1.1.3

>> U=eye(5)-diag(ones(4,1),1)	>> S=triu(ones(5))
U =	S =
1 -1 0 0 0	1 1 1 1 1
0 1 -1 0 0	0 1 1 1 1
0 0 1 -1 0	0 0 1 1 1
0 0 0 1 -1	0 0 0 1 1
0 0 0 0 1	0 0 0 0 1

```
>> U*S
ans =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
```

1.1.5

$$K_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \xrightarrow{\det(K_2)=3} K_2^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\det(K_3)=4} K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\det(K_4)=5} K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$K_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\det(K_5)=6} K_5^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

1.1.9

Problem (1.1.9).

Solution. Let c_i be the ith column of C_4. We simply must check that $c_i^T e = 0$, or equivalently,

$$C_4^T e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This verification can be done by a simple matrix multiplication. Alternatively, one can show that $u^T C_4^T e = 0$ for any u, which implies that $C_4^T e = 0$.

Next, we are asked to use *pinv* to solve the system

$$C_4 u = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

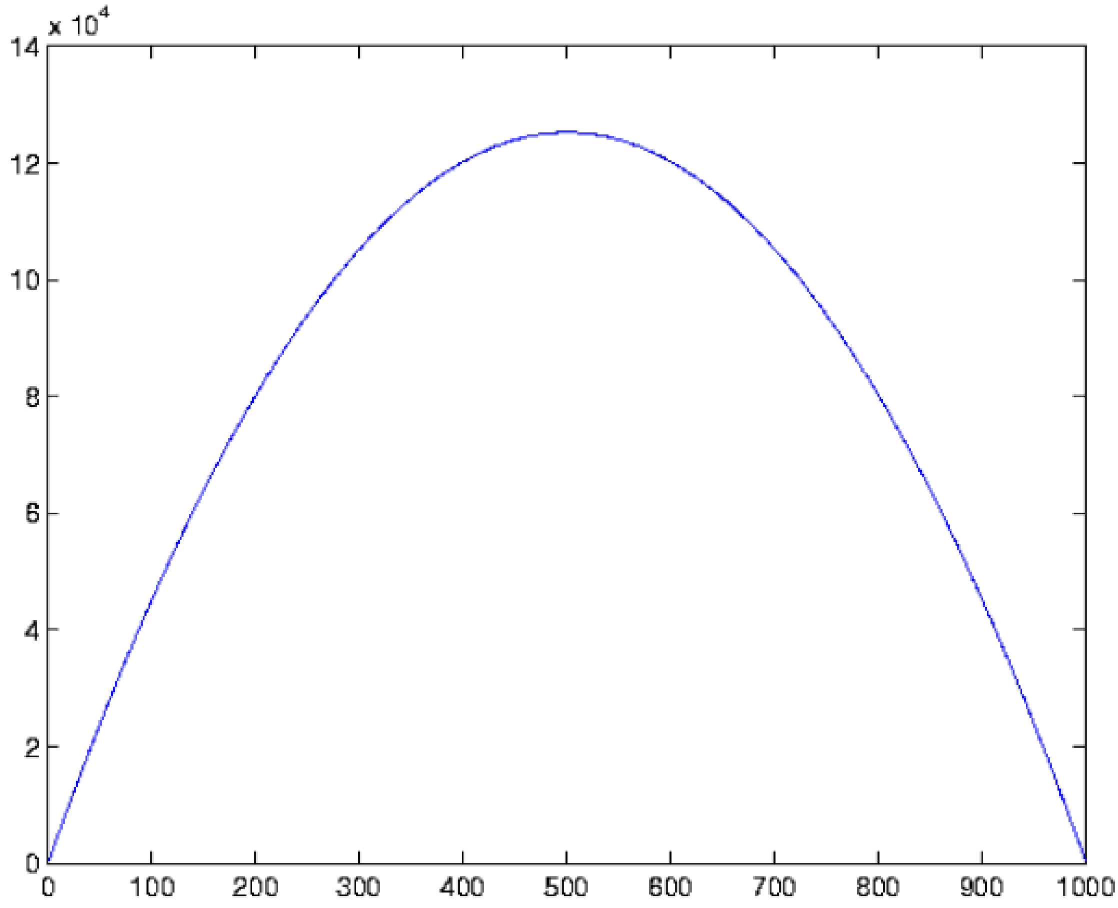
First, we define $C = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$ Using MatLab, we get $pinv\,(C) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{pmatrix}$. Thus, we have found a solution to a linear algebra problem involving a singular matrix. To see if the "\" command will also yield a solution, we again turn to MatLab. , we see that C\[1; 1; 1; 1]=$10^{16} \times \begin{pmatrix} 1.8014 \\ 1.8014 \\ 1.8014 \\ 1.8014 \end{pmatrix}$, a divergent result. Next, we see that C\[1; -1; 1; -1]=$\begin{pmatrix} -1 \\ -1.5 \\ -1 \\ -1.5 \end{pmatrix}$, which indeed solves our equation. As expected, the equation yields a solution when f is perpendicular to e.

Now we are asked to add a row of zeroes to the equation, and once again use "\" to solve. We will redefine C=[2 -1 0 -1;-1 2 -1 0;0 -1 2 -1;-1 0 -1 2;0 0 0 0], we see that

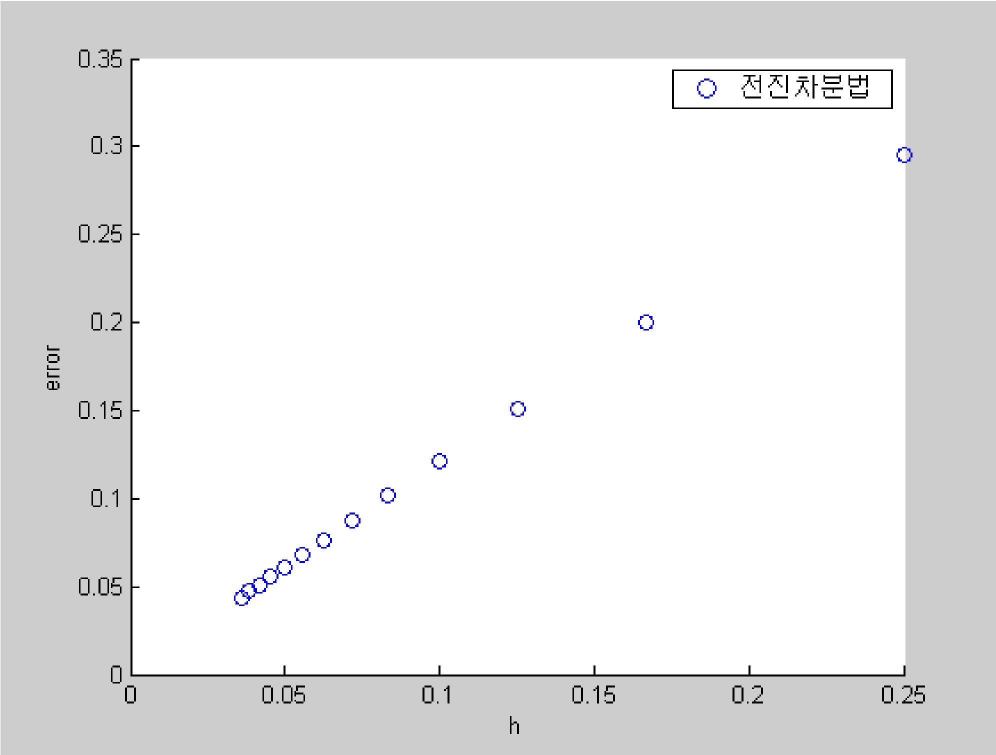
C\[1;1;1;1;0]=$10^-15 \times \begin{pmatrix} -0.0802 \\ -0.2355 \\ -0.5023 \\ 0 \end{pmatrix} \approx 0$, which also fails as a solution. However, C\[1;-1;1;-1;0]=$\begin{pmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{pmatrix}$, which is also a possible solution.

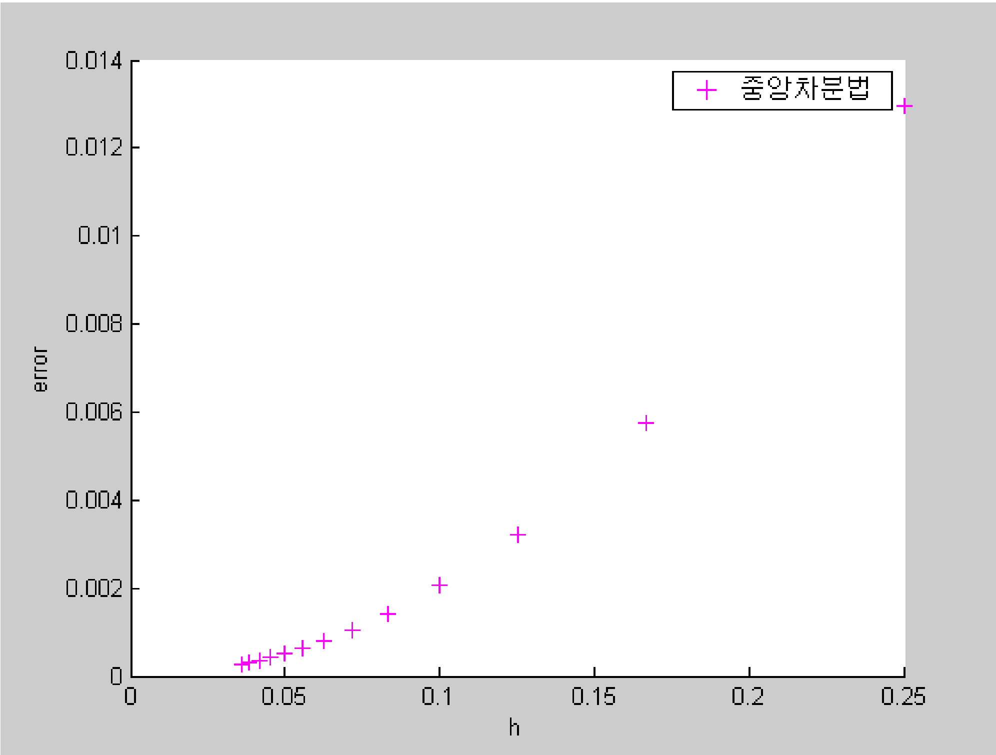
1.1.22

```
% problem set 1.1.20
n=1000;
e=ones(n,1);
K=spdiags([-e,2*e,-e],-1:1,n,n);
% A = SPDIAGS(B,d,m,n) creates an m-by-n sparse matrix from the
%    columns of B and places them along the diagonals specified by d
u=K\e;
plot(u);
```



MATLAB Experiment (pp.21)





1.2.14

14

Part (a).

Solve $-u'' = 12x^2$ with free-fixed conditions $u'(0) = 0$ and $u(1) = 0$. The complete solution involves integrating $f(x) = 12x^2$ twice, plus $Cx + D$.

Solving this without finite differences, but directly gives us

$$\begin{aligned} -u'' &= 12x^2 \\ u' &= -4x^3 + C, \text{ and } u'(0) = 0 = C \\ u &= -x^4 + D, \text{ and } u(1) = 0 = -1 + D \Rightarrow D = 1 \\ u &= -x^4 + 1 \end{aligned}$$

Part (b).

With $h = \frac{1}{n+1}$ and $n = 3, 7, 15$, compute the discrete $u_1, ..., u_n$ using T_n:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 12(ih)^2 \text{ with } u_0 = 0 \text{ and } u_{n+1} = 0$$

Compare u_i with the exact answer at the center point $x = ih = 1/2$. Is the error proportional to h or h^2?

The MATLAB code for constructing matrix T and vector v, and solving for u is

```
n=3; h=1/(n+1); Tn = toeplitz([2 -1 zeros(1,n-2)]); Tn(1,1) = 1;
v=[1:n]; for i=1:n; v(i) = 12*(v(i)*h)^2; end
u=(Tn/(h^2))\v'
```

For $n = 3$, the output from MATLAB is:

```
u =

    0.9375
    0.8906
    0.6563
```

For $x = ih = 1/2$, or $i = \frac{1/2}{1/4} = 2$, $u_2 = 0.8906$. The actual value at this point is $1-(1/2)^4 = 0.9375$, so the difference is $0.9375 - 0.8906 = 0.0469$, while $h = 0.25$.
For $n = 7$, the output from MATLAB is:

```
u =

    0.9844
    0.9814
    0.9668
    0.9258
    0.8379
    0.6768
    0.4102
```

For $x = ih = 1/2$, or $i = \frac{1/2}{1/8} = 4$, $u_4 = 0.9258$. The difference between this and the actual value is 0.0117, while $h = 0.125$.
And for $n = 15$, the output from MATLAB is:

```
u =

    0.9961
    0.9959
    0.9950
    0.9924
    0.9869
    0.9769
    0.9602
    0.9346
    0.8972
    0.8450
    0.7745
    0.6819
    0.5629
    0.4129
    0.2271
```

For $x = ih = 1/2$, or $i = \frac{1/2}{1/16} = 8$, $u_8 = 0.9346$. The difference between this and the actual value is 0.0029, while $h = 0.0625$.

It seems that the error is proportional to h^2. Dividing the error by h^2 for each of these different values of n gives us a factor of approximately 0.75. This makes sense, because this is what we calculated to be the error via Taylor series.

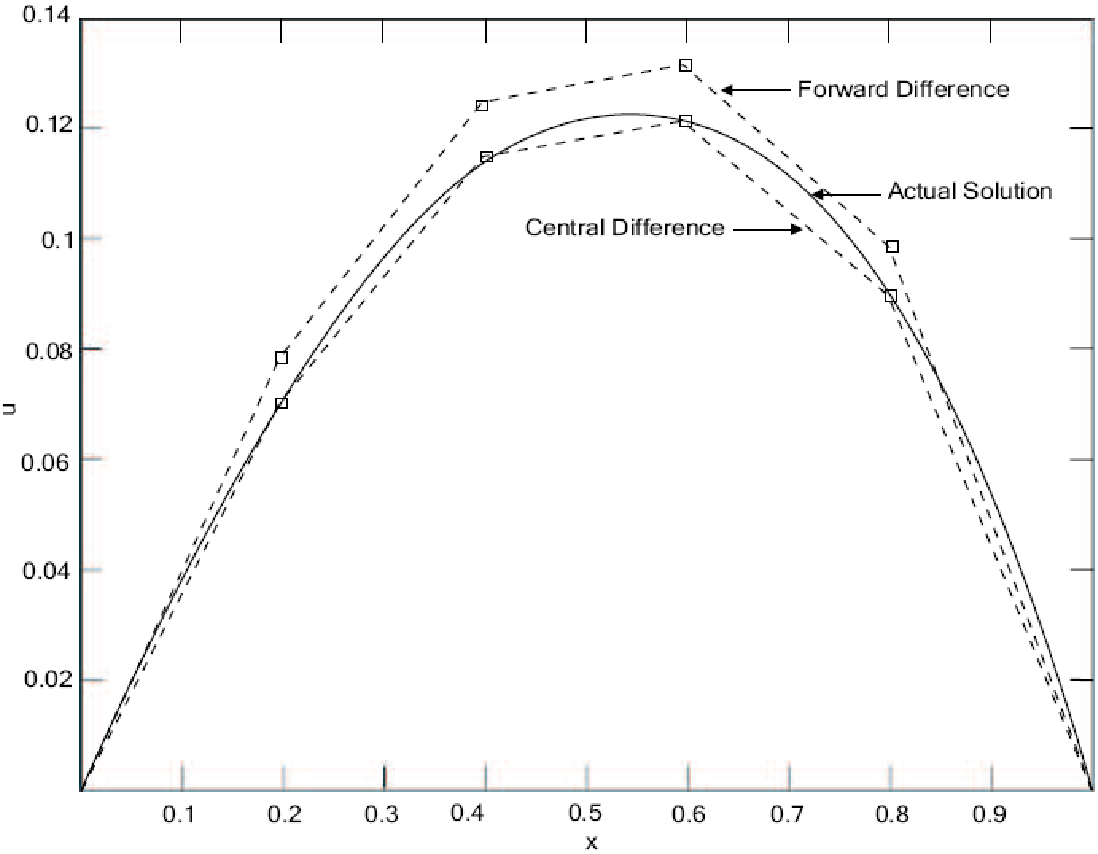
1.2.19

$$u(x) = x - \frac{1}{1-e}\left(1-e^{x}\right)$$

$$h = \frac{1}{5}$$

centered difference: $$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

forward difference: $$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



The centered finite second difference matrix with fixed-fixed boundary conditions should be a sum of the K matrix and the centered difference matrix Δ_0, so with $n = 4$ (and therefore $h = 1/5$) we have

$$K_0 = 25\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{bmatrix} = \begin{bmatrix} 50 & -20 & 0 & 0 \\ -30 & 50 & 0 & 0 \\ 0 & -30 & 50 & -20 \\ 0 & 0 & -30 & 50 \end{bmatrix}$$

Likewise the forward finite second difference matrix should be $K + \Delta_+$ so

$$K_+ = \begin{bmatrix} 45 & -20 & 0 & 0 \\ -25 & 45 & -20 & 0 \\ 0 & -25 & 45 & -20 \\ 0 & 0 & -25 & 45 \end{bmatrix}$$

The true $u(x)$ is $x + Ae^x + B$ for some A and B. We know $u(0) = u(1) = 0$ so plugging these in we get

$$A + B = 0$$

$$1 + Ae + B = 0$$

Using these two equations we get

$$A = \frac{1}{1-e} \qquad B = \frac{-1}{1-e}$$

The following MATLAB output gives the rest of the solution:

```
K = 25*toeplitz([2 -1 0 0]);
D1 = 5*(diag(ones(3,1),1) - diag(ones(3,1),1)'); % Centered difference
D2 = 5*(diag(ones(3,1),1) - diag(ones(4,1))); % Forward difference
K1 = K + D1; % Centered second difference
K2 = K + D2; % Forward second difference
u = K1\ ones(4,1)
u =
    0.0621      0.1052      0.1199      0.0919
U = K2\ ones(4,1)
U =
    0.0782      0.1258      0.1355      0.0975
e=exp(1);
x=.2:.2:.8;
f = x + (1-e)^(-1)*exp(x) + (e-1)^(-1);
f'
ans =
    0.0711      0.1138      0.1215      0.0868
```

1.2.21

$$u_0 - u_1 = -hu'(0) - \tfrac{1}{2}h^2u''(0) + \cdots = \tfrac{1}{2}h^2 f(0)$$

$u''(0) = f(0)$ and $u'(0) = 0$ so

$$\begin{aligned} u_0 - u_1 &= u(0) - u(h) \\ &= u(0) - \left(u(0) - \frac{1}{2}h^2 f(0) + O(h^3)\right) \\ &= \frac{1}{2}h^2 f(0) + O(h^3) \end{aligned}$$

1.3.2

(1)

$$K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{21}=\frac{-1}{2}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{32}=\frac{-1}{3/2}=-\frac{2}{3}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$$K_3 = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 3/4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1/2 & 1 & \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 3/4 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & 0 \\ & 1 & -2/3 \\ & & 1 \end{bmatrix} = LDL^T$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 2/3 & 1 \end{bmatrix}, D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}, \left(L^T\right)^{-1} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) i-th pivot of K: $\dfrac{i+1}{i}$

(3)

$$L_4 = \begin{bmatrix} 1 & & & \\ -1/2 & 1 & & \\ 0 & -2/3 & 1 & \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \xrightarrow{L_4L_4^{-1}=I,\,L_4^{-1}L_4=I} L_4^{-1} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/3 & 2/3 & 1 & \\ 1/4 & 2/4 & 3/4 & 1 \end{bmatrix}$$

(i, j) of $L_4^{-1} \to \dfrac{j}{i}$ (on and below the diagonal)

1.3.9

9) We are interested in Cholesky factorizing the matrices K_3, T_3, and B_3 using the MATLAB command chol. $A = \texttt{chol}(K)$ produces an upper triangular matrix A such that $K = A^T A$.

$$\texttt{chol}(K_3) = \texttt{chol}\left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} & \sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} & \sqrt{2/3} \\ 0 & 0 & \sqrt{4/3} \end{bmatrix}$$

$$\texttt{chol}(T_3) = \texttt{chol}\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\texttt{chol}(B_3)$ fails because B_3 is not a positive definite matrix, and the Cholesky factorization only works on positive definite matrices. To get around this problem we add the identity matrix multiplied by a small factor ϵ ($0 < \epsilon \ll 1$), using the MATLAB command eps*eye(3), to the matrix B_3 and try again.

$$\texttt{chol}(B_3 + \epsilon I_3) = \texttt{chol}\left(\begin{bmatrix} 1+\epsilon & -1 & 0 \\ -1 & 2+\epsilon & -1 \\ 0 & -1 & 1+\epsilon \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

1.3.11

```
K=ones(4)+eye(4)/100
[L U]=lu(K)
L'
% K=LU=LDL'--> U=DL'
D=U/L'
eig(K)
inv(K)
```

The matrix K is positive definite since all pivots and eigenvalues are positive.

1.3.13

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Therefore, the answer is 1 and -1. Then by multiplication, we get vectors $(1, 1)$, $(2, 1)$, $(3, 2)$, $(5, 3)$, $(8, 5)$, which give us the sequence

$$1, 1, 2, 3, 5, 8, ...$$

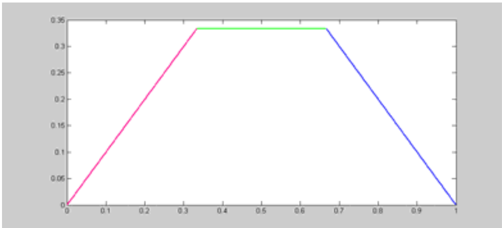
1.4.3

$$u(x) = -R(x-a) + (1-a)x = \begin{cases} (1-a)x & \text{for } x \le a \\ (1-x)a & \text{for } x \ge a \end{cases}$$

$$\text{when } a = \frac{1}{3}: \; u(x) = -R(x-\frac{1}{3}) + (1-\frac{1}{3})x = \begin{cases} (1-\frac{1}{3})x & \text{for } x \le \frac{1}{3} \\ (1-x)\frac{1}{3} & \text{for } x \ge \frac{1}{3} \end{cases}$$

$$\text{when } a = \frac{2}{3}: \; u(x) = -R(x-\frac{2}{3}) + (1-\frac{2}{3})x = \begin{cases} (1-\frac{2}{3})x & \text{for } x \le \frac{2}{3} \\ (1-x)\frac{2}{3} & \text{for } x \ge \frac{2}{3} \end{cases}$$

$$u(x) = \begin{cases} (1-\frac{1}{3})x + (1-\frac{2}{3})x & \text{for } x \le \frac{1}{3} \\ (1-x)\frac{1}{3} + (1-\frac{2}{3})x & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ (1-x)\frac{1}{3} + (1-x)\frac{2}{3} & \text{for } x \le \frac{2}{3} \end{cases}$$



1.4.8

8.

The difference $f(x) = \delta(x - \frac{1}{3}) - \delta\left(x - \frac{2}{3}\right)$ has zero total load, and $-u'' = f(x)$ can also be solved with periodic boundary conditions. Find a particular solution $u_{part}(x)$ and then complete solution $u_{part} + u_{null}$.

The complete solution is given by $R(x) + Cx + D$, where $R(x)$, the ramp, is the particular solution, and $Cx + D$ satisfies the homogeneous equation $u'' = 0$. In the previous problem, the piece-wise function incorporates both of these solutions — the ramp gives continuity and slope changes, while the homogeneous solution explains what happens within these regions. Once again, before $x = \frac{1}{3}$, our solution is of the form $u = Ax + B$. Between $x = \frac{1}{3}$ and $x = \frac{2}{3}$, our solution is of the same form, which we will call $u = Cx + D$. And, finally, after $x = \frac{2}{3}$, call our solution $Ex + F$. The periodic boundary condition we will write as $u'(0) = u'(1)$, and $u(0) = u(1)$.

$$u'(0) = u'(1) \Rightarrow A = E$$

$$u(0) = u(1) \Rightarrow B = A + F \text{ or } F = B - A$$

$$\text{No jump at } u = \frac{1}{3}: \; A\left(\frac{1}{3}\right) + B = C\left(\frac{1}{3}\right) + D$$

$$\text{No jump at } u = \frac{2}{3}:\ C\left(\frac{2}{3}\right) + D = E\left(\frac{2}{3}\right) + F = A\left(\frac{2}{3}\right) + B - A$$

$$\text{Slope drops by 1 at } u = \frac{1}{3}:\ A - C = 1 \text{ or } C = A - 1$$

$$\text{Slope increases by 1 at } u = \frac{2}{3}:\ A - C = 1$$

We can rewrite our jump condition at $u = \frac{1}{3}$ as

$$A\left(\frac{1}{3}\right) + B = (A-1)\left(\frac{1}{3}\right) + D$$

$$\Rightarrow A\left(\frac{1}{3}\right) + B = A\left(\frac{1}{3}\right) - \frac{1}{3} + D$$

$$\Rightarrow B = D - \frac{1}{3}$$

And at $u = \frac{2}{3}$:

$$(A-1)\left(\frac{2}{3}\right) + D = B - A\left(\frac{1}{3}\right)$$

$$\Rightarrow A\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right) + \cancel{D} = \cancel{D} - \frac{1}{3} - A\left(\frac{1}{3}\right)$$

$$\Rightarrow A = \frac{1}{3}$$

So our unknowns are given by

$$A = \frac{1}{3}$$

$$B = D - \frac{1}{3}$$

$$C = -\frac{2}{3}$$

$$D$$

$$E = \frac{1}{3}$$

$$F = B - A = D - \frac{2}{3}$$

Our equations are

$$u = \frac{1}{3}x + D - \frac{1}{3},\ \text{for } 0 < x < \frac{1}{3}$$

$$u = -\frac{2}{3}x + D,\ \text{for } \frac{1}{3} < x < \frac{2}{3}$$

$$u = \frac{1}{3}x + D - \frac{2}{3},\ \text{for } x > \frac{2}{3}$$

The null solution is

1.4.12

$u'''' = \delta(x)$

$C(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{x^3}{6}, & x \ge 0 \end{cases}$

$u(x) = C(x) + Ax^3 + Bx^2 + Dx + E \rightarrow u(x) = C(x) - \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6}$

$$\left.\begin{aligned} u(1) = 0 \rightarrow \frac{1}{6} + A + B + D + E = 0 \rightarrow -\frac{1}{12} - \frac{1}{4} + D + E = -\frac{1}{6} \\ u(-1) = 0 \rightarrow 0 - A + B - D + E = 0 \rightarrow \frac{1}{12} - \frac{1}{4} - D + E = 0 \end{aligned}\right\} \rightarrow \begin{cases} D = 0 \\ E = \frac{1}{6} \end{cases}$$

$u''(1) = 0 \rightarrow 1 + 6A + 2B = 0 \rightarrow A = -\frac{1}{12}$

$u''(-1) = 0 \rightarrow 0 - 6A + 2B = 0 \rightarrow B = 3A = -\frac{1}{4}$

1.5.3

3) We want to find the eigenvalues of K_5, and verify that they equal $(2-\sqrt{3}, 2-1, 2-0, 2+1, 2+\sqrt{3})$. This is done using MATLAB. First the decimal values for the eigenvalues are found using the MATLAB command `e = eig(K)`. We can compare these numbers with the numbers for the eigenvalues generated with the formula $\lambda_k = 2 - \cos(\frac{k\pi}{n+1})$, where $k = 1, 2, ..., n$ and in this case $n = 5$. These values are generated with the MATLAB command `e_expected = 2*ones(5,1) - 2*cos([1:5]*pi/6)'`. Taking the difference between `e` and `e_expected` in MATLAB, we get a column of zeros (within a tolerance of 1.0×10^{-15}), indicating that the two are equal.

1.5.4

e = eig (K5) = 2*ones(5,1)−2*cos([1:5]*pi/6)'

[Q, E] = eig (K5)

DST = Q*diag([−1 −1 1 −1 1])

JK = [1:5]'*[1:5]

sin(JK*pi/6)/sqrt(3)

$DST^T = DST^{-1}$

1.5.9

9) We now show that $K_3 = \Delta_-^T \Delta_-$ and $B_4 = \Delta_- \Delta_-^T$, where Δ_- is the 4×3 backward difference matrix given by

$$\Delta_- = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$K_3 = \Delta_-^T \Delta_- = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$B_4 = \Delta_- \Delta_-^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of K_3 are:

0.5858 = 2-2*cos(π/4)
2.0000 = 2-2*cos(2π/4)
3.4142 = 2-2*cos(3π/4)

1.5.18

Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k: $A^k = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}$

The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1=1$, $\lambda_2=3$ and eigenvectors $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvalue matrix, Λ, has eigenvalues λ_1 and λ_2 for the diagonal: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

and the kth power of Λ is Λ^k: $\begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}$

S has eigenvectors x_1 and x_2 for columns: $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and S^{-1} is: $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$S\Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1\cdot 1+1\cdot 0 & 1\cdot 0+1\cdot 3^k \\ -1\cdot 1+1\cdot 0 & -1\cdot 0+1\cdot 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 3^k \\ -1 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1\cdot 1+1\cdot 3^k & 1\cdot -1+1\cdot 3^k \\ -1\cdot 1+1\cdot 3^k & -1\cdot -1+1\cdot 3^k \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix} = A^k$$

1.5.23

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u \qquad u = Ce^{\lambda_1 t} x_1 + DCe^{\lambda_2 t} x_2$$

$$(A - \lambda I)x = 0$$

$$\det(A - \lambda I)x = \det\begin{pmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(1-\lambda) = 0 \qquad \rightarrow \qquad \lambda = 1 \qquad \lambda = 4$$

a) $\lambda = 1$

$$\begin{bmatrix} 4-1 & 3 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \qquad \rightarrow \qquad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) $\lambda = 4$

$$\begin{bmatrix} 4-4 & 3 \\ 0 & 1-4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \qquad \rightarrow \qquad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u(0) = C\begin{bmatrix} 1 \\ -1 \end{bmatrix} + D\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \rightarrow \qquad C = 2 \quad D = 3$$

$$\therefore u(t) = 2e^{t} x_1 + 3e^{4t} x_2$$

1.6.16

1.6.16. The eigenvalues of $A^{-1}$are the inverses of the eigenvalues of A, hence are also positive. Second proof: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite iff $a > 0, ac - b^2 > 0$. In particular, $c > 0$. Now $A^{-1} = \frac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$ has determinant $\frac{1}{ac-b^2} > 0$ and top left entry $c > 0$, hence is positive definite.

1.6.24

$$\frac{1}{2}\left(u-K^{-1}f\right)^T K\left(u-K^{-1}f\right)-\frac{1}{2}f^TK^{-1}f$$
$$=\frac{1}{2}\left[u^T-\left(K^{-1}f\right)^T\right]\left(Ku-f\right)-\frac{1}{2}f^TK^{-1}f$$
$$=\frac{1}{2}\left[u^TKu-u^Tf-\left(K^{-1}f\right)^T Ku+\left(K^{-1}f\right)^T f\right]-\frac{1}{2}f^TK^{-1}f$$
$$=\frac{1}{2}\left[u^TKu-u^Tf-\left[\left(Ku\right)^T\left(K^{-1}f\right)\right]^T+f^T\left(K^{-1}\right)^T f-f^TK^{-1}f\right]$$
$$=\frac{1}{2}\left[u^TKu-u^Tf-\left(u^TK^TK^{-1}f\right)^T\right]$$
$$=\frac{1}{2}u^TKu-u^Tf=P\left(u\right)$$

The long term $\frac{1}{2}\left(u-K^{-1}f\right)^T K\left(u-K^{-1}f\right)$ on the right hand side is always positive except when $u=K^{-1}f$

1.6.27

27) We are given that the matrices H (size $m\times m$) and K (size $n\times n$) are positive definite and matrices M and N are defined in block notation by

$$M=\begin{bmatrix}H & 0\\ 0 & K\end{bmatrix} \qquad N=\begin{bmatrix}K & K\\ K & K\end{bmatrix}$$

If we denote the upper triangular Gaussian eliminated forms of H and K as U_H and U_K respectively, then we can perform Gaussian elimination on matrices M and N and get

$$M\xrightarrow{\text{elimination}}\begin{bmatrix}U_H & 0\\ 0 & U_K\end{bmatrix} \qquad N\xrightarrow{\text{elimination}}\begin{bmatrix}U_K & U_K\\ 0 & 0\end{bmatrix}$$

So the pivots of M are composed of the pivots of H and the pivots of K. Since the pivots of both H and K are positive, the pivots of M are all positive and thus M is positive definite. The pivots of N are composed of the pivots of K and n zeros. Since N has positive and zero pivots, it is not positive definite but rather positive semi-definite.

The eigenvalues of M and N can also be connected to the eigenvalues of H and K. We define v_i^H and λ_i^H and to be the m eigenvectors and corresponding eigenvalues of H with $i=1,2,...,m$. We also define v_i^K and λ_i^K and to be the n eigenvectors and corresponding eigenvalues for K with $i=1,2,...,n$. Then the following observations can be made.

$$\begin{bmatrix}H & 0\\ 0 & K\end{bmatrix}\begin{bmatrix}v_i^H\\ 0\end{bmatrix}=\begin{bmatrix}Hv_i^H\\ 0\end{bmatrix}=\lambda_i^H\begin{bmatrix}v_i^H\\ 0\end{bmatrix} \qquad \begin{bmatrix}H & 0\\ 0 & K\end{bmatrix}\begin{bmatrix}0\\ v_i^K\end{bmatrix}=\begin{bmatrix}0\\ Kv_i^K\end{bmatrix}=\lambda_i^K\begin{bmatrix}0\\ v_i^K\end{bmatrix}$$

So the eigenvalues of M are composed of the eigenvalues of H and the eigenvalues of K. Also if we define e_i to be the column vector consisting of $(i-1)$ zeros followed by a one and then followed by $(n-i)$ zeros, then we can use it to find the eigenvalues of N.

$$\begin{bmatrix}K & K\\ K & K\end{bmatrix}\begin{bmatrix}e_i\\ -e_i\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}=0\begin{bmatrix}e_i\\ -e_i\end{bmatrix} \qquad \begin{bmatrix}K & K\\ K & K\end{bmatrix}\begin{bmatrix}v_i^K\\ v_i^K\end{bmatrix}=\begin{bmatrix}2Kv_i^K\\ 2Kv_i^K\end{bmatrix}=2\lambda_i^K\begin{bmatrix}v_i^K\\ v_i^K\end{bmatrix}$$

Since e_i is orthogonal to e_j for $i \neq j$, it is clear that $\begin{bmatrix} e_i \\ -e_i \end{bmatrix}$ for $i = 1, 2, ..., n$ are n linearly independent vectors. This means that zero is an eigenvalue for N with a multiplicity of n. The remaining eigenvalues come from the eigenvalues of K, but as seen above they are doubled. So the eigenvalues of N are 2 times the eigenvalues of K, as well as the eigenvalue zero with multiplicity n.

Finally, we want to construct the Cholesky of M, $\texttt{chol}(M)$, from $\texttt{chol}(H)$ and $\texttt{chol}(K)$. We let $A = \texttt{chol}(H)$, so that $H = A^T A$, and let $B = \texttt{chol}(K)$, so that $K = B^T B$. We then define a matrix C that is given in block notation by

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} \texttt{chol}(H) & 0 \\ 0 & \texttt{chol}(K) \end{bmatrix}$$

If we multiply the transpose of C with C, we find that it equals

$$C^T C = \begin{bmatrix} A^T & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^T A & 0 \\ 0 & B^T B \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} = M$$

So, $M = C^T C$, which is the Cholesky factorization. Thus, $\texttt{chol}(M) = C$, where C was defined above in terms of the $\texttt{chol}(H)$ and $\texttt{chol}(K)$.

1.7.6

$$U \Sigma V^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \sigma_1 \cos^2\alpha - \sigma_2 \sin^2\alpha & (\sigma_1 - \sigma_2)\sin\alpha\cos\alpha \\ (\sigma_1 - \sigma_2)\sin\alpha\cos\alpha & \sigma_1 \sin^2\alpha + \sigma_2 \cos^2\alpha \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det(A^T A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_{\max}\ (A^T A) = \frac{1}{2}(3 + \sqrt{5}) \rightarrow \|A\| = \sqrt{\frac{1}{2}(3 + \sqrt{5})} = \frac{1}{2}(1 + \sqrt{5})$$

1.7.18

$$A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}, A^TA=\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}, AA^T=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

$$\det\left(A^TA-\lambda I\right)=\lambda(1-\lambda)(\lambda-3)=0\rightarrow\Lambda=\begin{bmatrix}3&&\\&1&\\&&0\end{bmatrix}\rightarrow\Sigma=\begin{bmatrix}\sqrt{3}&&\\&1&\\&&0\end{bmatrix}$$

$$\lambda_1=3:\begin{bmatrix}-2&1&0\\1&-1&1\\0&1&-2\end{bmatrix}v_1=0\rightarrow v_1=\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}\rightarrow u_1=\frac{Av_1}{\sigma_1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lambda_2=1:\begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}v_2=0\rightarrow v_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\rightarrow u_2=\frac{Av_2}{\sigma_2}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\lambda_3=0:\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}v_3=0\rightarrow v_3=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\rightarrow u_3=\frac{Av_3}{\sigma_3}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$U\Sigma V^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0\\1&-1&0\end{bmatrix}\begin{bmatrix}\sqrt{3}&&\\&1&\\&&0\end{bmatrix}\begin{bmatrix}1/\sqrt{6}&1/\sqrt{2}&1/\sqrt{3}\\2/\sqrt{6}&0&-1/\sqrt{3}\\1/\sqrt{6}&-1/\sqrt{2}&1/\sqrt{3}\end{bmatrix}^T=A$$

1.7.28

matrix	$\lambda_{max}/\lambda_{min}$	Eq(19)
K_9	39.8635	40.5285
T_9	142.6689	

1.7.30

$$DIFF=\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}$$

```
>> [U, sigma, V] = svd(DIFF)
>> null(DIFF')
```