2.1.5

suppose all springs are identical with $c_i = c$

$$K = A^T C A = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \to K^{-1} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$w = Ce = CAu = CAK^{-1}f = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \to \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1.5mg \\ 0.5mg \\ -0.5mg \\ -1.5mg \end{bmatrix}$$

2.1.7

7) In the fixed-fixed case with three equal masses and originally four equal springs with spring constant equal to 1, we now weaken spring 2 so that $c_2 \to 0$. Now, the K matrix becomes

$$K = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

This matrix is still invertible because its determinant is 1. Solving $Ku = f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, we get $u = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$. To explain this answer physically, it is important to realize that by weaken spring 2, it splits

the problem into two decoupled problem. One problem is mass 1 hanging freely off spring 1. In this problem, we expect for the displacement to be $u_1 = 1$ because there is a force of 1 on the mass connected to a spring with spring constant equal to 1. The second problem is a free-fixed problem with two identical masses and two identical spring, but the problem is upside compared to the typical fixed-free spring-mass problem. So, when $u_2 = 3$ and $u_3 = 2$, it means that spring 4 is compressed by 2 by the two masses above it and spring 3 is compressed by 1 by the one mass above it, which makes sense physically.

2.2.5

2.2.5. (a) $\frac{d}{dt}\|u(t)\|^2 = 2(u_1u_1' + u_2u_2' + u_3u_3') = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = 0$. Having 0 derivative, $\|u(t)\|^2$ is constant, so $\|u(t)\|^2 = \|u(0)\|^2$.

(b) For any n we have $(A^n)^T = (A^T)^n$, hence $Q^T = \left(I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \ldots\right)^T = I + tA^T + \frac{t^2}{2!}(A^T)^2 + \frac{t^3}{3!}(A^T)^3 + \ldots = e^{tA^T} = e^{-At}$. Hence $QQ^T = e^{At}e^{-At} = I$, where the last equality follows from the formal power series identity $e^x e^{-x} = 1$.

2.2.6

6) The trapezoidal rule for $u' = Au$ is given by

$$(I - \frac{\Delta t}{2}A)U_{n+1} = (I + \frac{\Delta t}{2}A)U_n$$

If $A^T = -A$, then the trapezoidal rule will conserve the energy $||u||^2$. This can be proven by showing that $||U_{n+1}||^2 = ||U_n||^2$.

$$\begin{aligned}(I - \frac{\Delta t}{2}A)U_{n+1} &= (I + \frac{\Delta t}{2}A)U_n \\ U_{n+1} - U_n &= \frac{\Delta t}{2}A(U_{n+1} + U_n) \\ (U_{n+1}^T + U_n^T)(U_{n+1} - U_n) &= (U_{n+1}^T + U_n^T)\frac{\Delta t}{2}A(U_{n+1} + U_n) \\ U_{n+1}^T U_{n+1} + \cancel{U_{n+1}^T U_n} - \cancel{U_n^T U_{n+1}} - U_n^T U_n &= \frac{\Delta t}{2}\left[(U_{n+1} + U_n)^T A(U_{n+1} + U_n)\right] \\ ||U_{n+1}||^2 - ||U_n||^2 &= \frac{\Delta t}{2} v^T A v\end{aligned}$$

where we let $v = (U_{n+1} + U_n)$ and $U_{n+1}^T U_n$ was canceled by $U_n^T U_{n+1}$ since they are both equal scalars and the order of the inner product does not matter. Likewise, $v^T A v$ is a scalar and thus it is equal to its transpose $(v^T A v)^T = v^T A^T v$. However, since $A^T = -A$, this means that $v^T A v = -v^T A v$ and the only way that equality can be satisfied is if $v^T A v = 0$. Thus,

$$\begin{aligned}||U_{n+1}||^2 - ||U_n||^2 &= \frac{\Delta t}{2} v^T A v = 0 \\ ||U_{n+1}||^2 &= ||U_n||^2\end{aligned}$$

2.2.7

$$\lambda = \frac{1 + i\frac{h}{2}}{1 - i\frac{h}{2}} = \frac{1 - \frac{h^2}{4} + ih}{1 + \frac{h^2}{4}} \leftrightarrow \lambda = e^{ih} = \cos\ h + i\sin\ h \rightarrow \cos\ h = \frac{1 - \frac{h^2}{4}}{1 + \frac{h^2}{4}} \rightarrow h = \cos^{-1}\left(\frac{1 - \frac{h^2}{4}}{1 + \frac{h^2}{4}}\right)$$

$$\lambda^{32} = e^{i32h} = \cos 32h + i\sin 32h = \cos\theta + i\sin\theta \rightarrow \theta = 32h = 32\cos^{-1}\left(\frac{1 - \frac{h^2}{4}}{1 + \frac{h^2}{4}}\right)$$

$$h = \frac{2\pi}{32} \rightarrow \theta \approx 0.1957, \lambda^{32} = 0.9809 + 0.1945i$$

$$\left.\begin{aligned}\lambda &= \frac{1 + i\frac{h}{2}}{1 - i\frac{h}{2}} \xrightarrow{\frac{1+x}{1-x} = (1+x)(1+x+x^2+\cdots) = 1+2x+2x^2+2x^3+\cdots} \lambda = 1 + ih + i^2\frac{h^2}{2} + i^3\frac{h^3}{2^2} + \cdots \\ \lambda &= e^{ih} \xrightarrow{e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots} \lambda = 1 + ih + i^2\frac{h^2}{2!} + i^3\frac{h^3}{3!} + \cdots\end{aligned}\right\} \rightarrow \left\{\begin{aligned}&\text{the lowest power of h that} \\ &\text{the discrepancy occurs is three}\left(h^3\right)\end{aligned}\right.$$

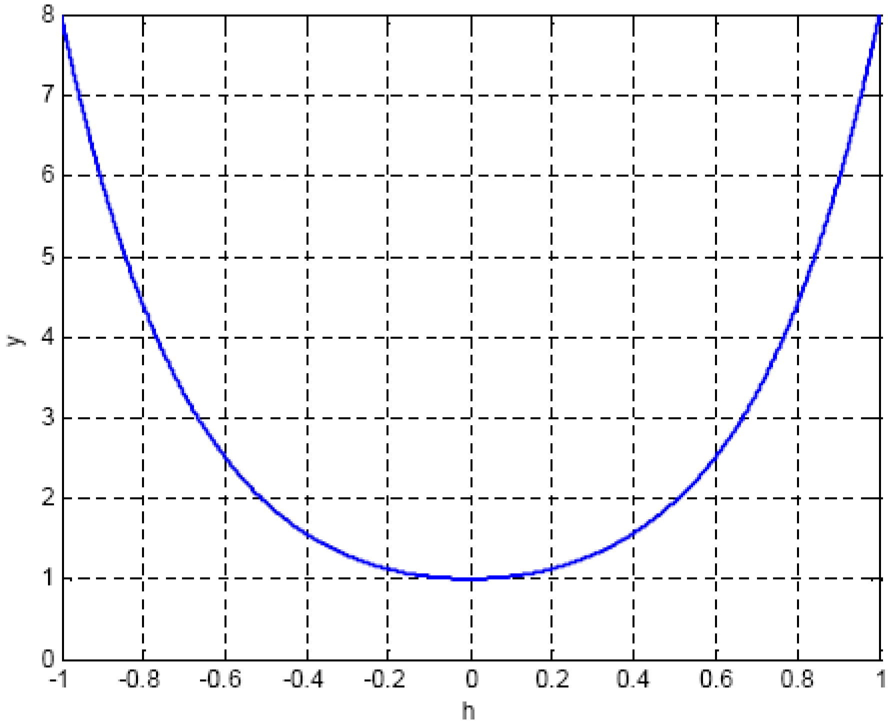
2.2.8

Forward Euler:

$$\left.\begin{aligned} U_{n+1} &= U_n + hV_n \\ V_{n+1} &= V_n - hU_n \end{aligned}\right\} \rightarrow U_{n+1}{}^2 + V_{n+1}{}^2 = \left(U_n + hV_n\right)^2 + \left(V_n - hU_n\right)^2 = \left(1+h^2\right)\left(U_n{}^2 + V_n{}^2\right)$$

$$h = \frac{2\pi}{32} \rightarrow \left(1+h^2\right)^{32} = 3.3552$$

$$\left(1+h^2\right)^{\frac{2\pi}{h}} = 1 \text{ as } h \rightarrow 0$$



Backward Euler:

$$\left.\begin{aligned} U_{n+1} &= U_n + hV_{n+1} \rightarrow U_{n+1} = \frac{1}{1+h^2}\left(U_n + hV_n\right) \\ V_{n+1} &= V_n - hU_{n+1} \rightarrow V_{n+1} = \frac{1}{1+h^2}\left(V_n - hU_n\right) \end{aligned}\right\}$$

$$\rightarrow U_{n+1}{}^2 + V_{n+1}{}^2 = \frac{\left(U_n + hV_n\right)^2 + \left(V_n - hU_n\right)^2}{\left(1+h^2\right)^2} = \frac{U_n{}^2 + V_n{}^2}{1+h^2}$$

2.3.3 QR experiment

V=fliplr(vander((0:49)/49))

A=V(:,1:12)

b=cos(0:.08:3.92)'

(1) A^TA₩(A^Tb)

(2) R₩(Q^Tb)

(3) modified Gram-Schmidt

(4) Householder code

(5) A₩b

(6) qr

Method 5 should yield the most accurate solutions. Why?

2.3.4 Gram-Schmidt

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$b_2 = a_2 - \left(q_1^T a_2\right) q_1 = \frac{1}{9}\begin{bmatrix} -17 \\ 10 \\ 14 \end{bmatrix} \rightarrow q_2 = \frac{b_2}{\|b_2\|} = \frac{1}{3\sqrt{65}}\begin{bmatrix} -17 \\ 10 \\ 14 \end{bmatrix}$$

$$A = QR \rightarrow R = Q^{-1}A = Q^T A = \begin{bmatrix} 3 & 4/3 \\ 0 & \sqrt{65}/3 \end{bmatrix}$$

2.3.5 Householder

$$a_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, r_1 = \begin{bmatrix} \|a_1\| = 3 \\ 0 \\ 0 \end{bmatrix}, w_1 = a_1 - r_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} -0.41 \\ 0.82 \\ 0.41 \end{bmatrix} \rightarrow H_1 = I - 2u_1u_1^T \rightarrow H_1A = \begin{bmatrix} 3 & 1.35 \\ 0 & -2.71 \\ 0 & -0.35 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 1.35 \\ -2.71 \\ -0.35 \end{bmatrix}, \|a_2\| = \|r_2\|, r_2 = \begin{bmatrix} 1.35 \\ -2.73 \\ 0 \end{bmatrix}, w_2 = a_2 - r_2 = \begin{bmatrix} 0 \\ 0.02 \\ -0.35 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0.06 \\ -1 \end{bmatrix} \rightarrow H_2 = I - 2u_2u_2^T \rightarrow H_2H_1A = Q^{-1}A = R$$

2.3.24

2.3/24 We would like to find the plane $b(x,y) = C + Dx + Ey$ that gives the best fit to the data $b = 0, 1, 3, 4$ and $(x,y) = (1,0), (0,1), (-1,0), (0,-1)$ respectively. The normal equation $A^T A\hat{u} = A^T b$ for $\hat{u} = (C, D, E)^T$ takes the form

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$

so that $(C, D, E) = (2, -3/2, -3/2)$ and we check that

$$b(0,0) = C = 2 = \mathrm{Av}(0,1,3,4).$$

2.4.7

2.4/7 The 3×4 incidence matrix that corresponds to the network in question is

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

and if $C = \mathrm{diag}(c_1, c_2, c_3)$ we compute

$$K = A^T C A = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1 + c_2 + c_3 \end{pmatrix}.$$

If we ground node 4 we form the reduced K_{red} by deleting the fourth row and the fourth column from K, so that $K_{\mathrm{red}} = \mathrm{diag}(c_1, c_2, c_3)$ and $\det K_{\mathrm{red}} = c_1 c_2 c_3$. Note that the unreduced K is singular, so $\det K = 0$!

2.4.9

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \to A^T A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$K = \left(A^T A\right)_{reduced} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = T_4 \to K^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$eig(K) \to \begin{cases} 1 \\ 0.1206 \\ 2.3473 \\ 3.5321 \end{cases}$$

$$\det K = 1$$

2.4.17

2.4/17 (a) The only non-zero entries of the matrix $A^TA = D - W$ are its diagonal entries and the off-diagonal ones that correspond to the presence of edges. Thus,

$$\#\text{ non-zero entries in } A^TA = 9 + 2 \times \#\text{ of edges} = 9 + 2 \times 12 = 33.$$

The number of the zero entries of A^TA is then $9^2 - 33 = 48$.

(b) $D = \text{diag}(2, 3, 2, 3, 4, 3, 2, 3, 2)$.

(c) The four -1's in the middle row of $-W$ correspond to the four other nodes that the center node connects to.

2.4.18

2.4/18 We want to solve $Ku = f$ where K is the 8×8 reduced matrix for the 3×3 grid from the previous problem (as we ground node $(3, 3)$, we obtain K by deleting the last row and column). The vector of current sources f is the 8×1 vector whose first entry is 1 and all others are zero. Using MATLAB we derive $u(1, 1) = u_1 = (K^{-1}f)_1 = 1.5$.

```
N=3;
B=toeplitz([2 -1 zeros(1,N-2)]);
B(1,1)=1;
B(N,N)=1;
L=kron(B,eye(N))+kron(eye(N),B);
K=L(1:8,1:8)
f=[1 0 0 0 0 0 0 0]
inv(K)*transpose(f)
```

we get $(1.5000, 1.0000, 0.7500, 1.0000, 0.7500, 0.5000, 0.7500, 0.5000)$. Therefore, $u(1, 1) = 1.5$.

2.7.3

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are 4 independent solution for $Au = 0$.

when $f \in$ ~~at~~ colum space of A^T, $A^T w = f$ has solution.

2.7.4

Truss D has 8 rows and 8 columns in the matrix A, because there are m=8 bars, and n=8 unknowns in the structure.

The underlying physics of column 1 multiply displacement u is the distribution of small displacement u_1^H to the elongation of each bar, as indicated in the following equation:

$Au = e$, in which $a_{i1}u_1^H = e_{1-u_1^H}$, so we just need to find out the contribution of u_1^H to the elongations of the 8 bars, then we can get the column 1.

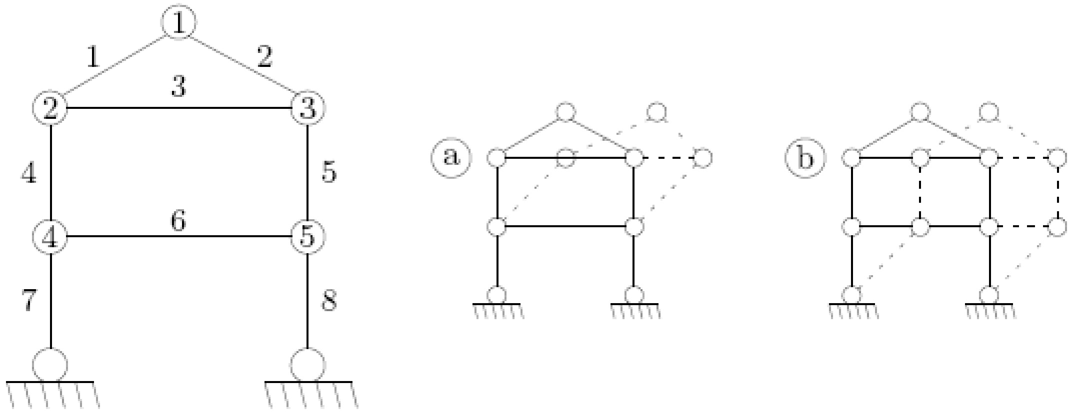
Column1=

$$[\cos 45^\circ \quad \cos 90^\circ \quad \cos 135^\circ \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T = \left[\frac{\sqrt{2}}{2} \quad 0 \quad -\frac{\sqrt{2}}{2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]^T$$

Although A is 8 by 8, it has dependent columns, so there is nonzero solution to $Au = 0$, and $A^T w = 0$. Physically, if we remove any of the 8 bars out of the structure, then the stiffness matrix A becomes 7 by 8, and it has the same nonzero solution with the previous one.

The mechanism solution to $Au = 0$ is the truss rotates as whole around node 5.

2.7.5 n − m = 2(5)-8 = 2 independent solutions (mechanisms) → the truss is unstable (not positive definite) → A^TA must be positive semidefinite since A has dependent columns



2.7.6

m = 4 and n = 6 → 2 mechanisms

There are m=4 bars and n=6 unknowns in truss F, so there should be n-m=2 mechanism motions for it. We can find a solution to $Au = 0$, however, the best method is using picture.
Mechanism 1: Fix node 1, and the bar 2, 3 and 4 can move.
Mechanism 2: Fix node 3, and the bar 1, 2, and 3 can move.
The two mechanisms are independent, so they gives two orthogonal solutions to $Au = 0$.
Stable solution: we have to add at least 2 bars to make it stable, just connect any two non-adjacent nodes in the five nodes (node 1, 2, 3 and the supports nodes).
Then the stiffness matrix A becomes singular and invertible, as it has independent columns, and it has only 1 solution to $A^T w = f$.

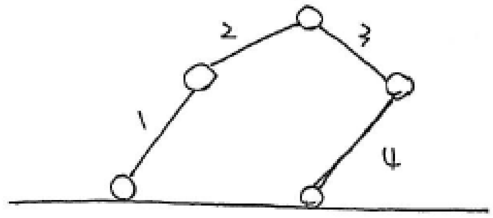
6. Truss F has 5 nodes (with 2 of them fixed)

has 4 bars

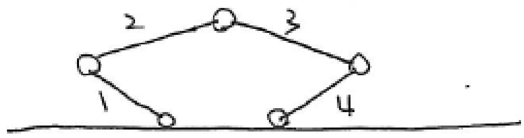
So A is of size 4×6 since A does not allow rigid motion, it has 2 mechanisms.

~~Solution 1.~~

mechanism 1.

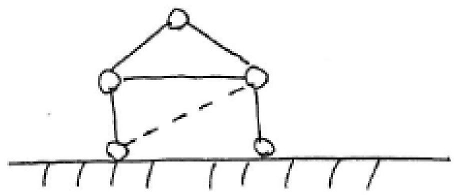


mechanism 2



---

one possible way to make truss stable is as following:



Now $A^T w = f$ has a unique solution.