol{x}) \text{ for } x \in (0, L)$$

$\alpha(\boldsymbol{x})$: result from the material properties and geometry

$f(\boldsymbol{x})$: depend on source or loads

$u(\boldsymbol{x})$: solution to be determined, dependent variable

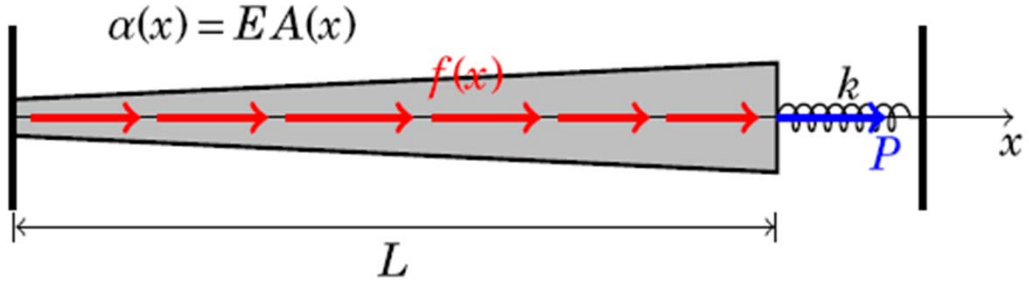
– The domain of this 1D problem is an interval (0,L), and the points x = 0 and x = L are the boundary points where boundary conditions are imposed, e.g.,

$$\text{BCs: } \begin{cases} u(0) = 0 & \text{(Dirichlet b.c.)} \\ \left[-\alpha(\boldsymbol{x})\frac{du(\boldsymbol{x})}{dx} + ku(\boldsymbol{x})\right]_{x=L} = P & \text{(Robin b.c.)} \end{cases}$$

P: given boundary value

Variational statement of the problem

– This mathematical model may describe the problem of the axial deformation of a nonuniform bar under an axial load, fixed stiffly at one end, and subjected to an elastic spring and a force at the other end.



– This BVP problem is equivalent to minimizing the following functional:

$$\Pi(u) = \int_0^L \left[\frac{\alpha}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{k}{2}\left[u(L)\right]^2 - Pu(L)$$

– This functional describes the total potential energy of the bar, and so the problem solution ū ensures the minimum of total potential energy.
– The necessary condition for the minimum of Π

$$\delta\Pi(u) = B(u,\delta u) - L(\delta u) = 0 \quad \text{or} \quad B(u,\delta u) = L(\delta u)$$

$$B(u,v) = \int_0^L \alpha \frac{du}{dx}\frac{dv}{dx}dx + ku(L)v(L)$$

$$L(u) = \int_0^L fudx + Pu(L)$$

– The essential boundary condition of the problem is provided by the geometric constraint, u(0) = 0, and must be satisfied by ϕ_0(x)

Approximation

Applying the Ritz approximation and minimizing the functional gives:

$$\frac{\partial \Pi}{\partial c_i} = \int_0^L \left[\alpha \frac{d\phi_i}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} + \frac{d\phi_0}{dx} \right) - f\phi_i \right] dx$$

$$+ k\phi_i(L) \left(\sum_{j=1}^{N} c_j \phi_j(L) + \phi_0(L) \right) - P\phi_i(L) \quad (i = 1, \ldots, N)$$

This is the system of equations for the Ritz parameters:

$$\sum_{j=1}^{N} A_{ij} c_j = b_i \quad (i = 1, \ldots, N)$$

$$A_{ij} = \int_0^L \alpha \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx + k\phi_i(L)\phi_j(L)$$

$$b_i = -\int_0^L \left[\alpha \frac{d\phi_i}{dx} \frac{d\phi_0}{dx} - f\phi_i \right] dx - k\phi_i(L)\phi_0(L) + P\phi_i(L)$$

Solution

- The problem data: $\alpha(x) = EA(x) = \underbrace{\alpha_0}_{EA_0}\left(2 - \frac{x}{L}\right),\ f(x) = f_0,\ k = 0$
- The approximation functions:

$$\phi_0(x) = 0,\quad \phi_j(x) = x^j \quad \text{for } j = 1, \ldots, N$$

- The approximate solutions:

(1) $N = 1$: $\tilde{u}^{(1)} = c_1 x$

(2) $N = 2$: $\tilde{u}^{(2)} = c_1 x + c_2 x^2$

► $N = 1$: $\tilde{u}^{(1)} = c_1 x$			
$A_{11} = \frac{3}{2}\alpha_0 L$		$b_1 = \frac{1}{2} f_0 L^2 + PL$	$c_1 = \frac{f_0 L + 2P}{3\alpha_0}$
► $N = 2$: $\tilde{u}^{(2)} = c_1 x + c_2 x^2$			
$A_{11} = \frac{3}{2}\alpha_0 L$	$A_{12} = \frac{4}{3}\alpha_0 L^2$	$b_1 = \frac{1}{2} f_0 L^2 + PL$	$c_1 = \frac{7 f_0 L + 6P}{13\alpha_0}$
$A_{21} = A_{12}$	$A_{22} = \frac{5}{3}\alpha_0 L^3$	$b_2 = \frac{1}{3} f_0 L^3 + PL^2$	$c_2 = \frac{-3 f_0 L + 3P}{13\alpha_0 L}$