3.1 Differential Equations

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \xrightarrow[A^T = -\frac{d}{dx}]{A = \frac{d}{dx}} A^T C A u = f$$

$$\begin{cases} \text{fixed: } u(0) = 0 \\ \text{free: } w(1) = 0 \end{cases}$$

$$f(x) = \begin{cases} 1 \text{ (constant)} \\ \cos \omega x \text{ (periodic)} \\ \delta(x-a) \text{ (at one point, at one instant of time)} \end{cases}$$

Hanging Bar: stretching

$$\underbrace{u(x)}_{u(0)=0} \xrightarrow[\text{starin-displacement}]{A=\frac{d}{dx}} e(x) \xrightarrow[\text{stress-starin}]{C} \underbrace{w(x)}_{w(1)=c(1)u'(1)=0} \xrightarrow{A^T=-\frac{d}{dx}} f(x)$$

$$-\frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) = f(x) \text{ with } u(0)=0,\ w(1)=0$$

$$-\frac{dw(x)}{dx} = f(x) \rightarrow w(x) = -\int_0^x f(s)ds + C$$

$$w(1) = -\int_0^1 f(s)ds + C = 0$$

$$w(x) = -\int_0^x f(s)ds + \int_0^1 f(s)ds = \int_x^1 f(s)ds$$

$$c(x)\frac{du(x)}{dx} = w(x) \rightarrow u(x) = \int_0^x \frac{w(s)}{c(s)}ds$$

Hanging Bar: Example

$f(x)$	$c(x)$	@ $x=0$	@ $x=1$
f_0	c_0	$u = 0$	$w = 0$
$2-x$	$2-x$	$u = 0$	$u = 0$
$f_0\delta(x-x_0)$		$u = 0$	$w = 0$

A=d/dx $\leftrightarrow$ A^T=-d/dx

$$A = \underbrace{\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}}_{\text{backward difference}} \leftrightarrow A^T = \underbrace{\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}}_{-(\text{forward difference})}$$

$$\underbrace{e^T w}_{\text{energy}} = \underbrace{u^T f}_{\text{work}} \leftrightarrow (Au)^T w = u^T \left(A^T w\right)$$

$$(e,w) = \int_0^1 e(x)\,w(x)\,dx \rightarrow (Au,w) = \int_0^1 \frac{du(x)}{dx} w(x)\,dx = \underbrace{\left[u(x)\,w(x)\right]_0^1}_{=0} - \int_0^1 u(x)\frac{dw(x)}{dx}\,dx$$

A (for u)	A^T (for w)
u(0) = 0	w(1) = 0
u(0) = u(1) = 0	no conditions
no conditions	w(0) = w(1) = 0

Galerkin's Method

Finite difference: strong form of the differential equation

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \quad \text{with fixed-free or fixed-fixed}$$

Finite element: weak form with test functions $v(x)$

$$\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}dx = \int_0^1 f(x)v(x)dx \quad \text{for all } v(x)$$

$$U(x) = \sum_{i=1}^{n} U_i \underbrace{\phi_i(x)}_{\text{trial functions}} \rightarrow \int_0^1 c(x)\left(\sum_{j=1}^{n} U_j \frac{d\phi_j}{dx}\right)\frac{dV_i}{dx}dx = \int_0^1 f(x)V_i(x)dx \rightarrow KU = F$$

$$K_{ij} = \int_0^1 c(x)\frac{dV_i}{dx}\frac{d\phi_j}{dx}dx, \; F_i = \int_0^1 f(x)V_i(x)dx \xleftarrow{\text{Advantages}} \begin{cases} \text{1. point load: } f(x) = \delta(x-a) \\ \text{2. step function } c(x) \end{cases}$$

(1) Choose the ϕ_i and V_i: one unknown for each ϕ, one equation for each V

(2) Compute exact or approximate integrals K_{ij} and F_i: if $\phi_i = V_i$ then $K_{ij} = K_{ji}$

(3) The weak form becomes $KU = F$. The FEM approximation is $U(x) = \sum_{i=1}^{n} U_i \phi_i(x)$.

Linear Finite Elements: examples

c(x)	f(x)	w(x)	u(x)
1	1		
Jump from 2 to 4 @ x = 1/3	δ(x-1/2)		
any	any		

	accuracy		approximation
u(x)	second-order	U(x)	piecewise linear
u'(x)	first-order	U'(x)	piecewise constant

- How to reduce error?
  - Refine the mesh: many meshpoints
  - Increase the degree of the finite elements

3.2 Cubic Finite Elements

Elastic rod stretching	Beam bending
u'' = δ(x)	u'''' = δ(x)
u = A + Bx	u = A + Bx +Cx^2 + Dx^3
jump in the slope u'	jump in the slope u'''

$$\left(U_i, U_{i+1}\right) \rightarrow a + bx + cx^2 + dx^3 \leftarrow \left(U_i^d, U_i^s, U_{i+1}^d, U_{i+1}^s\right)$$

$$U = \sum_{i=1}^{N} \left(U_i^d \phi_i^d + U_i^s \phi_i^s\right): \ 2N \text{ trial functions (Hermite cubics)}$$

– These trial and test functions can still be used for u'''' = f
– Error u(x) – U(x) is of order h^4
– Element stiffness matrix K_i over each interval: assemble into the global stiffness matrix K

4^{th}-order Equations: Beam Bending

$$\underbrace{u(x)}_{\text{displacement}} \xrightarrow{A=\frac{d^2}{dx^2}} \underbrace{e(x)}_{\text{curvature}} \xrightarrow{C} \underbrace{M(x)}_{\text{moment}} \xrightarrow[\text{balance}]{A^T=\frac{d^2}{dx^2}} f(x)$$

$$(Au, M) = \left(u, A^T M\right) \rightarrow \int_0^1 \frac{d^2u}{dx^2} M dx = \int_0^1 u \frac{d^2M}{dx^2} M dx$$

A (for u)	A^T (for M)	
u = 0	M = 0	simply supported
u = 0, du/dx = 0		clamped (fixed)
	M = 0, dM/dx = 0	free
du/dx = 0	dM/dx = 0	sliding clamped

$$\frac{d^2}{dx^2}\left(c\frac{d^2u}{dx^2}\right) = f(x) \xrightarrow{\text{weak form}} \begin{cases} \int_0^1 \frac{d^2}{dx^2}\left(c(x)\frac{d^2u}{dx^2}\right) v(x) dx = \int_0^1 f(x) v(x) dx \\ \int_0^1 c(x)\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx = \int_0^1 f(x) v(x) dx \end{cases}$$

Cubic Splines for Interpolation

Interpolation: exact fit $y(x_i) = y_i$ for n points

polynomial of high degree	$y(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$
piecewise polynomial: linear	$y(x) = y_i \dfrac{x - x_{i+1}}{x_i - x_{i+1}} + y_{i+1} \dfrac{x - x_i}{x_i - x_{i+1}}$

- Splines: cubic pieces with good accuracy $O(h^4)$ to achieve continuous slopes and second derivatives
  - Unknown: 2n (y_i, s_i)
  - Known: n conditions + (n-2) continuity + 2 BCs (clamped, not-a-knot)
- B-splines
  - basis for the whole (n+2)-dimensional space of splines

Finite Differences

$$\left.\begin{array}{l}\left(\Delta^2 u\right)_{i+1} = u_{i+2} - 2u_{i+1} + u_i \\ \left(\Delta^2 u\right)_i = u_{i+1} - 2u_i + u_{i-1}\end{array}\right\} \to \Delta^3 u = u_{i+2} - 3u_{i+1} + 3u_i - u_{i-1}$$

$$\left.\begin{array}{l}\left(\Delta^2 u\right)_i = u_{i+1} - 2u_i + u_{i-1} \\ \left(\Delta^2 u\right)_{i-1} = u_i - 2u_{i-1} + u_{i-2}\end{array}\right\} \to \Delta^3 u = u_{i+1} - 3u_i + 3u_{i-1} - u_{i-2}$$

$$\to$$

$$\left(\Delta^4 u\right)_i = u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}$$

$$\frac{d^4}{dx^4} \approx \frac{1}{h^4}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}^2 = \frac{1}{h^4}\begin{bmatrix} 5 & -4 & 1 & & \\ -4 & 6 & -4 & 1 & \\ 1 & -4 & 6 & -4 & 1 \\ & 1 & -4 & 6 & -4 \\ & & & -4 & 5 \end{bmatrix}$$

3.3 Gradient and Divergence

$$\text{gradient}(\nabla) = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}, \quad \text{divergence}(\nabla \cdot) = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}$$

$$\underbrace{u(x,y)}_{\substack{\text{potential} \\ u=u_0(x,y)}} \xrightarrow{A=\nabla} \underbrace{v(x,y)}_{\text{velocity}} \xrightarrow{C} \underbrace{w(x,y)}_{\substack{\text{flow rate} \\ w\cdot n=F_0(x,y)}} \xrightarrow{A^T=-\nabla\cdot} \underbrace{f(x,y)}_{\text{source}}$$

$$A^T CAu = f \rightarrow -\text{div}(c\,\text{grad}\,u) = -\nabla\cdot(c\nabla u) = f$$

$$\rightarrow \begin{cases} -\dfrac{\partial}{\partial x}\left(c\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(c\dfrac{\partial u}{\partial y}\right) = f(x,y)\text{: Poisson equation} \\ f = 0 \text{ and } c = \text{constant}, A^T Au = 0 \rightarrow -\nabla\cdot(\nabla u) = 0 \rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0\text{: Laplace equation} \end{cases}$$

Key Ideas

- The gradient extends the derivative du/dx to a two-variable function u(x,y)
  - Derivative of u in the direction of any unit vector (n_1, n_2)

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_1 + \frac{\partial u}{\partial y} n_2 = \nabla u \cdot n \perp \left(u\left(x, y\right) = \text{constant: level curves} \right)$$

- Zero divergence ↔ Kirchhoff's Current Law
  - Flow in = flow out
  - Incompressible flow with no sources and so sinks

$$\text{div } w = \nabla \cdot w = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0 \leftrightarrow A^T w = 0$$

- Laplace's equation has polynomial solutions that are easy to find

$$\left. \begin{array}{l} v = \nabla u \\ \nabla \cdot w = 0 \end{array} \right\} \xrightarrow{\text{Gauss-Green formula}} \nabla^T = -\nabla \cdot$$

A = gradient

$$e = Au \rightarrow \begin{cases} \text{linear algebraist: } e \text{ must be in the column space of } A \\ \text{Kirchhoff: the sum of potential differences around any loop must be zero} \\ \text{(Kirchhoff's Voltage Law)} \end{cases}$$

$$v = Au \xrightarrow{\text{continuous case}} v = \nabla u : \text{ irrotational velocity field}$$

$$\rightarrow \begin{cases} \dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y} = 0 \text{ at each point (zero vorticity)} \xrightarrow{\text{3D}} \text{curl } v = 0 \\ \text{potential function } u(x, y): v_1 = \dfrac{\partial u}{\partial x}, v_2 = \dfrac{\partial u}{\partial y} \\ \displaystyle\int v_1 dx + v_2 dy = 0 \text{ around every loop (zero circulation)} \end{cases}$$

$$\int_C v_1 dx + v_2 dy = \iint_R \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) dxdy : \text{ Stoke's theorem}$$

$A^T = -$divergence

$$A = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \rightarrow A^T = \begin{bmatrix} \left(\frac{\partial}{\partial x}\right)^T & \left(\frac{\partial}{\partial y}\right)^T \end{bmatrix} = -\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \leftrightarrow \nabla^T = -\nabla \cdot$$

Kirchhoff's Current Law: flow in equals flow out $\rightarrow A^T w = 0$

flow vector w is source-free if mass is conserved

$$\rightarrow \begin{cases} \text{div } w = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0 \text{ at each point (zero divergence)} \\ \rightarrow \text{stream function } s(x, y)\text{: } w_1 = \frac{\partial s}{\partial y}, w_2 = -\frac{\partial s}{\partial x} \\ \int w_1 dy - w_2 dx = 0 \text{ through every loop (zero flux)} \end{cases}$$

$$\int_C w_1 dy - w_2 dx = \iint_R \text{div } w dx dy : \text{ Divergence theorem}$$

Example

	gradient	zero divergence
v = (2x, 2y)		
V = (2x, -2y)		
w = (2y, -2x)		
v = w = (2y, 2x)		

Divergence Theorem

$$\text{flow is} \begin{cases} \perp \text{ to the equipotential curves: } u(x,y)=c \\ || \text{ to the streamlines: } s(x,y)=c \end{cases}$$

$$\underbrace{\iint_R (\text{div } w)\, dxdy}_{\text{total source in}} = \underbrace{\int_B \Big(\underbrace{w \cdot n}_{\text{flow rate out}} \Big) ds}_{\text{total flux out}} \leftrightarrow \iint_R \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right) dxdy = \int_B (w_1 dy - w_2 dx)$$

$$\underbrace{\iint_R \left(\frac{\partial u}{\partial x} w_1 + \frac{\partial u}{\partial y} w_2 \right) dxdy}_{\text{integration by parts}} = -\iint_R \left(u\frac{\partial w_1}{\partial x} + u\frac{\partial w_2}{\partial y} \right) dxdy + \int_B u(w_1 dy - w_2 dx)$$

$$\underbrace{\iint_R (\nabla u \cdot w)\, dxdy}_{(Au)^T w} = \underbrace{\iint_R u(-\nabla \cdot w)\, dxdy}_{u^T (A^T w)} + \underbrace{\int_B uw \cdot n ds}_{\iint_R \text{div}(uw) dxdy} \rightarrow \text{Gauss-Green formula}$$

$$\text{div}(uw) = \frac{\partial}{\partial x}(uw_1) + \frac{\partial}{\partial y}(uw_2) = \left(\frac{\partial u}{\partial x} w_1 + \frac{\partial u}{\partial y} w_2 \right) + u\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right) = \nabla u \cdot w + u(\nabla \cdot w)$$

divergence theorem for $(uw) \rightarrow$ Gauss-Green formula

3D

- Which vector fields v(x,y,z) are gradients?

$$v = \nabla u : \; v_1 = \frac{\partial u}{\partial x}, v_2 = \frac{\partial u}{\partial y}, v_3 = \frac{\partial u}{\partial z} \rightarrow \text{integral around a closed loop will be zero}$$

$$\text{test: } \frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial y}, \frac{\partial v_3}{\partial y} = \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} = \frac{\partial v_3}{\partial x} \leftrightarrow \text{curl } v = 0$$

- Which vector fields w(x,y,z) have zero divergence?

$$\left(\text{div } w = 0 : \; \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z} = 0 \right) \rightarrow \text{flux} = \text{zero}, \; \iiint \left(\text{div } w\right) dxdydz = \iint \left(w \cdot n\right) dS$$

$$\text{test: div}\left(\text{curl } S\right) = 0 \leftarrow w = \text{curl } S = \nabla \times S = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial s_3}{\partial y} - \frac{\partial s_2}{\partial z} \\ \frac{\partial s_1}{\partial z} - \frac{\partial s_3}{\partial x} \\ \frac{\partial s_2}{\partial x} - \frac{\partial s_1}{\partial y} \end{bmatrix}$$

Summary

$$\begin{cases} \text{A gradient field comes from a potential:} \\ v = \nabla u \text{ when curl } v = 0 \rightarrow \text{curl } \left(\nabla u\right) = 0 \end{cases}$$

$$\begin{cases} \text{A source-free field comes from a stream function:} \\ \text{div } w = 0 \text{ when } w = \text{curl } S \rightarrow \text{div } \left(\text{curl } S\right) = 0 \end{cases}$$

$$\rightarrow \left(\text{curl } \left(\text{grad}\right) = 0\right)^T \xrightarrow{\left(\text{grad}\right)^T = -\text{div}, \left(\text{curl}\right)^T = \left(\text{curl}\right)} \text{div } \left(\text{curl}\right) = 0$$

3.4 Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\underbrace{\left.\begin{matrix} u(x,y) \\ u(r,\theta) \end{matrix}\right\}}_{\substack{\text{potential} \\ \text{temperature}}} \xrightarrow{\text{Hilbert transform}} \underbrace{\left\{\begin{matrix} s(x,y) \\ s(r,\theta) \end{matrix}\right.}_{\substack{\text{stream function} \\ \text{heat flow}}}$$

$$z = x + iy = re^{i\theta} \rightarrow z^n = \left(x+iy\right)^n = r^n e^{in\theta}$$

$$\begin{cases} u_n = \mathrm{Re}\left[z^n\right] = \mathrm{Re}\left[\left(x+iy\right)^n\right] = \mathrm{Re}\left[r^n e^{in\theta}\right] \\ s_n = \mathrm{Im}\left[z^n\right] = \mathrm{Im}\left[\left(x+iy\right)^n\right] = \mathrm{Im}\left[r^n e^{in\theta}\right] \end{cases}$$

Cauchy-Riemann Equations (1)

[potential u and stream function s]

physically, the flow velocity $v = \nabla u$ is along the streamlines $s =$ constant

geometrically, the equipotentials $u =$ constant are perpendicular to those streamlines

mathematically, u and s both come from the same analytic function

$$\underbrace{f(x+iy)}_{\text{analytic function}} = \underbrace{u(x,y)}_{\text{[harmonic}} + i\,\underbrace{s(x,y)}_{\text{functions]}} = f(z) = \sum_{n=0}^{\infty} c_n z^n$$

converged power series $\rightarrow$ solution of Laplace's equation: $u(x,y)$ and $s(x,y)$

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz} = \left\{ \begin{array}{l} x \rightarrow 0 : \dfrac{1}{i}\dfrac{\partial f}{\partial y} \\[2ex] y \rightarrow 0 : \dfrac{\partial f}{\partial x} \end{array} \right\} \rightarrow \frac{\partial}{\partial y} f(x+iy) = i\frac{\partial}{\partial x} f(x+iy)$$

$$\rightarrow \frac{\partial}{\partial y}(u+is) = i\frac{\partial}{\partial x}(u+is) \rightarrow \frac{\partial u}{\partial x} = \frac{\partial s}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}$$

Cauchy-Riemann Equations (2)

- Streamlines are perpendicular to equipotentials
- The derivatives of u across a curve equals the derivative of s along the curves
- Streamlines pass through any and every curve from P to Q
- The total flow between the points is s(Q)–s(P)

Poisson's Equation in a Square

$$-u_{xx} - u_{yy} = f(x, y)$$

eigencvectors: $u_{mn} = (\sin m\pi x)(\sin n\pi y) \rightarrow -u_{xx} - u_{yy} = (m^2 + n^2)\pi^2 u = \lambda_{mn} u$

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} u_{mn} \rightarrow u(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{b_{mn}}{\lambda_{mn}} (\sin m\pi x)(\sin n\pi y)$$

$u(x, y) = 0$ on the boundary of the square $\rightarrow \begin{cases} \sin m\pi x = 0 \text{ if } x = 0 \text{ or } 1 \\ \sin n\pi y = 0 \text{ if } y = 0 \text{ or } 1 \end{cases}$

Conformal Mapping

- A change from x, y to X, Y based on an analytical function F(z)
- Not circular boundary? → change variables

$$F(x+iy) = X(x,y) + iY(x,y)$$

$$U\left(X(x,y), Y(x,y)\right) = \mathrm{Re}\left[F(X+iY)\right] = \mathrm{Re}\left[f\left(F(x+iy)\right)\right] = u(x,y)$$

- Important conformal mappings

$$Z = e^{z} = e^{x+iy} = e^{x}e^{iy} : \text{ infinte strip}\left(0 \le y \le \pi\right) \to \text{upper half plane}$$

$$Z = \frac{az+b}{cz+d} : \text{ circle}\left(z=0\right) \to \text{circle}\left(Z = b/d\right)$$

$$Z = \frac{1}{2}\left(z + \frac{1}{z}\right) : \begin{cases} \text{circle}\left(|z| = r\right) \to \text{ellipse} \\ \text{circle}\left(|z| = 1\right) \to -1 \le Z \le 1 \end{cases}$$

3.4 Laplace Equation

- Complex variables
- Fourier series
- Finite differences
- Finite elements

3.5 Finite Differences

$$4U_{i,j} - U_{i,j-1} - U_{i-1,j} + U_{i+1,j} - U_{i,j+1} = h^2 f\left(ih, jh\right)$$

$$\begin{bmatrix} \cdots & \underbrace{-1\ \ 0\ \ 0\ \ 0\ \ \text{-}1}_{w=N} & 4 & \underbrace{-1\ \ 0\ \ 0\ \ 0\ \ \text{-}1}_{w=N} & \cdots \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & . & . & . \\ & & -1 & 2 \end{bmatrix} \quad K2D = \begin{bmatrix} K+2I & -I & & \\ -I & K+2I & -I & \\ & . & . & . \\ & & -I & K+2I \end{bmatrix} = \text{kron}\left(I, K\right) + \text{kron}\left(K, I\right)$$

$$\rightarrow \underbrace{\left(K2D\right)}_{\text{sparse}} U = F$$

- Elimination in a good order
  - Elimination fills in the zeros inside the band
- Fast Poisson solver
- Odd-even reduction

Poisson Solver

- Use eigenvalues and eigenvectors to solve
- Faster than elimination

$$(1)\ F = a_1 y_1 + \cdots + a_N y_N = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = Sa \rightarrow a = S^{-1}F$$

$$(2)\ \text{divide } a_k \text{ by } \lambda_k : \Lambda^{-1}a = \Lambda^{-1}S^{-1}F$$

$$(3)\ U = \left(\frac{a_1}{\lambda_1}\right) y_1 + \cdots + \left(\frac{a_N}{\lambda_N}\right) y_N = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \begin{bmatrix} a_1/\lambda_1 \\ \vdots \\ a_N/\lambda_N \end{bmatrix} = S\Lambda^{-1}a = S\Lambda^{-1}S^{-1}F$$

$$\xleftarrow{K = S\Lambda S^{-1}} U = K^{-1}F$$

	eigenvectors		elimination
1D	$N \log_2 N$	>	cN
2D	$N^2 \log_2 (N^2)$	<	N^4

Discrete Sine Transform

$$\begin{cases} S_{jk} = \sin \dfrac{jk\pi}{N+1} = \operatorname{Im} \omega^{jk} = \operatorname{Im}\left(e^{i2\pi jk/M}\right) = \operatorname{Im}\left(e^{i\pi jk/(N+1)}\right) \\ \|y_i\|^2 = \dfrac{N+1}{2} \\ DST = S\sqrt{\dfrac{2}{N+1}} : \text{ orthonormal } \rightarrow DST = DST^{-1} = DST^T \end{cases}$$

$$\left\{ \lambda_k = 2 - 2\cos \frac{k\pi}{N+1}, \ \sum_{k=1}^{n} \lambda_k = \operatorname{trace}(K_n), \ \prod_{k=1}^{n} \lambda_k = \det(K_n) = N+1 \right.$$

Fast Poisson Solvers: 2D

$$\left.\begin{aligned} y_{kl} &= \sin\frac{ik\pi}{N+1}\sin\frac{jl\pi}{N+1}:\ \text{separable} \\ \lambda_{kl} &= \lambda_k + \lambda_l = \left(2-2\cos\frac{k\pi}{N+1}\right)+\left(2-2\cos\frac{l\pi}{N+1}\right) \end{aligned}\right\} \leftarrow (K2D)\, y_{kl} = \lambda_{kl}\, y_{kl}$$

$$(K2D)U = F \rightarrow \begin{cases} F_{i,j} = \sum\sum a_{kl}\sin\frac{ik\pi}{N+1}\sin\frac{jl\pi}{N+1} \leftrightarrow \text{AM=DST*FM*DST} \\ U_{i,j} = \sum\sum \frac{a_{kl}}{\lambda_{kl}}\sin\frac{ik\pi}{N+1}\sin\frac{jl\pi}{N+1} \leftrightarrow \text{UM=DST*}\left(\frac{\text{AM.}}{\text{LM}}\right)\text{*DST} \end{cases}$$

Cyclic Odd-Even Reduction

$$\langle 1D \rangle \begin{cases} \left.\begin{array}{ll} (i-1): -U_{i-2} + 2U_{i-i} - U_i & = F_{i-1} \\ (i): \quad -U_{i-1} + 2U_i - U_{i+1} & = F_i \\ (i+1): \quad -U_i + 2U_{i+1} - U_{i+2} & = F_{i+1} \end{array}\right\} \\ \xrightarrow{(i-1)+2(i)+(i+1)} -U_{i-2} + 2U_i - U_{i+2} = F_{i-1} + 2F_i + F_{i+1} : \text{ half-size system} \end{cases}$$

$$\langle 2D \rangle \begin{cases} K2D = \begin{bmatrix} A & -I & & \\ -I & A & -I & \\ & \cdot & \cdot & \cdot \\ & & -I & A \end{bmatrix} \text{ with } A = K + 2I \\ \rightarrow -IU_{i-2} + \left(A^2 - 2I\right)U_i - IU_{i+2} = F_{i-1} + AF_i + F_{i+1} \rightarrow \text{bandwidth doubles} \end{cases}$$

$$\langle \text{red-blakc ordering} \rangle \begin{bmatrix} D_b & R \\ B & D_r \end{bmatrix} \begin{bmatrix} u_b \\ u_r \end{bmatrix} = \begin{bmatrix} f_b \\ f_r \end{bmatrix}$$

Neumann Conditions: ∂u/∂n = 0

1D	$\frac{du}{dx} = 0$ (free end): $K \rightarrow B$
2D	$D = diag\left(\begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 & \frac{1}{2} \end{bmatrix}\right)$ $B2D = kron(D, B) + kron(B, D)$ $(N = 3) : B2D = \begin{bmatrix} B/2 & & \\ & B & \\ & & B/2 \end{bmatrix} + \begin{bmatrix} D & -D & \\ -D & 2D & -D \\ 0 & -D & D \end{bmatrix}$

3.6 Finite Element Method

– Curved boundary, unstructured mesh

- Write the equation in its weak form, integrated with test functions $v(x,y)$
- Subdivide the region into triangles or quadrilaterals
- Choose N simple trial functions $\phi_j(x,y)$ and look for $U = U_1\phi_1 + \ldots + U_N\phi_N$
- Produce N equations KU = F from test functions V_j
- Assemble the stiffness matrix K and the load vector F. Solve KU = F.

Strong / Weak Form

$$\text{stong form}: A^T CAu = f \xrightarrow{\text{when } C=I} -u_{xx} - u_{yy} = f$$

$$\text{weak form}: \left(A^T CAu\right)^T v = f^T v \rightarrow (CAu)^T (Av) = f^T v$$

$$\xrightarrow{\text{when } C=I} \iint \left(-u_{xx} - u_{yy}\right) v\,dxdy = \iint f v\,dxdy$$

$$\rightarrow -\int_B \left(u_x v dy + u_y v dx\right) + \iint \left(u_x v_x + u_y v_y\right) dxdy = \iint f v\,dxdy$$

$$\xrightarrow[\text{Divergence Theorem}]{\text{Gauss-Green formula}} \iint \left(u_x v_x + u_y v_y\right) dxdy = \int \frac{\partial u}{\partial n} v\,ds + \iint f v\,dxdy$$

$$\rightarrow (Au)^T (Av) = \iint (\nabla u) \cdot (\nabla v) = \iint \left(u_x v_x + u_y v_y\right) dxdy = \int \frac{\partial u}{\partial n} v\,ds + \iint f v\,dxdy$$

$$\xrightarrow[\text{Essential B.C.}]{\text{Laplce's Equation}} \iint \left(u_x v_x + u_y v_y\right) dxdy = 0 \quad \text{for all admissible } v$$

Trial / Test Functions

$$U\left(x,y\right)=\sum_{j=1}^{N}U_j\phi_j\left(x,y\right)\rightarrow\iint\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dxdy=\iint fvdxdy$$

$$\iint\left[\left(\sum_{j=1}^{N}U_j\frac{\partial\phi_j}{\partial x}\right)\frac{\partial V_i}{\partial x}+\left(\sum_{j=1}^{N}U_j\frac{\partial\phi_j}{\partial y}\right)\frac{\partial V_i}{\partial y}\right]dxdy=\iint fV_i dxdy\rightarrow KU=F$$

$$K_{ij}=\iint\left(\frac{\partial\phi_j}{\partial x}\frac{\partial V_i}{\partial x}+\frac{\partial\phi_j}{\partial y}\frac{\partial V_i}{\partial y}\right)dxdy$$

Pyramid Functions

$u_{xx} + u_{yy} = 0$ inside a square

boundary values: around the whole square of side $2h$

subdomain division: 8 triangles $\leftrightarrow$ five-point Laplacian

$U(x, y) = a + bx + cy$: linear approximation

Element Matrices and Vectors: Triangle

$$K_{ij} = \iint \left(\frac{\partial \phi_j}{\partial x} \frac{\partial V_i}{\partial x} + \frac{\partial \phi_j}{\partial y} \frac{\partial V_i}{\partial y} \right) dxdy \xrightarrow{V_i = \phi_i, U = \sum_{j=1}^{N} U_j \phi_j}$$

$$\iint \left(U_x^2 + U_y^2 \right) dxdy = \sum_{i=1}^{N} \sum_{i=1}^{N} U_i U_j \iint \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dxdy$$

$$= U^T \left(K_T \right) U = \begin{bmatrix} U_i & U_{i+1} & U_{i+2} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} U_i \\ U_{i+1} \\ U_{i+2} \end{bmatrix}$$

$$K_e = \begin{bmatrix} c_2 + c_3 & -c_3 & -c_2 \\ -c_3 & c_1 + c_3 & -c_1 \\ -c_2 & -c_1 & c_1 + c_2 \end{bmatrix} \text{ with } c_i = \frac{1}{2 \tan \theta_i}$$

$$\iint_e f \left(\sum_{i=1}^{N} U_i \phi_i \right) dxdy = \sum_{i=1}^{N} U_i \iint_e f \phi_i dxdy = U^T \left(F_e \right)$$

$$\iint_e f(x, y) U(x, y) dxdy \xrightarrow{\text{one-point integration}} f(P) \left(\frac{U_i + U_{i+1} + U_{i+2}}{3} \right) (\text{area of } e)$$

Boundary Conditions

$-u'' = 1$ with fixed-free B.C. $\left[u(0) = 0 \text{ and } u'(1) = 0\right]$

$\rightarrow \int u_x v_x dx = \int 1 v dx$

Essential / Dirichlet / Fixed	Natural / Neumann / Free
$u(0) = A$ on U $v(0) = 0$ on V $cu'v = 0$ at $x = 0$	$w(1) = cu'(1) = G$ no conditions on $u'(1)$ or $U'(1)$ no conditions on $v'(1)$ or $V'(1)$ $cu'v = Gv(1) \rightarrow$ move into F

Element Matrices in 2D

$$\phi_i = a_i + b_i x + c_i y \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{P\text{: position}} \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{C}$$

$$\xrightarrow[(0,0)(1,0)(0,1)]{\text{standard triangle}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{cases} \phi_1 = 1 - x - y \\ \phi_2 = x \\ \phi_3 = y \end{cases}$$

$$\left(K_e\right)_{ij} = \iint \left(\frac{\partial \phi_i}{\partial x} \frac{\partial V_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial V_j}{\partial y} \right) dxdy = \underbrace{(area)}_{\frac{\det(P)}{2}} \left(b_i b_j + c_i c_j\right) = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Bilinear Q_1 elements: $U = a + bx + cy + dxy$

Quadratic P_2 elements: $U = a + bx + cy + dxy + ex^2 + fy^2$

Mass Matrix

$$\begin{cases} -u_{xx} - u_{yy} + u = 0 \rightarrow \left(K + M\right)U = 0 \\ -u_{xx} - u_{yy} = \lambda u \rightarrow KU = \lambda MU \\ u_{xx} + u_{yy} = u_{tt} \rightarrow MU'' + KU = 0 \end{cases}$$

$$\iint U^2 dxdy = \iint \left(\sum_{i=1}^{N} U_i \phi_i\right)\left(\sum_{j=1}^{N} U_j \phi_j\right) dxdy = \sum_{i=1}^{N}\sum_{i=1}^{N} U_i U_j \iint \phi_i \phi_j dxdy$$

$$\text{1D: } U = U_0 + \frac{U_1 - U_0}{h} x \rightarrow \int_0^h U^2 dx$$

$$\text{2D: } U = U_1 + \left(U_1 - U_0\right) x + \left(U_3 - U_1\right) y \rightarrow \int_0^h \int_0^h U^2 dxdy$$