3.1.10

$c(x)=1,\ f(x)=2$

$$Ku=f \rightarrow \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

3.1.11

By $-u'' = x$, u has the form of

$$u = -1/6x^3 + C_1 x + C_2.$$

By the condition of $u(0)$ and $u(1)$, we have $C_1 = 1/6$ and $C_2 = 0$. So we have

$$u = -1/6x^3 + 1/6x.$$

Then we have

$$F_1 = \int_0^1 x\phi_1(x)dx = \frac{1}{9}$$

and

$$F_2 = \int_0^1 x\phi_2(x)dx = \frac{2}{9}.$$

Also we have

$$K_{11} = \int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} = 6,$$

$$K_{12} = K_{21} = \int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} = -3$$

and

$$K_{22} = \int_0^1 \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} = 6.$$

So the equations $KU = F$ becomes

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix},$$

which has the solution $u_1 = 0.049$ and $u_2 = 0.062$.

By Matlab, the largest difference between the exact and approximate solutions happens at $x = 0.84$ with an error 0.0116.

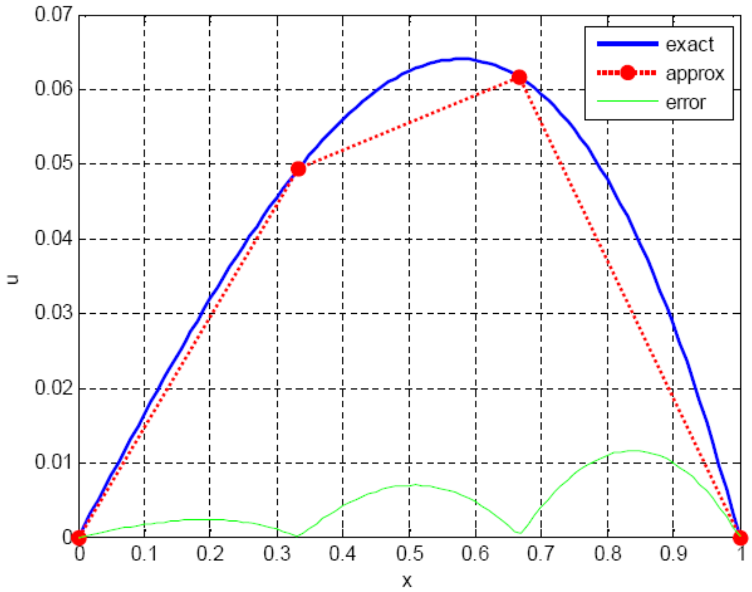
[exact solution]

$$u = -\frac{1}{6}x^3 + \frac{1}{6}x$$

[approximate solution: $V_i = \phi_i,\ c(x) = 1,\ f(x) = x$]

$$Ku = f \rightarrow \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 2/9 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0.0494 \\ 0.0617 \end{bmatrix}$$

largest error: 0.0116 @ x=0.84



3.1.12. We will use the same ϕ_i as in the previous problem. Again, because of symmetry, we only need to compute $M_{11} = \int_0^{1/4}(4x)^2dx + \int_{1/4}^{1/2}(1-4x)^2dx = \frac{1}{6}$, $M_{12} = \int_{1/4}^{1/2}(4x)(1-4x)dx = \frac{1}{24}$. The desired matrix is

$$M = \begin{bmatrix} \frac{1}{6} & \frac{1}{24} & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & \frac{1}{24} & \frac{1}{6} \end{bmatrix}.$$

3.1.14. Integrating $\frac{d}{dx}(w(x)v(x)) = w(x)\frac{dv}{dx} + v(x)\frac{dw}{dx}$ from 0 to 1, and setting $w(x) = c(x)\frac{du}{dx}$, we obtain $\left[c(x)v(x)\frac{du}{dx}\right]_0^1 = \int_0^1 d\left(c(x)\frac{du}{dx}\right)v(x)dx + \int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}dx$, i.e. equation (21).

The area under ϕ_5 is

$$\int_h^{2h} \phi_5(x)dx = \frac{h}{6}\phi_5(h) + \frac{4h}{6}\phi_5(\frac{3h}{2}) + \frac{h}{6}\phi_5(2h).$$

By the condition that $\phi_5(h) = \phi_5(2h) = 0$ and $\phi_5(\frac{3h}{2}) = 1$, we have

$$\int_h^{2h} \phi_5(x)dx = \frac{4h}{6} = \frac{2}{9}.$$

3.2.8. The displacement and slope functions centered at ih are (cf. page 245) $\phi_i^d = \left(\left|\frac{x-ih}{h}\right| - 1\right)^2 \left(2\left|\frac{x-ih}{h}\right| + 1\right), \phi_i^s = \left(\left|\frac{x-ih}{h}\right| - 1\right)^2 \left|\frac{x-ih}{h}\right|$. Doing calculations, $(\phi_i^d)'' = \frac{12}{h^3}|x-ih| - \frac{6}{h^2}, (\phi_i^s)'' = \frac{6}{h^3}|x-ih| - \frac{4}{h^2}$. In our problem, $h = \frac{1}{3}, i = 0, 1$ so $(\phi_0^d)'' = 324|x| - 54, (\phi_1^d)'' = 324|x - \frac{1}{3}| - 54, (\phi_0^s)'' = 162|x| - 36, (\phi_1^s)'' = 162|x - \frac{1}{3}| - 36$.

3.2.9. We integrate the 16 pairwise products of $(\phi_0^d)'' = 324|x| - 54, (\phi_1^d)'' = 324|x - \frac{1}{3}| - 54, (\phi_0^s)'' = 162|x| - 36, (\phi_1^s)'' = 162|x - \frac{1}{3}| - 36$ from 0 to 1 to obtain

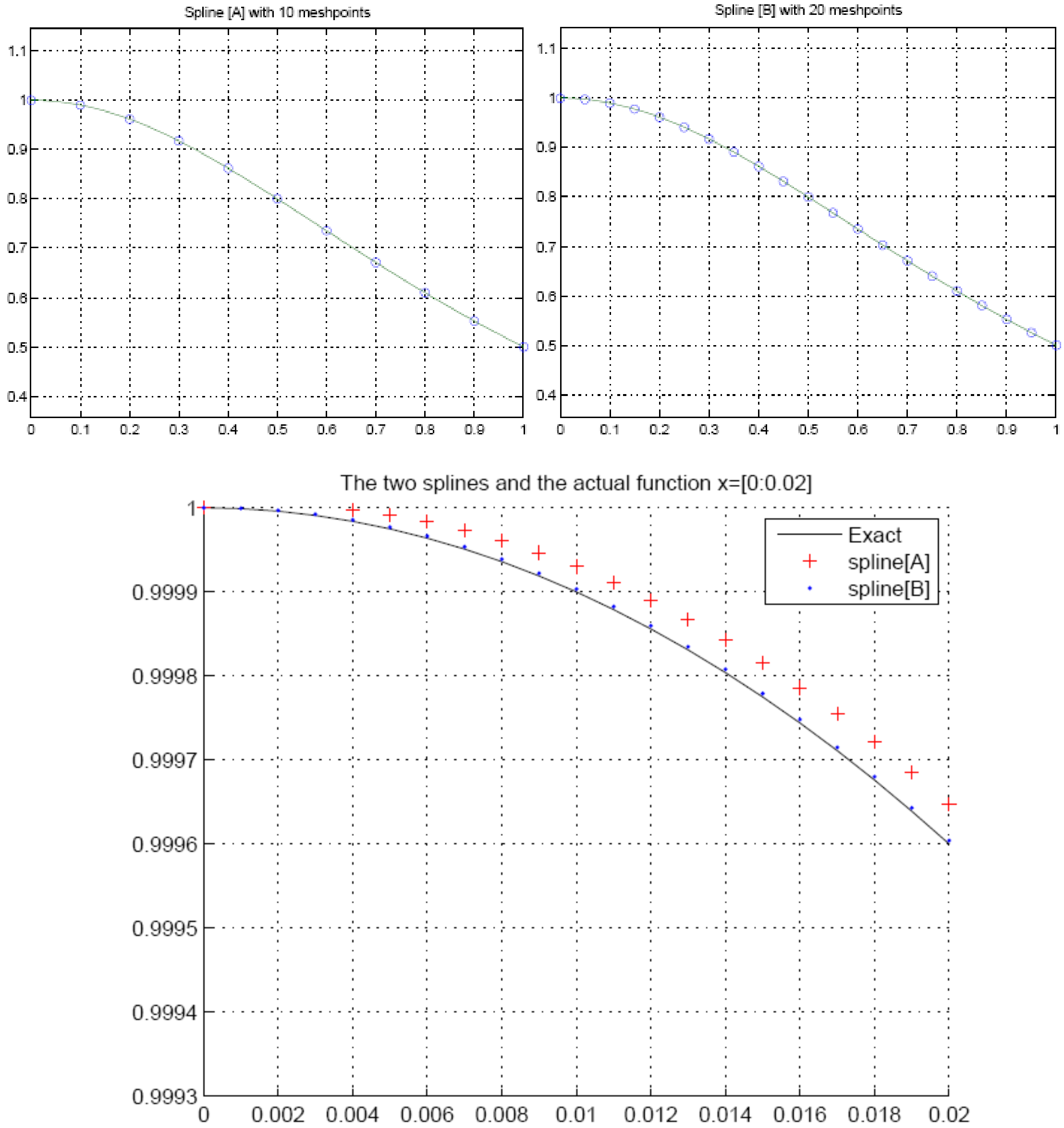
$$K_e = \begin{bmatrix} 324 & 162 & -324 & -162 \\ 162 & 108 & -162 & -54 \\ -324 & -162 & 648 & 162 \\ -162 & -54 & 162 & 216 \end{bmatrix}$$

Note that by symmetry reasons, K_e is symmetric, and its (3,3) entry is twice the (1,1) entry, the (4,4) entry is the same as (2,2), and (1,2) is same as (3,4) and opposite of (2,3).

3.2.10. By symmetry, we can easily assemble the symmetric 8x8 matrix K from K_e in problem 9:

$$K = \begin{bmatrix} 324 & 162 & -324 & -162 & 0 & 0 & 0 & 0 \\ 162 & 108 & -162 & -54 & 0 & 0 & 0 & 0 \\ -324 & -162 & 648 & 162 & -324 & -162 & 0 & 0 \\ -162 & -54 & 162 & 216 & -162 & -54 & 0 & 0 \\ 0 & 0 & -324 & -162 & 648 & 162 & -324 & -162 \\ 0 & 0 & -162 & -54 & 162 & 216 & -162 & -54 \\ 0 & 0 & 0 & 0 & -324 & -162 & 324 & 162 \\ 0 & 0 & 0 & 0 & -162 & -54 & 162 & 108 \end{bmatrix}$$

3.2.17 $f(x) = \frac{1}{1+x^2}$


Spline [A] with 10 meshpoints
Spline [B] with 20 meshpoints
The two splines and the actual function x=[0:0.02]
Exact
spline[A]
spline[B]

3.3.1. For $v = (1,0) = w$, we solve $\frac{du}{dx} = 1$, $\frac{du}{dy} = 0$ to get $u(x,y) = x + C$ and $-\frac{ds}{dx} = 0$, $\frac{ds}{dy} = 1$ to get $s(x,y) = y + C$. Thus the equipotentials are vertical lines ($x = C$) and the streamlines are horizontal lines ($y = C$).

3.3.5. For $v_1 = -y, v_2 = 0$ Stoke's law becomes $\int_C -y dx = \int_C v_1 dx + v_2 dy = \int\int_R \left(\frac{dv_2}{dx} - \frac{dv_1}{dy}\right) dx dy = \int\int_R 1 dx dy = Area(R)$. The ellipse is parametrized by $x = a\cos t, y = b\sin t, 0 \le t \le 2\pi$. Hence its area is $\int_C -y dx = \int_0^{2\pi} -(b\sin t)(-a\sin t)dt = \pi ab$.

3.3.8

[Divergence Theorem]

$$\iint_S \text{div grad } u \; dxdy = \iint_S \text{div } (2x,0) \; dxdy = \iint_S 2 \; dxdy = 2 * area(S) = 8$$

$$\int_C n \cdot \text{grad } u \; ds = \int_C (2x,0)\cdot n \; ds = 2\int_{-1}^{1} (2x,0)\cdot(1,0)\, dy = 2\int_{-1}^{1} 2dy = 8$$

along the top edges of the square where $n = (0,1)$, $\int_C (2x,0)\cdot n \; ds = 0$

3.3.9

$$u(x,y) = y \to s(x,y) = -x \to (\nabla u)^T (\nabla s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0$$

$$u(x,y) = x - y \to s(x,y) = x + y \to (\nabla u)^T (\nabla s) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$u(x,y) = \log\left(x^2 + y^2\right)^{1/2} \to s(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \to (\nabla u)^T (\nabla s) = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \end{bmatrix} \begin{bmatrix} \frac{-y^2}{x^2+y^2} \\ \frac{xy}{x^2+y^2} \end{bmatrix} = 0$$

3.3.10

10) $v = (2xy, x^2 - y^2) = \nabla u$.

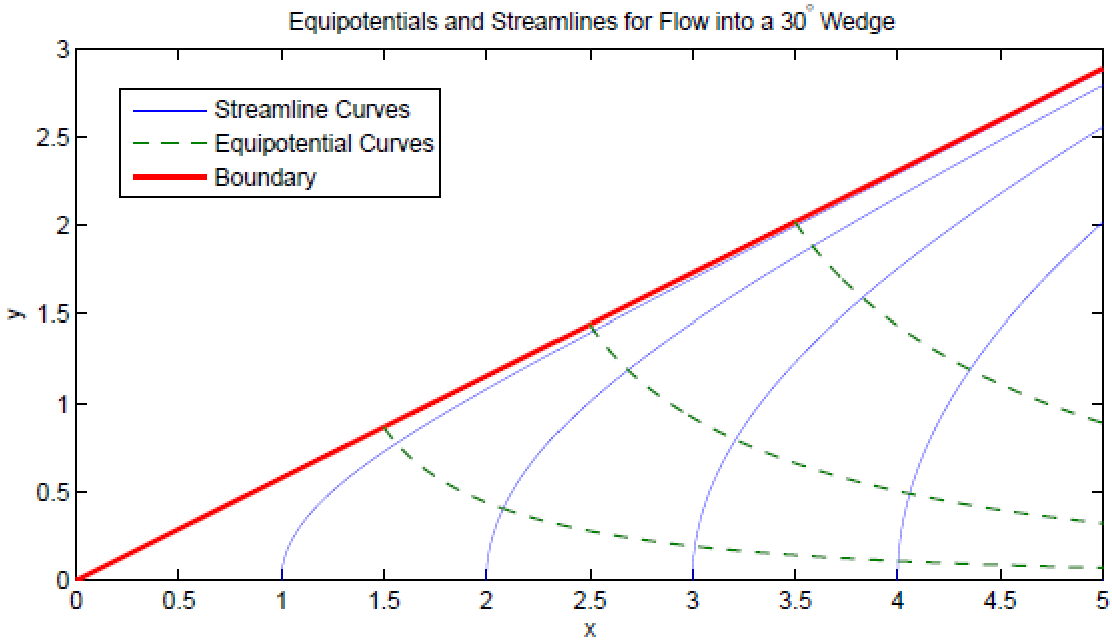
$$\frac{\partial s}{\partial y} = \frac{\partial u}{\partial x} = 2xy \qquad -\frac{\partial s}{\partial x} = \frac{\partial u}{\partial y} = x^2 - y^2$$

$$u(x,y) = x^2 y + f_1(y) \qquad u(x,y) = x^2 y - \frac{1}{3}y^3 + f_2(x)$$

$$s(x,y) = xy^2 + f_3(x) \qquad s(x,y) = xy^2 - \frac{1}{3}x^3 + f_4(y)$$

Equating the two equations for $u(x,y)$ and $s(x,y)$, and letting arbitrary constants be 0, we find $f_1(y) = -\frac{1}{3}y^3$, $f_2(x) = 0$, $f_3(x) = -\frac{1}{3}x^3$, and $f_4(x) = 0$. So, $\boxed{u(x,y) = x^2 y - \frac{1}{3}y^3}$ and $\boxed{s(x,y) = xy^2 - \frac{1}{3}x^3}$. The equipotentials $(u(x,y) = c)$ and streamlines $(s(x,y) = c)$ are shown in the figure below for flow into a 30° wedge. To verify that flow does not pass through the upper boundary, we can show that $v \cdot n = 0$ for $y = x/\sqrt{3}$. On the upper boundary, $n = (-\sin(30°), \cos(30°)) = (-1/2, \sqrt{3}/2)$, and $v = (2x(x/\sqrt{3}),\ x^2 - (x/\sqrt{3})^2) = x^2(2/\sqrt{3}, 2/3)$.

$$v \cdot n = -\frac{1}{\not 2}x^2\frac{\not 2}{\sqrt{3}} + \frac{\sqrt{3}}{\not 2}x^2\frac{\not 2}{3}$$
$$= -\frac{x^2}{\sqrt{3}} + \frac{x^2}{\sqrt{3}} = 0 \checkmark$$



3.4.4

$$u = r\cos\theta + r^{-1}\cos\theta \rightarrow \begin{cases} \dfrac{\partial u}{\partial r} = \cos\theta - r^{-2}\cos\theta \rightarrow \dfrac{\partial^2 u}{\partial r^2} = 2r^{-3}\cos\theta \\ \dfrac{\partial u}{\partial \theta} = -r\sin\theta - r^{-1}\sin\theta \rightarrow \dfrac{\partial^2 u}{\partial \theta^2} = -r\cos\theta - r^{-1}\cos\theta \end{cases}$$

$$[\text{Laplace's equation}]\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$z = x + iy = re^{\theta} \quad \text{where } x = r\cos\theta, y = r\sin\theta$$

$$f(z) = z + z^{-1} = \text{Re}[f(x+iy)] + \text{Im}[f(x+iy)] = u(x,y) + is(x,y)$$

$$u(x,y) = \frac{x^3 + x(y^2+1)}{x^2+y^2} = x + \frac{x}{x^2+y^2}$$

$$v = (u_x, u_y) = \left(1 + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}\right)$$

$$\left.\begin{array}{l} x^2+y^2 = 1 \\ n = (x,y) \end{array}\right\} \rightarrow v\cdot n = (2-2x^2, -2xy)\cdot(x,y) = 0$$

3.4.5

5) $u = \ln(r)$ and $U = \ln(r^2)$ both satisfy Laplace's equation. In fact, $U = \ln(r^2) = 2\ln(r) = 2u$. So, it is enough to show that u satisfies Laplace's equation, because then U automatically will as well. This is shown below.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{1}{r}\left(\frac{1}{r}\right) + 0 = -\frac{1}{r^2} + \frac{1}{r^2} = 0$$

u and U would have been obtained by selecting the complex functions $f(z) = \ln(z) = u + is$ and $F(z) = \ln(z^2) = U + iS$ where $z = re^{i\theta}$. $f(z) = \boxed{u + is = (\ln(r)) + i(\theta)}$ and $F(z) = \boxed{U + iS = (\ln(r^2)) + i(2\theta)}$. The Cauchy-Riemann equations require that ${}^{\partial U}/_{\partial x} = {}^{\partial S}/_{\partial y}$ and ${}^{\partial U}/_{\partial y} = -{}^{\partial S}/_{\partial x}$. This is verified below.

$$U(x,y) = \ln(x^2+y^2) \qquad \frac{\partial U}{\partial x} = \frac{2x}{(x^2+y^2)} \qquad \frac{\partial U}{\partial y} = \frac{2y}{(x^2+y^2)}$$

$$S(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right) \qquad \frac{\partial S}{\partial y} = 2\left[1 + \left(\frac{y}{x}\right)^2\right]^{-1}\left(\frac{1}{x}\right) = \frac{2x}{(x^2+y^2)} \qquad \frac{\partial S}{\partial x} = 2\left[1 + \left(\frac{y}{x}\right)^2\right]^{-1}\left(-\frac{y}{x^2}\right) = -\frac{2y}{(x^2+y^2)}$$

3.4.17

17) We want to verify that $u_k(x,y) = \sin(\pi kx)\sinh(\pi ky)/\sinh(\pi k)$ for $k = 1, 2, ...$ solves Laplace's equation and satisfies the boundary conditions on a unit square $u(x,1) = \sin(\pi kx)$ and $u(x,0) = u(0,y) = u(1,y) = 0$. First, we check if $u_k(x,y)$ satisfies Laplace's equation:

$$\frac{\partial^2 u_k}{\partial x^2} = \frac{\sinh(\pi ky)}{\sinh(\pi k)}\left(-\pi^2 k^2 \sin(\pi kx)\right) \qquad \frac{\partial^2 u_k}{\partial y^2} = \frac{\sin(\pi kx)}{\sinh(\pi k)}\left(\pi^2 k^2 \sinh(\pi kx)\right)$$

$$\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial y^2} = \frac{\pi^2 k^2}{\sinh(\pi k)}\left[-\sin(\pi kx)\sinh(\pi ky) + \sin(\pi kx)\sinh(\pi kx)\right] = 0 \checkmark$$

Then, we check if $u_k(x,y)$ satisfies the boundary conditions:

$$u_k(0,y) = \frac{\sin(0)\sinh(\pi ky)}{\sinh(\pi k)} = 0 \checkmark \qquad u_k(1,y) = \frac{\sin(\pi k)\sinh(\pi ky)}{\sinh(\pi k)} = 0 \checkmark$$

$$u_k(x,0) = \frac{\sin(\pi kx)\sinh(0)}{\sinh(\pi k)} = 0 \checkmark \qquad u_k(x,1) = \frac{\sin(\pi kx)\cancel{\sinh(\pi k)}}{\cancel{\sinh(\pi k)}} = \sin(\pi kx) \checkmark$$

Extra Problem) Solutions to Laplace's Equation $\nabla^2\Psi = 0$ come in pairs u_i and s_i. Four of these pairs are listed below for both 2D Cartesian coordinates and 2D Polar coordinates.
In 2D Cartesian coordinates, $u_i(x,y)$ and $s_i(x,y)$ both satisfying $\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0$:

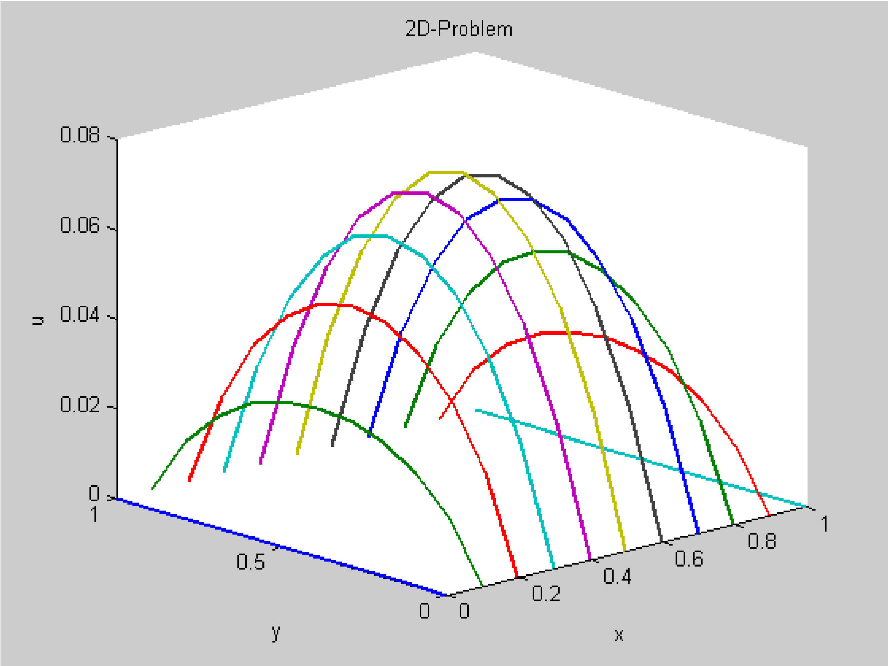
$$\begin{aligned} u_1(x,y) &= x & s_1(x,y) &= y \\ u_2(x,y) &= x^2 - y^2 & s_2(x,y) &= 2xy \\ u_3(x,y) &= x^3 - 3xy^2 & s_3(x,y) &= 3x^2y - y^3 \\ u_4(x,y) &= x^4 - 6x^2y^2 + y^4 & s_4(x,y) &= 4x^3y - 4xy^3 \end{aligned}$$

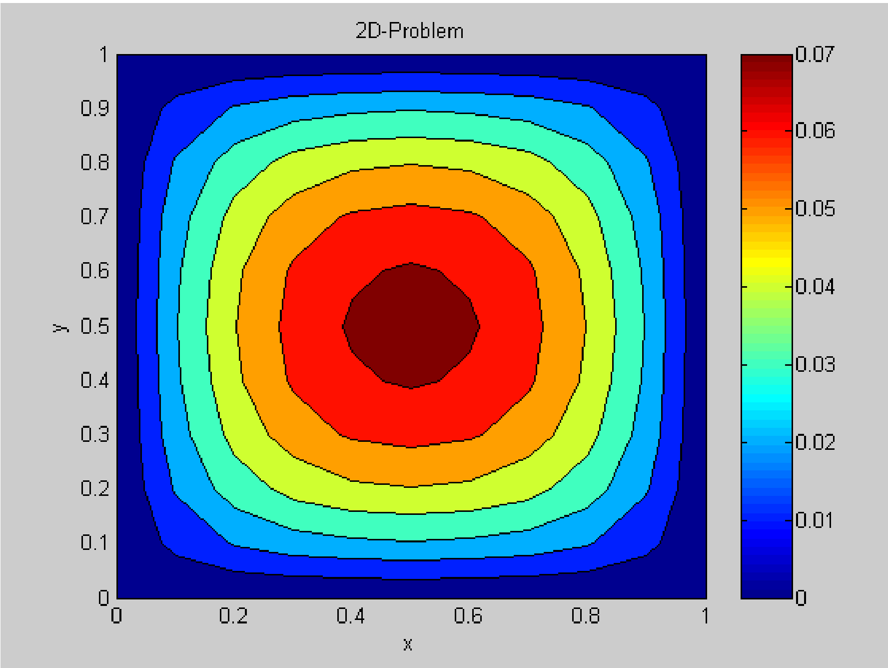
In 2D Polar coordinates, $u_i(r,\theta)$ and $s_i(r,\theta)$ both satisfying $\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = 0$:

$$\begin{aligned} u_1(r,\theta) &= r\cos(\theta) & s_1(r,\theta) &= r\sin(\theta) \\ u_2(r,\theta) &= r^2\cos(2\theta) & s_2(r,\theta) &= r^2\sin(2\theta) \\ u_3(r,\theta) &= r^3\cos(3\theta) & s_3(r,\theta) &= r^3\sin(3\theta) \\ u_4(r,\theta) &= r^4\cos(4\theta) & s_4(r,\theta) &= r^4\sin(4\theta) \end{aligned}$$

3.4.19

$$u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{mn}}{\lambda_{mn}}(\sin m\pi x)(\sin n\pi y)$$





3.5.1

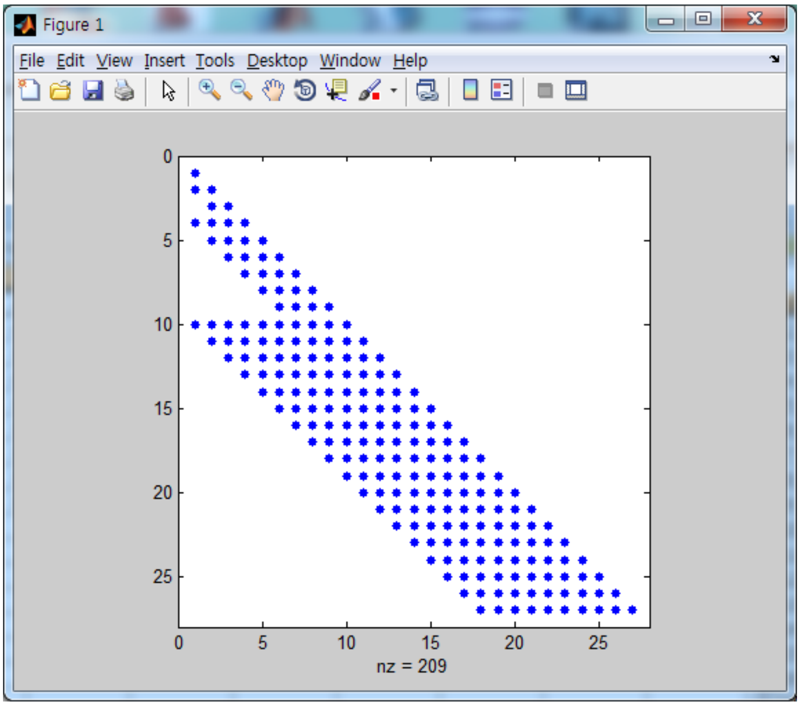
The 7-point Laplace difference equation in 3D has +6 along the diagonal and six -1's on interior row.

Thus, $K3D = -\Delta_x^2 - \Delta_y^2 - \Delta_z^2$ has size of N^3.

Create K3D by the kron command from K2D and I2D = kron(I, I) in the xy plane using K and I of size N in the z direction.

```
K = toeplitz([2 -1 zeros(1,5)])
I = eye(7)
K2D = kron(I,K) + kron(K,I)
I2D = kron(I,I)
K3D = kron(I,K2D) + kron(K,I2D)
```

3.5.2



3.5.8

3.5.8. The ij block of $C = kron(A, B)$ is the n by n matrix $A_{ij}B$. Hence the ij block of C^T is $A_{ji}B^T$, i.e. $C^T = kron(A^T, B^T)$. We now check that $kron(A, B)kron(A^{-1}, B^{-1}) = I$:
The ij block of the pabove product is

$$A_{i1}B(A^{-1})_{1j}B^{-1} + ... + A_{in}B(A^{-1})_{nj}B^{-1} = \left(A_{i1}(A^{-1})_{1j} + ... + A_{n1}(A^{-1})_{nj}\right) BB^{-1} = \delta_{ij}I_n$$

Thus the diagonal blocks are I_n, and the off-diagonal blocks are 0, i.e. $kron(A, B)kron(A^{-1}, B^{-1}) = I_{n^2}$.

3.6.4

$$K_e = \begin{bmatrix} c_2+c_3 & -c_3 & -c_2 \\ -c_3 & c_1+c_3 & -c_1 \\ -c_2 & -c_1 & c_1+c_2 \end{bmatrix} \text{ with } c_i = \frac{1}{2\tan\theta_i}$$

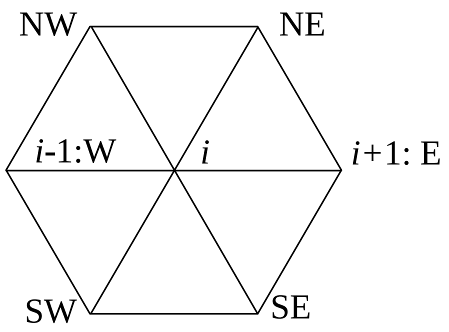
$$\begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \begin{bmatrix} c_2+c_3 & -c_3 & -c_2 \\ -c_3 & c_1+c_3 & -c_1 \\ -c_2 & -c_1 & c_1+c_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$= c_3\left(U_1^2 - 2U_1U_2 + U_2^2\right) + c_2\left(U_1^2 - 2U_1U_3 + U_3^2\right) + c_1\left(U_2^2 - 2U_2U_3 + U_3^2\right)$$

$$= \frac{1}{2}\left[\frac{\left(U_2-U_1\right)^2}{\tan\theta_3} + \frac{\left(U_3-U_1\right)^2}{\tan\theta_2} + \frac{\left(U_3-U_2\right)^2}{\tan\theta_1}\right]$$

$$\theta_1 = \theta_2 = \theta_3 = 60° \to K_e = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} & & & & & & & i-1 & i & i+1 & & & & & & & \\ \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & -1 & 6 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots \\ & & NW & NE & & & & W & & E & & & & SW & SE & & \end{bmatrix}$$



3.6.5

```
m=3;
n=3;
[p,t,b]=squaregrid(m,n);
[K,F]=assemble(p,t);
[Kb,Fb]=dirichlet(K,F,b) ;
```

b：1 2 3 4 7 6 9 8

Kb：(5,5) 4

3.6.7

3.6/7 Let us first find the function U in the lower triangle. $U(x,0) = a+bx+dx^2$ is a quadratic in x which is supposed to vanish for $x = 0, \frac{1}{2}, 1$. This happens only when $a = b = d = 0$. Similarly,

$$U(1,y) = a + b + d + (c+e)y + fy^2$$

is a quadratic that vanishes for $y = 0, \frac{1}{2}, 1$. So,

$$a+b+d = 0 \quad c+e = 0 \quad f = 0.$$

Thus, $U(x,y) = cy(1-x)$. From $U(1/2,1/2) = 1$ we find $c = 4$:

$$U(x,y) = 4y(1-x) \qquad \text{in the lower triangle.}$$

Using the reflection symmetry about the diagonal $x = y$, one easily obtains

$$U(x,y) = 4x(1-y) \qquad \text{in the upper triangle.}$$

3.6.11

3.6/11 **(a)** If $v_1 = (1,1)$, $v_2 = (-1,1)$, $v_3 = (-1,-1)$ and $v_4 = (1,-1)$ and we want $U(v_i) = U_i$, then

$$\begin{aligned} U(x,y) &= U_1\frac{(x+1)(y+1)}{4} + U_2\frac{(x-1)(y+1)}{-4} \\ &+ U_3\frac{(x-1)(y-1)}{4} + U_4\frac{(x+1)(y-1)}{-4} \\ &= \frac{1}{4}((U_1+U_2+U_3+U_4) + (U_1-U_2-U_3+U_4)x \\ &+ (U_1+U_2-U_3-U_4)y + (U_1-U_2+U_3-U_4)xy) \end{aligned}$$

(b) From (a) we know

$$b = U_1 - U_2 - U_3 + U_4 \qquad c = U_1 + U_2 - U_3 - U_4.$$

Thus,

$$\begin{pmatrix} \partial U/\partial x \\ \partial U/\partial y \end{pmatrix}(0,0) = \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}.$$

(c) It's a straightforward check to see that $(1,1,1,1)^T$ and $(1,-1,1,-1)^T$ are in the nullspace of G.