Curves

- Hermite Curve
- Bezier Curve
- B-Spline Curve
- Nonuniform Rational B-Spline (NURBS) Curve

$$\mathbf{U} = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{t}_0 \\ \mathbf{t}_1 \end{bmatrix}, \ \mathbf{R} = \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix}$$

$$\mathbf{r}(t) = \begin{cases} \mathbf{UA} \leftarrow \text{parametric curve} \\ \mathbf{UCS} \leftarrow \text{Hermite curve} \\ \mathbf{UMR} \leftarrow \text{Bezier curve} \\ \mathbf{UNR} \leftarrow \text{Uniform B-spline curve} \end{cases}$$

Curve Model: Parametric Polynomial

- 초월함수(sin, cos, log, ...)를 쓰지 않고 다항식을 씀

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

- 자유곡선의 표현에 적합, CAD/CAM 시스템에서 가장 널리 이용
- Parametric polynomial curve의 종류
  - Power basis Polynomial Curve
  - Ferguson (Hermite) Curve
  - Bezier Curve
  - B-Spline Curve
  - 이들은 모두 상호 변환 가능함

Parametric Cubic Curve (3차)

- Scalar form

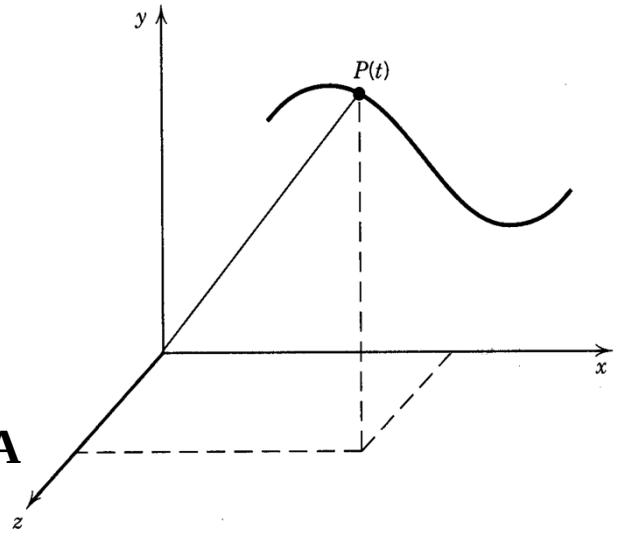
$$\begin{cases} x(u) = a_{0x} + a_{1x}u + a_{2x}u^2 + a_{3x}u^3 \\ y(u) = a_{0y} + a_{1y}u + a_{2y}u^2 + a_{2y}u^3 \\ z(u) = a_{0z} + a_{1z}u + a_{2z}u^2 + a_{3z}u^3 \end{cases} \qquad 0 \le u \le 1$$

- Vector form

$$\mathbf{r}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 = \sum_{i=0}^{3} u^i \mathbf{a}_i$$

- Matrix form

$$\mathbf{r}(u) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \mathbf{UA}$$



  - 장점: 계산 속도가 빠르다.
  - 단점: 계수 $[\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$의 의미를 파악하기 어렵다.

Hermite (Furguson) Curve (1)

- Polynomial 곡선식 $\mathbf{r}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 \quad (0 \le u \le 1)$

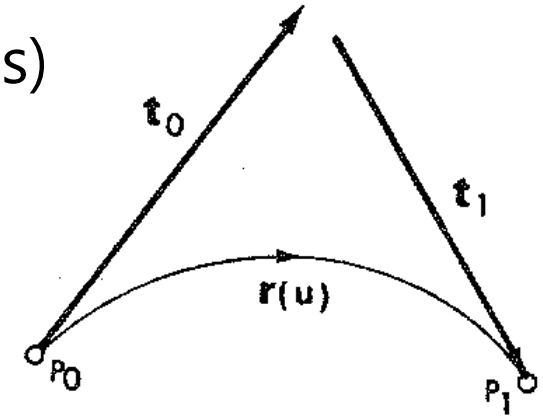
  $\mathbf{r}'(u) = \mathbf{a}_1 + 2\mathbf{a}_2 u + 3\mathbf{a}_3 u^2$

- 곡선 양단에서의 끝점(P_0, P_1)과 접선벡터(t_0, t_1)를 대입

$$\begin{cases} \mathbf{r}(0) = \\ \mathbf{r}(1) = \\ \mathbf{r}'(0) = \\ \mathbf{r}'(1) = \end{cases}$$

- a_i 에 대하여 풀면 (algebraic coefficients)

$$\begin{cases} \mathbf{a}_0 = \\ \mathbf{a}_1 = \\ \mathbf{a}_2 = \\ \mathbf{a}_3 = \end{cases}$$



Hermite (Furguson) Curve (2)

- 위의 식들을 대입하여 r(u)로 표시

$$\mathbf{r}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 \quad (0 \le u \le 1)$$

$$=$$

$$=$$

geometric coefficients

$$\begin{cases} \mathbf{P}_0 = \mathbf{P}(0) \\ \mathbf{P}_1 = \mathbf{P}(1) \\ \mathbf{t}_0 = \mathbf{P}'(0) \\ \mathbf{t}_1 = \mathbf{P}'(1) \end{cases}$$

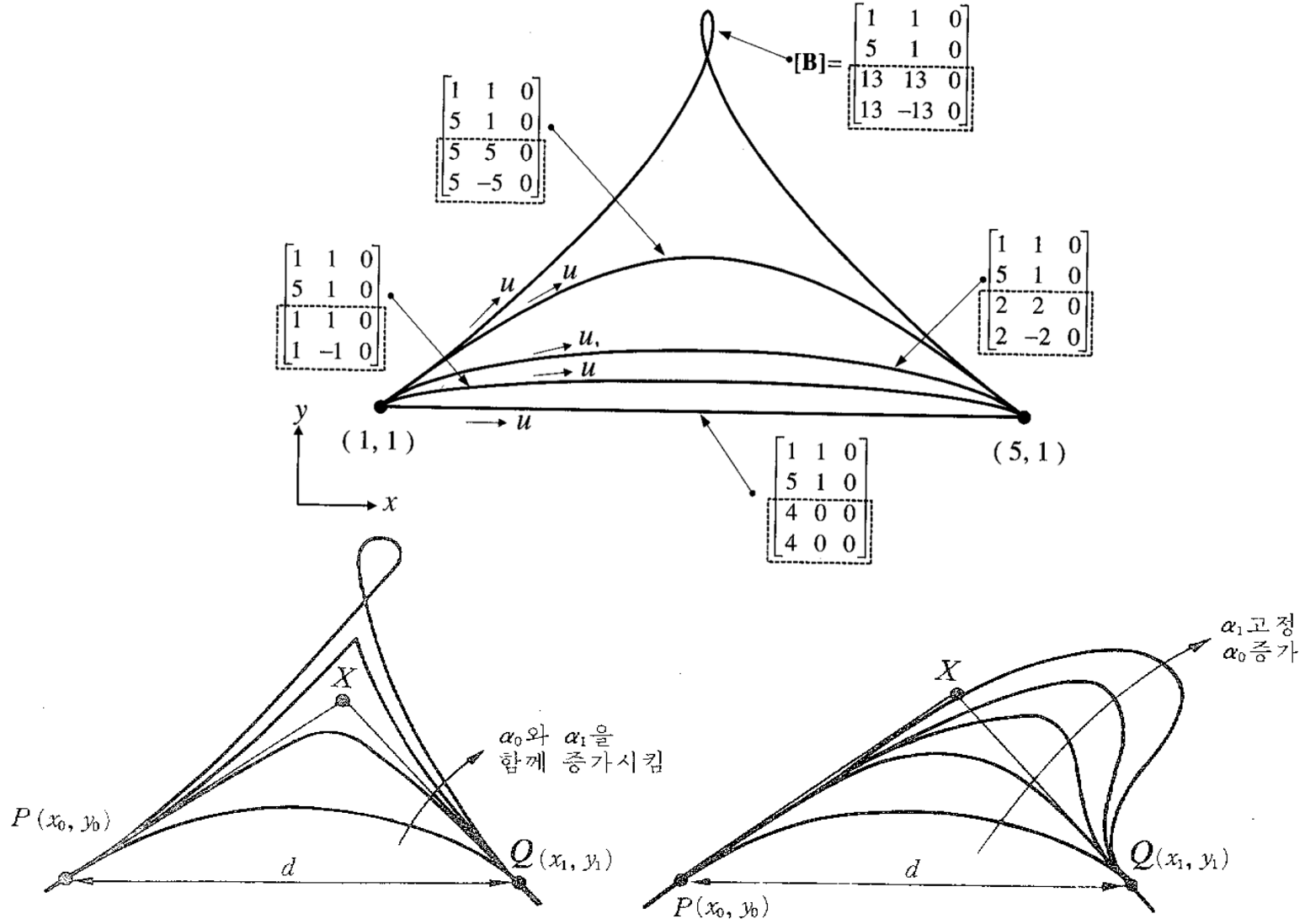
blending functions

$$\begin{cases} f_1(u) = 1 - 3u^2 + 2u^3 \\ f_2(u) = 3u^2 - 2u^3 \\ f_3(u) = u - 2u^2 + u^3 \\ f_4(u) = -u^2 + u^3 \end{cases}$$

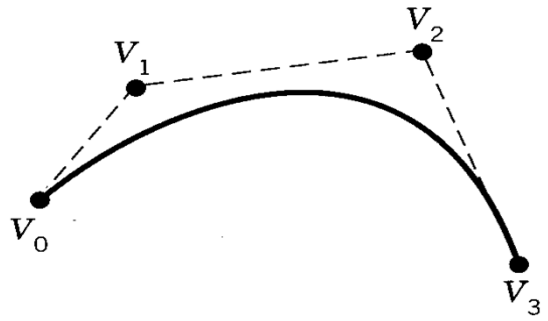
- Matrix form

$$\mathbf{r}(u) = [1 \quad u \quad u^2 \quad u^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{t}_0 \\ \mathbf{t}_1 \end{bmatrix} = \mathbf{UCS}$$

접선벡터의 영향


[B]=
1 1 0
5 1 0
13 13 0
13 −13 0
1 1 0
5 1 0
5 5 0
5 −5 0
1 1 0
5 1 0
1 1 0
1 −1 0
1 1 0
5 1 0
2 2 0
2 −2 0
1 1 0
5 1 0
4 0 0
4 0 0
u
u
u
u
u
y
x
(1, 1)
(5, 1)
X
α0와 α1을
함께 증가시킴
P (x0, y0)
Q (x1, y1)
d
α1 고정
α0 증가
X
Q(x1, y1)
d
P(x0, y0)

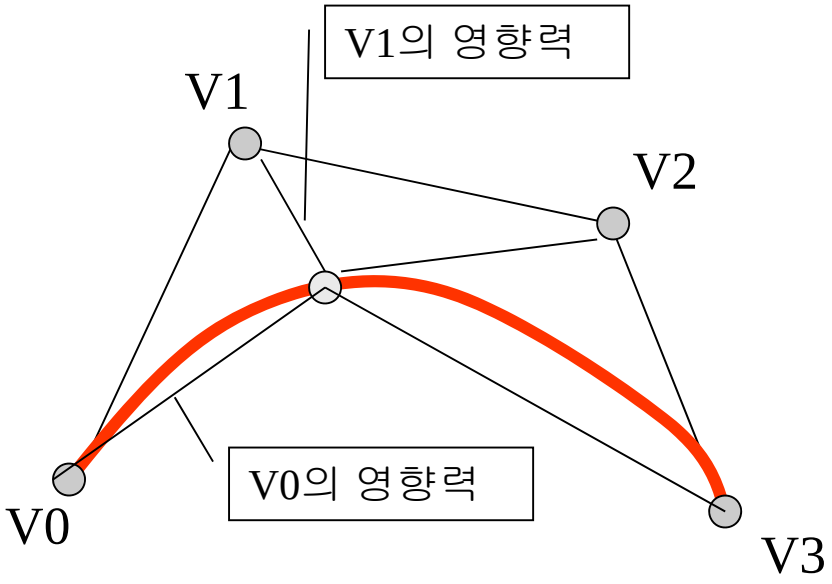
Bezier Curves

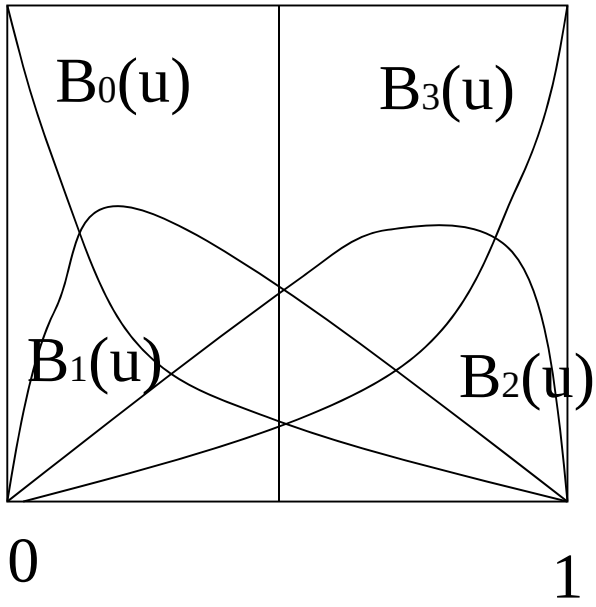


- History of Bezier Curve
  - Bezier designed the Bezier curve
    - in the early 1960s
    - at Renault, French automobile company
    - UNISURF: surface modeler used by Renault since 1972 to design auto-bodies
  - de Casteljau at Citroen also designed the Bezier curve at the same time with Bezier's
- Features of Bezier Curve
  - Pass through the first and last control points
  - Tangent to the lines joining the first two and last two control points
  - No oscillation

Bezier Curve의 정의

- 4개의 조정점으로부터 영향력의 정도를 나타내는 블렌딩 함수 (Bernstein Blending function)를 이용하여 하나의 Bezier Curve를 정의





Cubic Bezier Curve

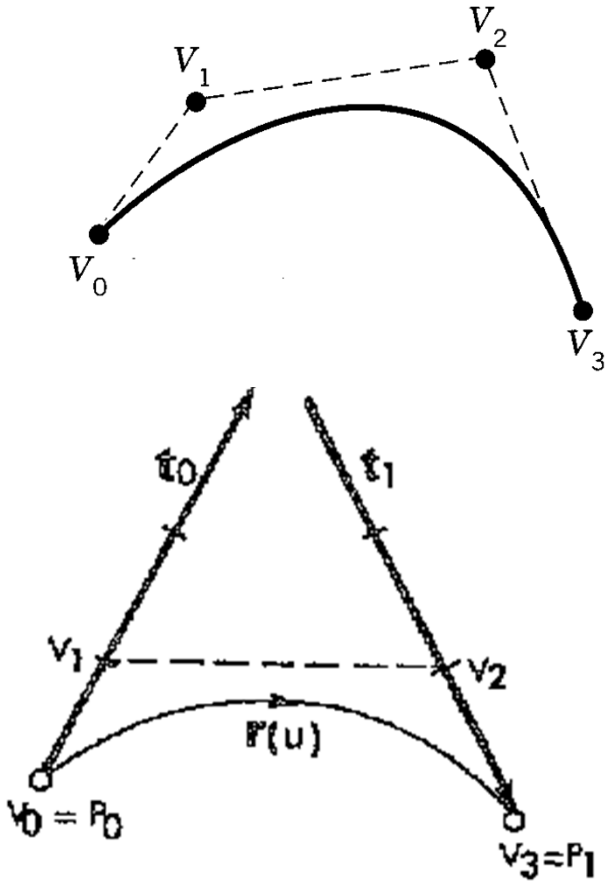
- 점 4개로 하나의 Bezier Curve를 정의

$$\mathbf{r}(u) = (1-u)^3\mathbf{V}_0 + 3u(1-u)^2\mathbf{V}_1 + 3u^2(1-u)\mathbf{V}_2 + u^3\mathbf{V}_3$$
$$= \mathbf{UMR} \qquad (0 \le u \le 1)$$

$$\mathbf{U} = [1 \quad u \quad u^2 \quad u^3]$$

$$\mathbf{M} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

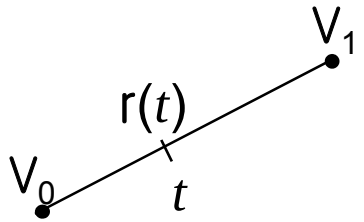
$$\mathbf{R} = \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} : \text{control points}$$



Bernstein Basis 에 의한 Bezier Curve

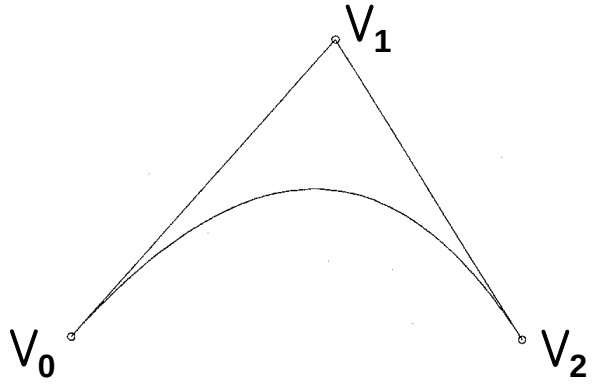
- 선형 Bezier 선

$$\mathbf{r}(t) = (1-t)\mathbf{V}_0 + t\mathbf{V}_1$$



- 2차 Bezier 곡선

$$\mathbf{r}(u) = (1-u)^2\mathbf{V}_0 + 2(1-u)u\mathbf{V}_1 + u^2\mathbf{V}_2$$



- n차 Bezier curve의 식

$$\mathbf{r}(u) = \sum_{i=0}^{n} \mathrm{B}_i^n(u)\mathbf{V}_i, \quad 0 \le u \le 1$$

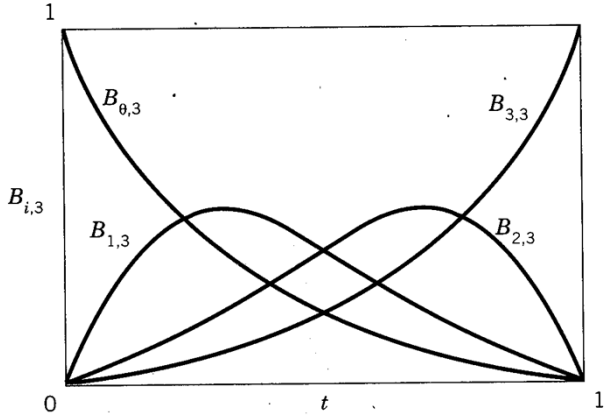
$$\mathrm{B}_i^n(u) = \binom{n}{i}(1-u)^{n-i}u^i = \frac{n!}{i!(n-i)!}(1-u)^{n-i}u^i \text{ : Bernstein basis function}$$

Bezier Curve의 일반식

$$B_{i,n}(u) = \binom{n}{i}(1-u)^{n-i}\,u^{i} \quad 0 < u \le 1$$

- Blending function of Cubic Bezier Curve (degree 3)

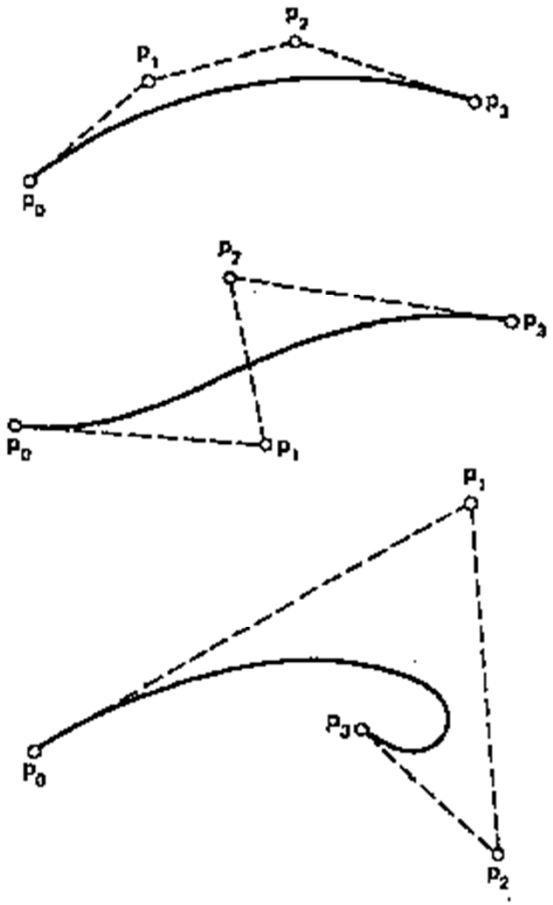
$$\begin{cases} B_{0,3}(u) = \dfrac{3!}{0!3!}(1-u)^{3}\,u^{0} = (1-u)^{3} \\ B_{1,3}(u) = \dfrac{3!}{1!2!}(1-u)^{2}\,u^{1} = 3u(1-u)^{2} \\ B_{2,3}(u) = \dfrac{3!}{2!1!}(1-u)u^{2} = 3u^{2}(1-u) \\ B_{3,3}(u) = \dfrac{3!}{3!0!}(1-u)^{0}\,u^{3} = u^{3} \end{cases}$$

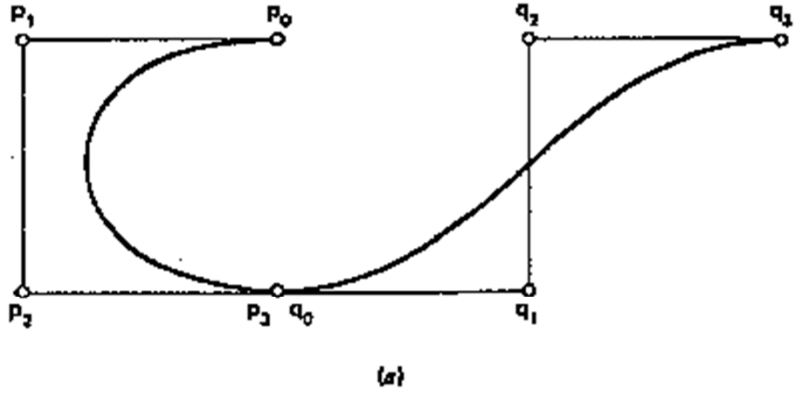


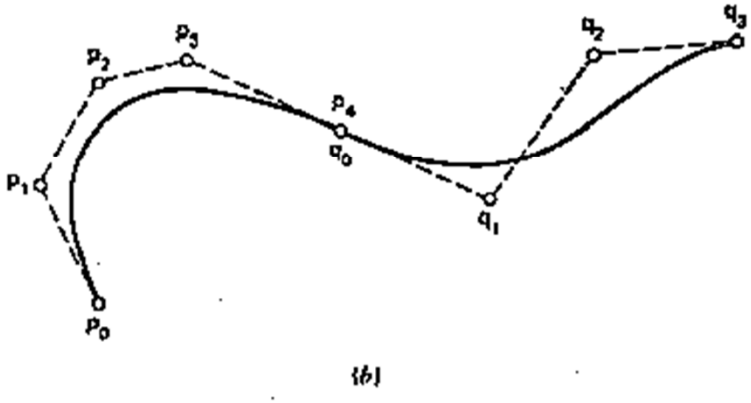
- Normalizing Property

$$(1-u)^{3} + 3u(1-u)^{2} + 3u^{2}(1-u) + u^{3} = 1$$

Bezier Curve의 예



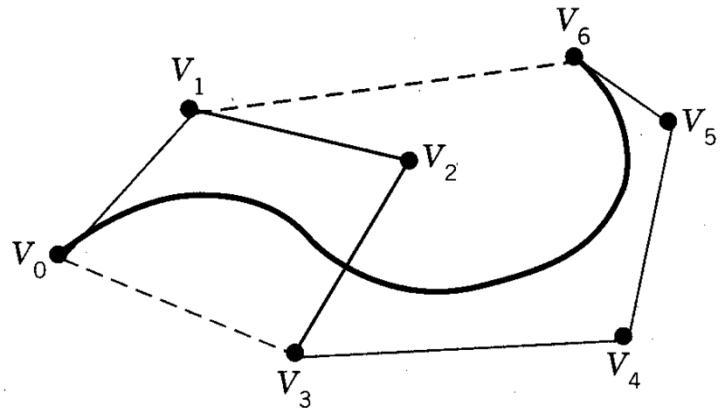


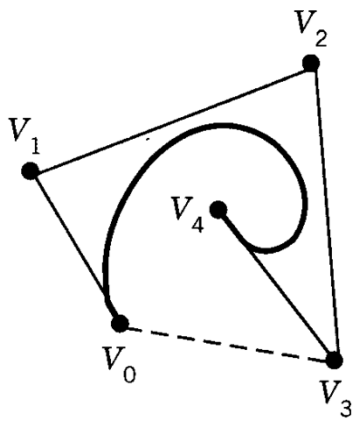


Composite Bezier curves

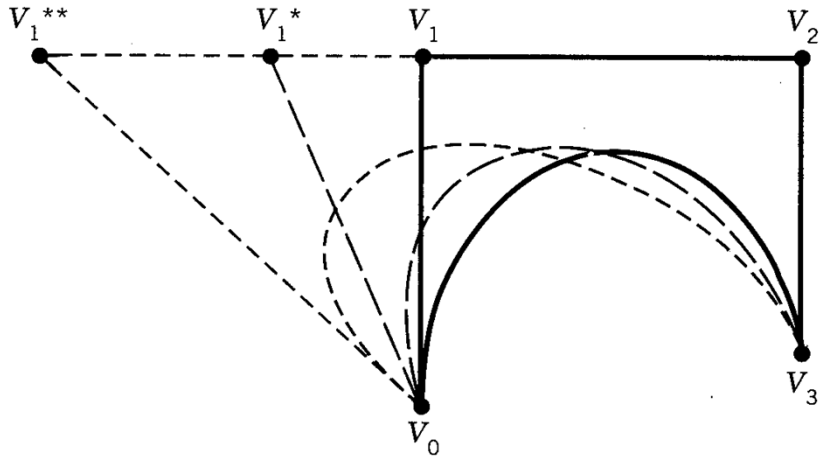
Bezier Curve의 성질 (1)

- Convex Hull Property



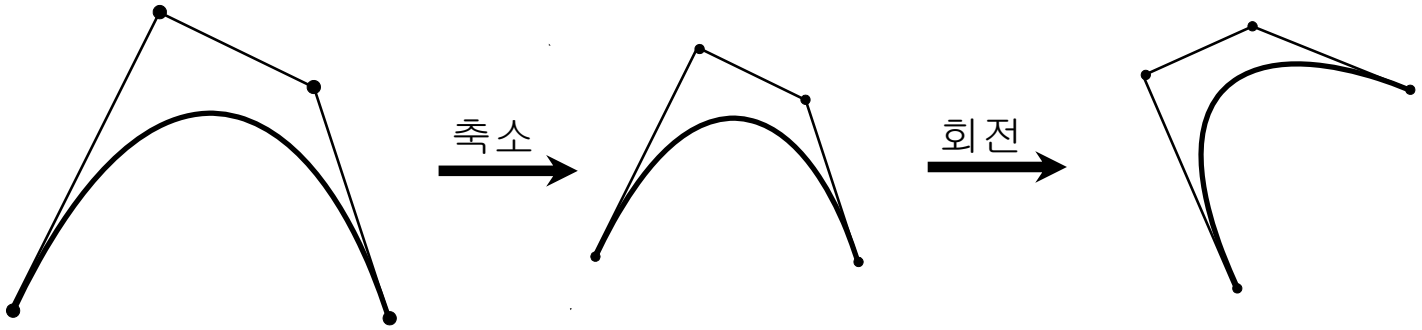


- Effect of moving control points

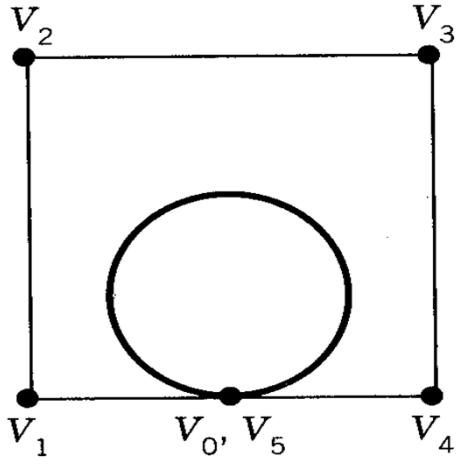


Bezier Curve의 성질 (2)

- Affine Invariance
  - Control point를 transformation하면 곡선도 같이 transform된다



- Closed loop : first and last control points are coincide

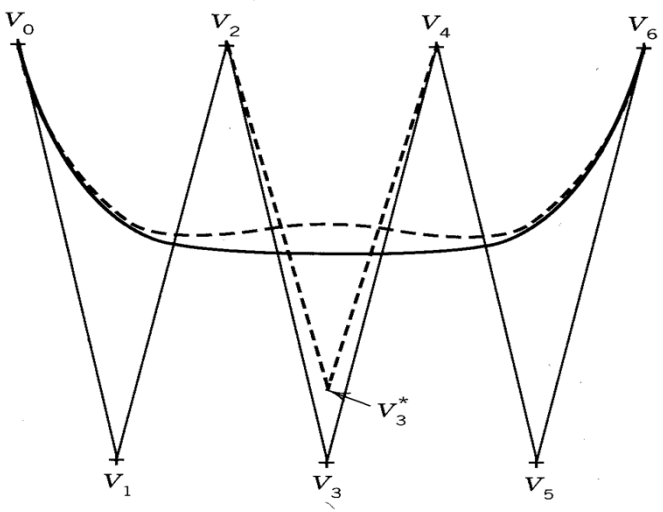


Bezier Curve의 성질 (3)

- 조정점의 개수와 곡선의 차수가 직결되어 모든 조정점이 곡선의 형상에 영향을 줌.

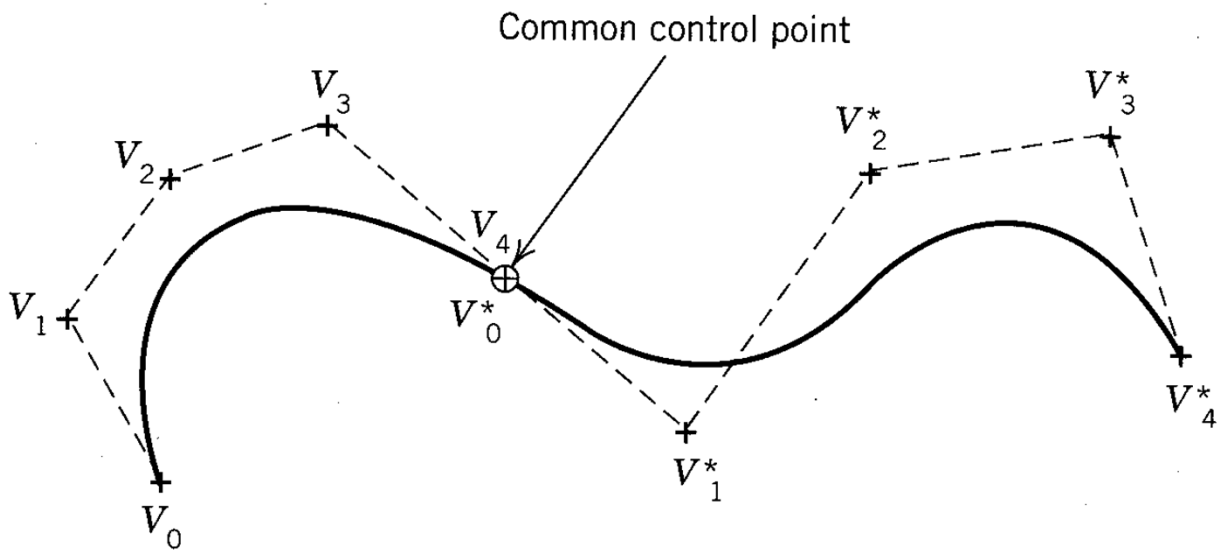
$$\mathbf{r}(u) = \sum_{i=0}^{n} \mathrm{B}_i^n(u)\mathbf{V}_i, \quad 0 \le u \le 1, \quad \mathrm{B}_i^n(u) = \binom{n}{i}(1-u)^{n-i}u^i$$

  - 많은 조정점을 이용할 경우, 곡선식의 차수도 올라가게 되어 계산량이 증가되며 곡선이 진동하는 문제가 야기됨
  - 모든 조정점이 곡선의 형상에 영향을 주기 때문에 곡선의 일부분을 변형시키면 나머지 부분도 예상치 못한 변화가 발생할 수 있음



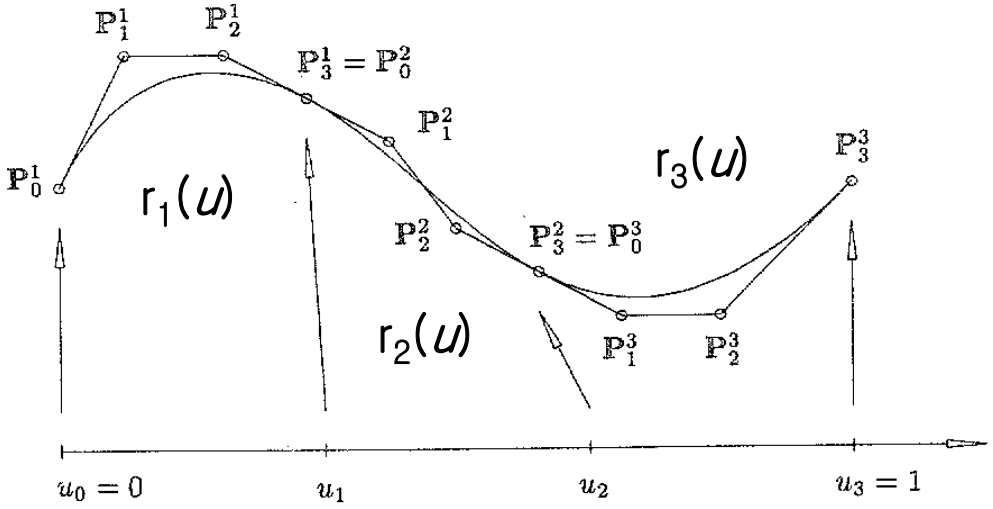
Composite Bezier Curve

- Piecewise Bezier curve in case of a large number of control points
  - C^1 continuity: three control points around the intersection are colinear



Composite Curve (복합 곡선)

- 복잡한 자유곡선을 표현하는 두가지 방법
  - 곡선의 차수 증가 → 원하지 않는 굴곡 발생
  - 여러 개의 곡선 결합으로 표현 → 대부분의 CAD/CAM system 이용
- Composite Curve: piecewisely defined continuous curve



{ $r_1(u)$, $r_2(u)$, $r_3(u)$ } : composite curve
$r_1(u)$, $r_2(u)$, $r_3(u)$: curve segment

Continuity (연속 조건)

- Parametric Continuity

C^0 연속

$$\mathbf{r}_1(u_1) = \mathbf{r}_2(u_1)$$

C^1 연속

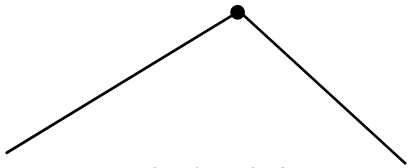
$$\frac{d}{du}\mathbf{r}_1(u_1) = \frac{d}{du}\mathbf{r}_2(u_1)$$

C^2 연속

$$\frac{d^2}{du^2}\mathbf{r}_1(u_1) = \frac{d^2}{du^2}\mathbf{r}_2(u_1)$$

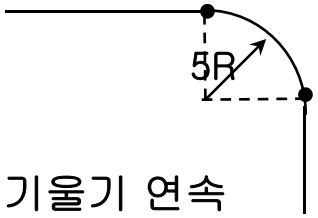
- Geometric Continuity

G^0 연속



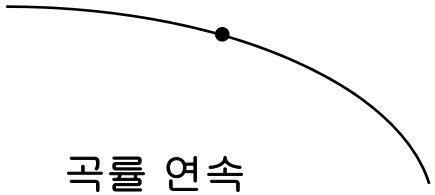
위치 연속

G^1 연속



기울기 연속

G^2 연속

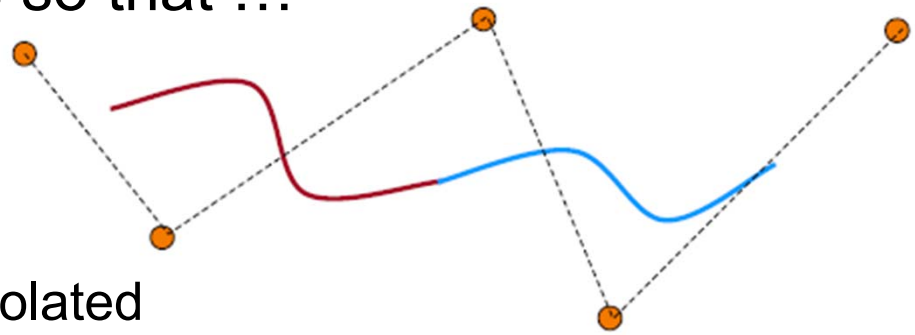


곡률 연속

- G^x 연속은 curve의 재매개변수화에 의해 C^x 연속으로 바꿀 수 있다.

Uniform Cubic B-splines (1)

- Choose blending functions so that …
  - Cubic polynomials
  - C^2 continuity
  - Local control
  - Points not necessarily interpolated



- Derivation
  - Three continuity conditions for each joint J_i …
    - Position, derivatives and second derivatives of two curves are equal at J_i
  - Also, local control implies …
    - Each joint is affected by small set of (4) points

Uniform Cubic B-splines (2)

$$Q_1(u) = b_0(u)V_0 + b_1(u)V_1 + b_2(u)V_2 + b_3(u)V_3$$
$$Q_2(u) = b_0(u)V_1 + b_1(u)V_2 + b_2(u)V_3 + b_3(u)V_4$$

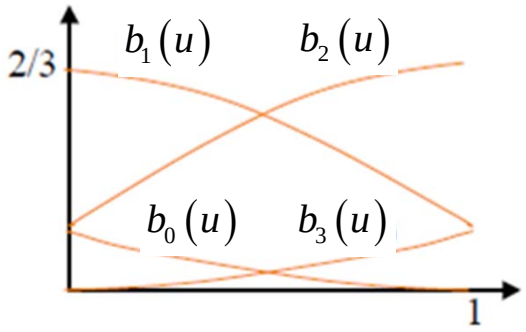
(15 continuity constraints)

$$Q_1(1) = Q_2(0) \qquad Q_1'(1) = Q_2'(0) \qquad Q_1''(1) = Q_2''(0)$$

$$\rightarrow \begin{cases} b_0(1) = 0 \\ b_1(1) = b_0(0) \\ b_2(1) = b_1(0) \\ b_3(1) = b_2(0) \\ 0 = b_3(0) \end{cases} \rightarrow \begin{cases} b_0'(1) = 0 \\ b_1'(1) = b_0'(0) \\ b_2'(1) = b_1'(0) \\ b_3'(1) = b_2'(0) \\ 0 = b_3'(0) \end{cases} \rightarrow \begin{cases} b_0''(1) = 0 \\ b_1''(1) = b_0''(0) \\ b_2''(1) = b_1''(0) \\ b_3''(1) = b_2''(0) \\ 0 = b_3''(0) \end{cases}$$

(1 normality constraint)

$$b_0(0) + b_1(0) + b_2(0) + b_3(0) = 1$$



$$\left.\begin{aligned} b_0(u) &= a_{00} + a_{01}u + a_{02}u^2 + a_{03}u^3 \\ b_1(u) &= a_{10} + a_{11}u + a_{12}u^2 + a_{13}u^3 \\ b_2(u) &= a_{20} + a_{21}u + a_{22}u^2 + a_{23}u^3 \\ b_3(u) &= a_{30} + a_{31}u + a_{32}u^2 + a_{33}u^3 \end{aligned}\right\}$$

$$\rightarrow \begin{cases} b_0(u) = \frac{1}{6} - \frac{1}{2}u + \frac{1}{2}u^2 - \frac{1}{6}u^3 \\ b_1(u) = \frac{2}{3} + 0u - u^2 + \frac{1}{2}u^3 \\ b_2(u) = \frac{1}{6} + \frac{1}{2}u + \frac{1}{2}u^2 - \frac{1}{6}u^3 \\ b_3(u) = 0 + 0u + 0u^2 + \frac{1}{6}u^3 \end{cases}$$

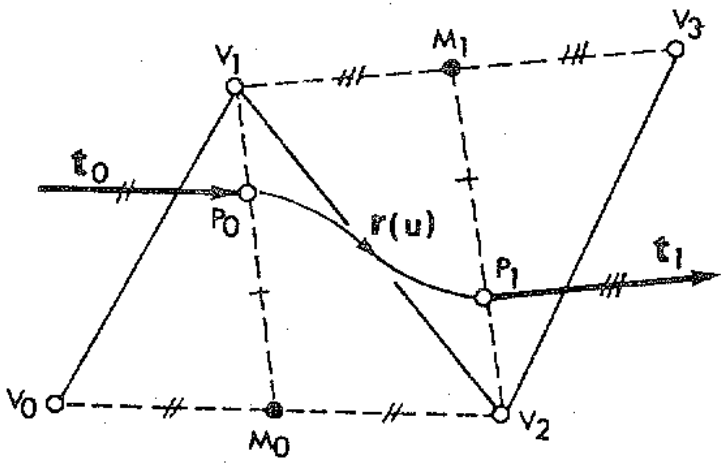
Uniform Cubic B-Spline Curve (3)

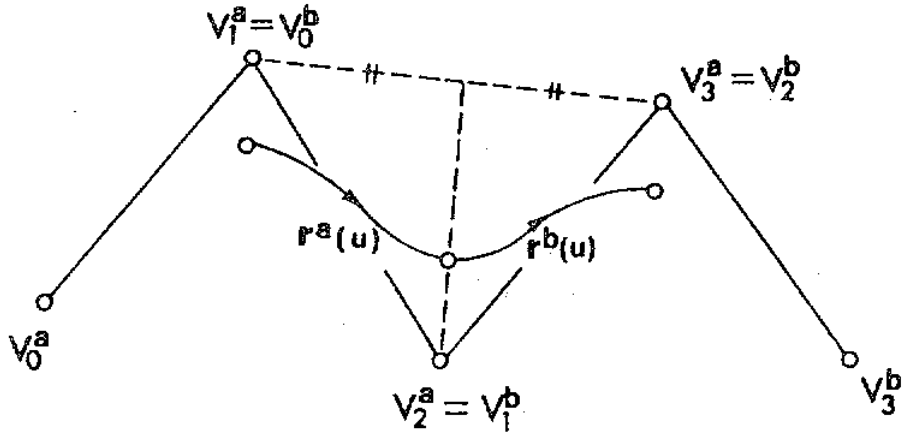
$$\mathbf{r}(u) = b_0(u)\mathbf{V}_0 + b_1(u)\mathbf{V}_1 + b_2(u)\mathbf{V}_2 + b_3(u)\mathbf{V}_3$$

$$= \left(\frac{1}{6} - \frac{1}{2}u + \frac{1}{2}u^2 - \frac{1}{6}u^3\right)\mathbf{V}_0 + \left(\frac{2}{3} + 0u - u^2 + \frac{1}{2}u^3\right)\mathbf{V}_1$$

$$+ \left(\frac{1}{6} + \frac{1}{2}u + \frac{1}{2}u^2 - \frac{1}{6}u^3\right)\mathbf{V}_2 + \left(0 + 0u + 0u^2 + \frac{1}{6}u^3\right)\mathbf{V}_3$$

$$= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} = \mathbf{UNR} \quad (0 \le u \le 1)$$





B-Spline Curve의 특성

- 조정점의 개수와 다항식의 차수가 서로 독립적이다.
  - 설계자가 원하는 차수를 직접 정할 수 있음.
  - Bezier curve에서는 조정점의 개수 = 차수 +1
- 국부적인 형상 조정이 가능하다.
  - 모든 블렌딩 함수는 매개변수 u의 전체 범위 중 각각 서로 다른 일정 범위에서만 값을 갖도록 함.
  - Bezier curve에서는 블렌딩 함수가 u의 전체범위에서 값을 가짐 → 형상이 전체적으로 바뀜
- Degree가 3차 이상(order는 4차이상)이면 2차 미분 연속이 보장됨

B-spline Curve의 정의

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(u) \quad (t_0 \le u \le t_{n+k})$$

$\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n\}$: control point

$N_{i,k}(u)$: blending function of degree $(k-1)$

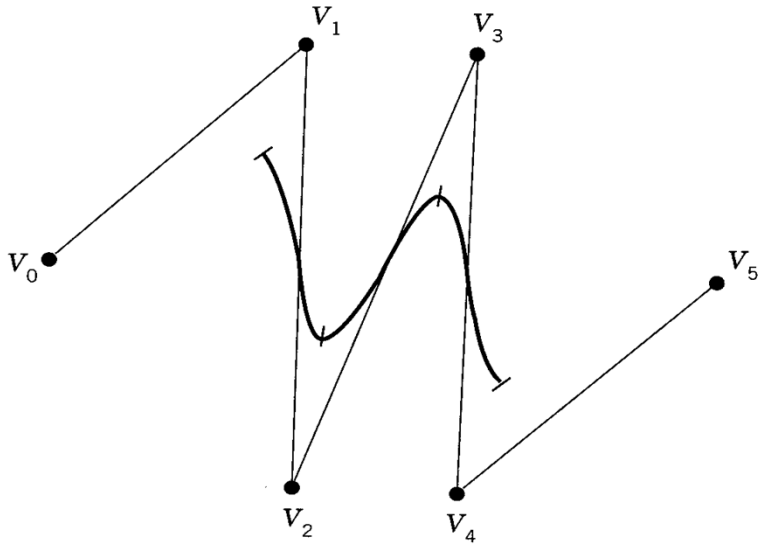
$$N_{i,k}(u) = \frac{(u - t_i)}{t_{i+k-1} - t_i} N_{i,k-1}(u) + \frac{(t_{i+k} - u)}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(u)$$

Cox-de Boor (blending) function

$$N_{i,1}(u) = \begin{cases} 1 & t_i \le u \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

t_i : u의 범위 내에 존재하는 매듭값(knot value)

k : order $(\text{degree} = k-1)$

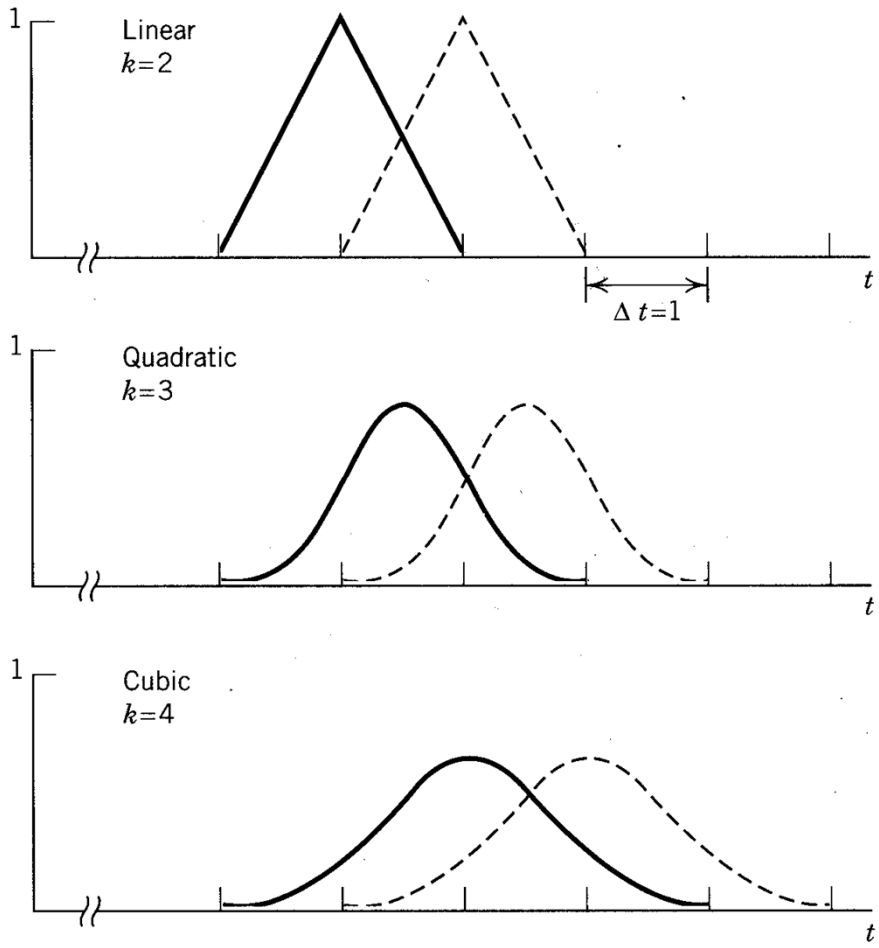


B-spline of order *k* in the *i*-th span is the weighted average of the B-splines of order (*k*-1) in the *i*-th and (*i*+1)-th spans

- Convex hull property
- Normalizing property

$$\sum_{i=0}^{n} N_{i,k}(u) = 1$$

B-spline Curve의 Blending Functions



Knot Vector

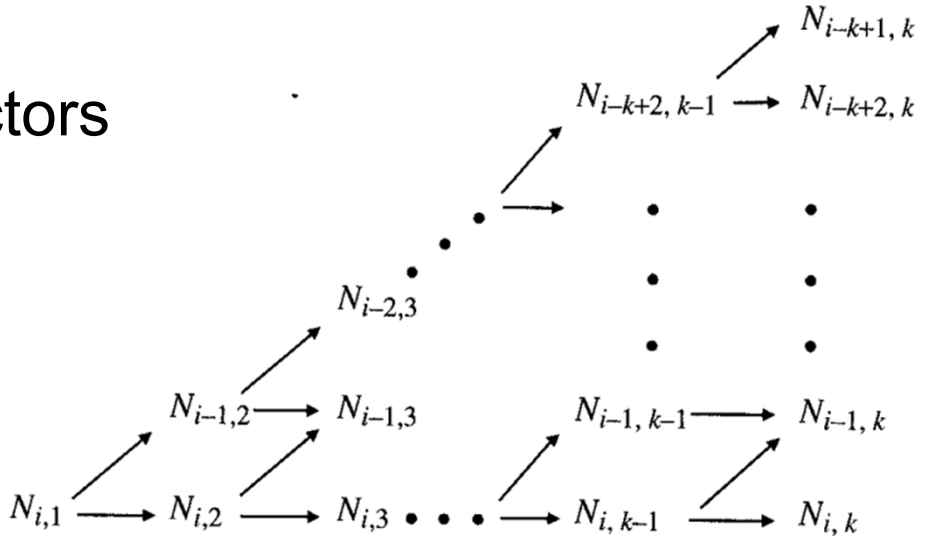
- Relationship of parameters

$$\underbrace{(m+1)}_{\text{\# of knots}} = \underbrace{(n+1)}_{\text{\# of control points}} + \underbrace{k}_{\text{order of curve}}$$

$$m = n + k$$

$$\underbrace{N_{0,k}(u), \ldots, N_{n,k}(u)}_{(n+1)\text{ blending functions}} \leftarrow \underbrace{\{t_0, \ldots, t_{n+k}\}}_{(n+k+1)\text{ knot values}}$$

- Classification of knot vectors
  - Uniform / periodic
  - Nonperiodic
  - Nonuniform



Uniform/Periodic (1)

- Uniform knot vector has equispaced t_i values
  - Let (t_i - t_{i-1}) = a
  - [0 1 2 3 4] with a = 1
  - [-0.5 0.0 0.5 1.0 1.5] with a = 0.5

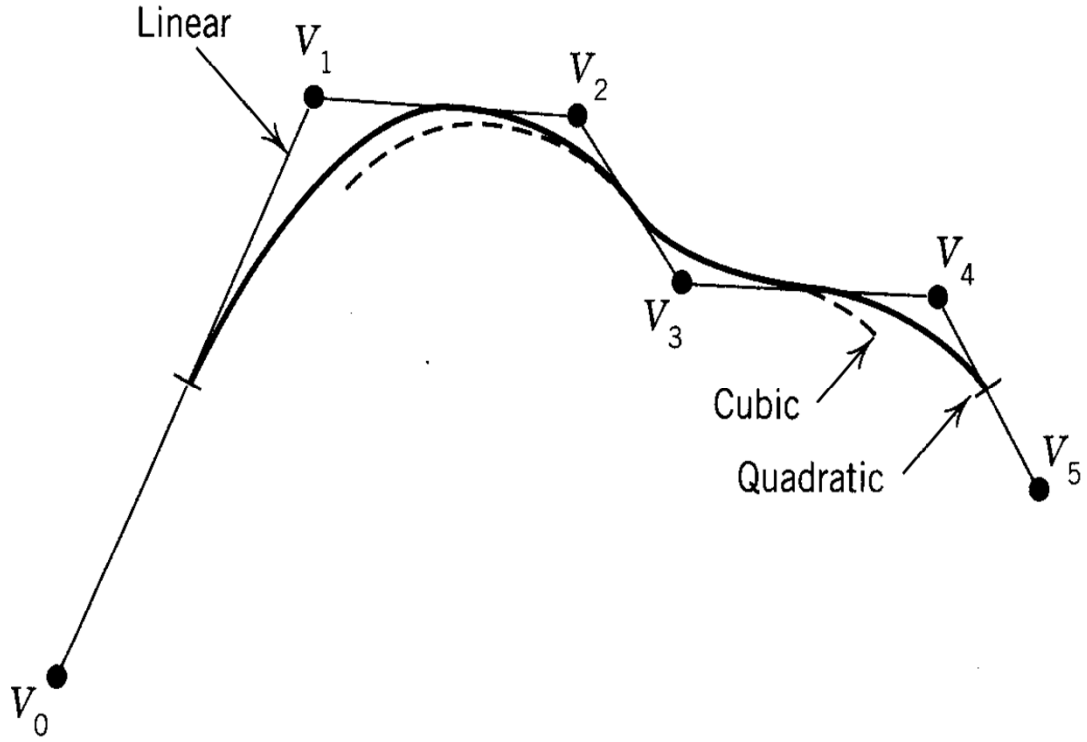
$$\begin{cases} n = 5 \\ \#\ \text{of control points} = n + 1 = 6 \end{cases}$$

Degree ($k-1$)	Order (k)	Knot Vector ($m = n + k$)	Parameter Range $(k-1) \le t \le (n+1)$
1	2	[0 1 2 3 4 5 6 7]	$1 \le t \le 6$
2	3	[0 1 2 3 4 5 6 7 8]	$2 \le t \le 6$
3	4	[0 1 2 3 4 5 6 7 8 9]	$3 \le t \le 6$

- Normalized in the range of [0 to 1]
  - [0 ¼ ½ ¾ 1]

Uniform/Periodic (2)

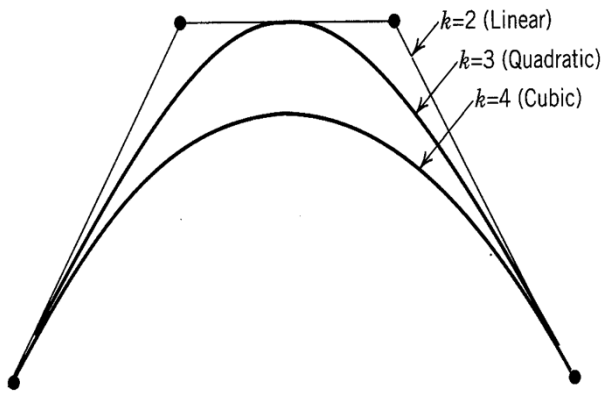
- Uniform B-splines of various degrees



Nonperiodic (1)

- Nonperiodic or Open Knot Vector
  – Has repeated knot values at the ends with multiplicity equal to the order of the function k and internal knots equally spaced

Order (k)	*No. of knots* ($m = n + k$)	*Nonperiodic knot vector*
2	6	$[\underbrace{0\ 0}_{k}\ 1\ 2\ \underbrace{3\ 3}_{k}]$
3	7	$[\underbrace{0\ 0\ 0}_{k}\ 1\ \underbrace{2\ 2\ 2}_{k}]$
4	8	$[\underbrace{0\ 0\ 0\ 0}_{k}\ \underbrace{1\ 1\ 1\ 1}_{k}]$



$$\begin{cases} n = 3 \\ \#\ \text{of control points} = n+1 = 4 \end{cases}$$

- General expression

$$\underbrace{t_0,\ldots,t_{n+k}}_{(n+k+1)\text{ knot values}} \rightarrow \begin{cases} \text{periodic: } t_i = i-k, \quad 0 \le i \le n+k \\ \text{nonperiodic: } t_i = \begin{cases} 0 & 0 \le i < k \\ i-k+1 & k \le i \le n \\ n-k+2 & n < i \le n+k \end{cases} \end{cases}$$

Nonperiodic (2)

- No loss of parameter range
  - Curve interpolates the first and last control points
  - $0 \le u \le n-k+2$
- Bezier curve: special case of nonperiodic B-spline
  - If no. of control points (n+1) = order (k) and a nonperiodic uniform knot vector is used
  - Cubic B-spline with 4 control points and a knot vector [0 0 0 0 1 1 1 1] ➔ Cubic Bezier curve

$$[\underbrace{0 \quad 0 \quad \cdots \quad 0}_{k} \quad \underbrace{1 \quad 1 \quad \cdots \quad 1}_{k}]$$

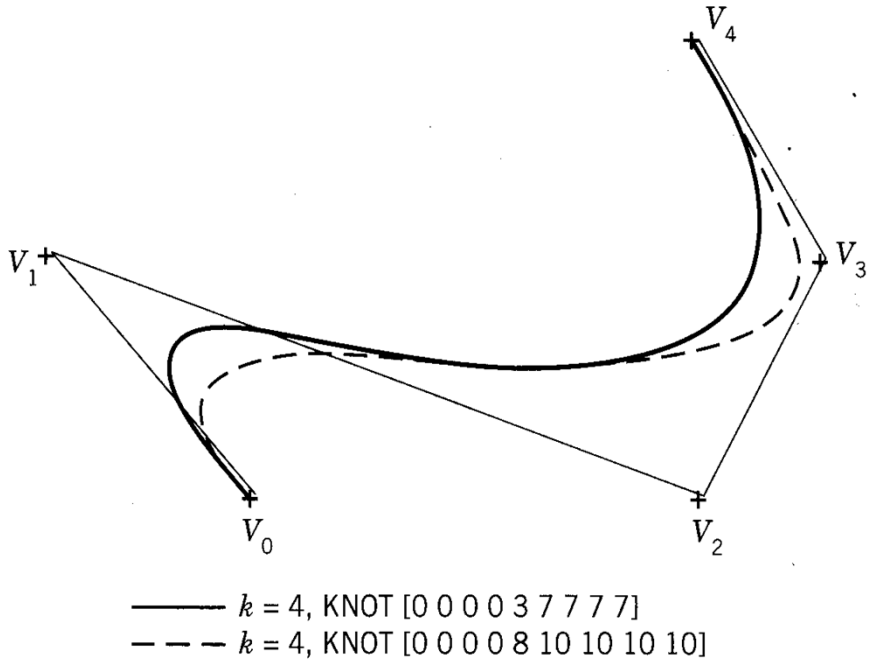
Nonperiodic (3)

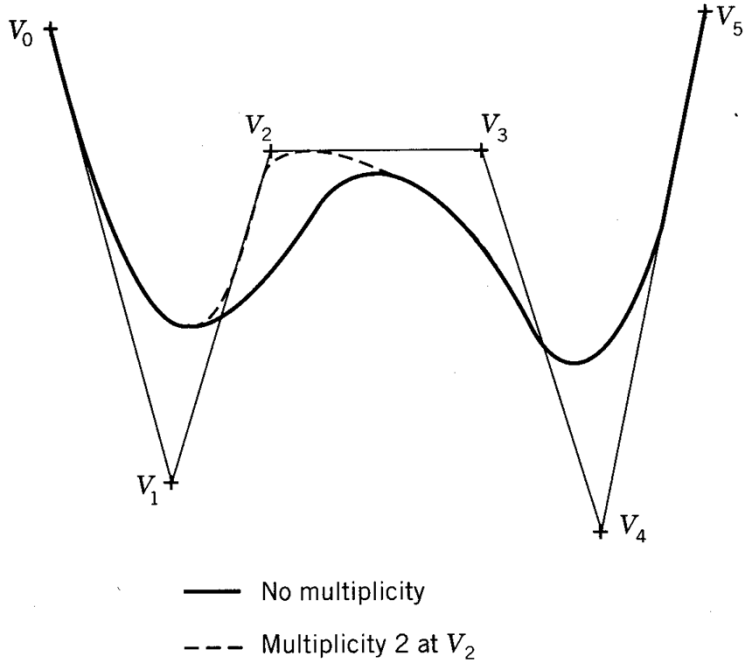
- Multiple interior knot values or unequal spacing

  [0 1 2 3 3 4]

  [0.0 0.20 0.55 0.75 1.0]

- Effect of multiplicity of control points
  - Generate a span of zero length
  - C^{k-m-2} continuity at t_i
  - m (≤ k-2) is the multiplicity of interior knot value





B-spline Curve의 예 (1)

P_0, P_1, P_2의 조정점을 갖고 order (k)가 3인 비주기적 B-spline 곡선

비 주기 매듭값 t_i는 다음과 같음.

$$t_0 = 0, t_1 = 0, t_2 = 0, t_3 = 1, t_4 = 1, t_5 = 1$$

k=1에 해당되는
블렌딩 함수 $N_{i,1}$을 유도.

$$N_{0,1} = \begin{cases} 1 & t_0 \le u < t_1 \\ 0 & otherwise \end{cases}$$

$$N_{1,1} = \begin{cases} 1 & t_1 \le u < t_2 \\ 0 & otherwise \end{cases}$$

$$N_{2,1} = \begin{cases} 1 & t_2 \le u < t_3 \\ 0 & otherwise \end{cases}$$

$$N_{3,1} = \begin{cases} 1 & t_3 \le u < t_4 \\ 0 & otherwise \end{cases}$$

$$N_{4,1} = \begin{cases} 1 & t_4 \le u < t_5 \\ 0 & otherwise \end{cases}$$

k=2에 해당되는
블렌딩 함수 $N_{i,2}$을 유도.

$$N_{0,2} = \frac{(u-t_0)N_{0,1}}{t_1-t_0} + \frac{(t_2-u)N_{1,1}}{t_2-t_1} = \frac{uN_{0,1}}{0} + \frac{(-u)N_{1,1}}{0} = 0$$

$$N_{1,2} = \frac{(u-t_1)N_{1,1}}{t_2-t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = \frac{uN_{1,1}}{0} + \frac{(1-u)N_{2,1}}{1} = (1-u)$$

$$N_{2,2} = \frac{(u-t_2)N_{2,1}}{t_3-t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = \frac{uN_{2,1}}{1} + \frac{(1-u)N_{3,1}}{0} = u$$

$$N_{3,2} = \frac{(u-t_3)N_{3,1}}{t_4-t_3} + \frac{(t_5-u)N_{4,1}}{t_5-t_4} = \frac{(u-1)N_{3,1}}{0} + \frac{(1-u)N_{4,1}}{0} = 0$$

B-spline Curve의 예 (2)

k=3에 해당되는 블렌딩 함수 $N_{i,3}$을 유도.

$$N_{0,3} = \frac{(u-t_0)N_{0,2}}{t_2-t_0} + \frac{(t_3-u)N_{1,2}}{t_3-t_1} = \frac{uN_{0,2}}{0} + \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3} = \frac{(u-t_1)N_{1,2}}{t_3-t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3} = \frac{(u-t_2)N_{2,2}}{t_4-t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} = u^2$$

다음 식을 위의 값을 이용하여 정리하면 다음과 같다.

$$P(u) = \sum_{i=0}^{2} P_i N_{i,k}(u) \qquad (t_2 =)0 \le u \le 1(= t_3)$$

$$P(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$$

Rational Curves

- General Meaning
  - Functions are obtained by the “ratio” of two polynomials
  - This representation makes use of the concept of homogeneous coordinates
  - 한 꼭지점이 곡선에 미치는 영향의 양을 결정할 수 있음
- General Form

	Bezier	B-Spline
Nonrational (Integral)	$Q(t) = \sum_{i=0}^{n} B_{in}(t) V_i$	$P(t) = \sum_{i=0}^{n} N_{ik}(t) V_i$
Rational	$Q(t) = \dfrac{\sum_{i=0}^{n} B_{in}(t) w_i V_i}{\sum_{i=0}^{n} B_{in}(t) w_i}$	$P(t) = \dfrac{\sum_{i=0}^{n} N_{ik}(t) w_i V_i}{\sum_{i=0}^{n} N_{ik}(t) w_i}$

Rational Polynomial Curve

- Unit circle
  - Implicit form: $x^2 + y^2 = 1$
  - Parametric form
    - $\mathbf{r}(u) = (\cos(u), \sin(u), 0)$
    - rational polynomial form

[Half Angle Formula]

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A}$$

$$t = \tan(u/2) \rightarrow \mathbf{r}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0\right)$$

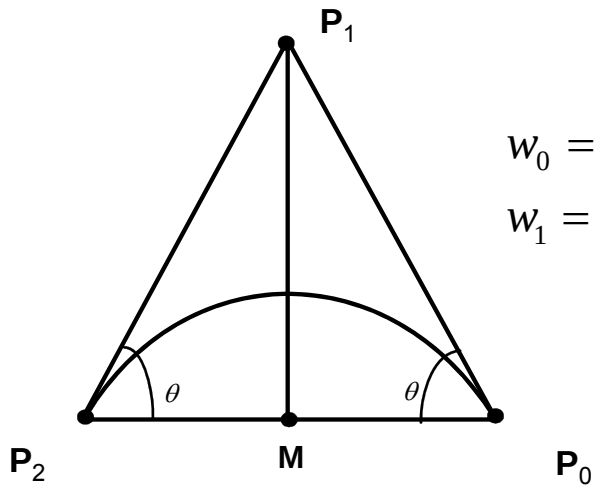
    - polynomial form in homogeneous coordinates

$$\mathbf{R}(t) = (xw, yw, zw, w)$$
$$= (2t, 1-t^2, 0, 1+t^2)$$

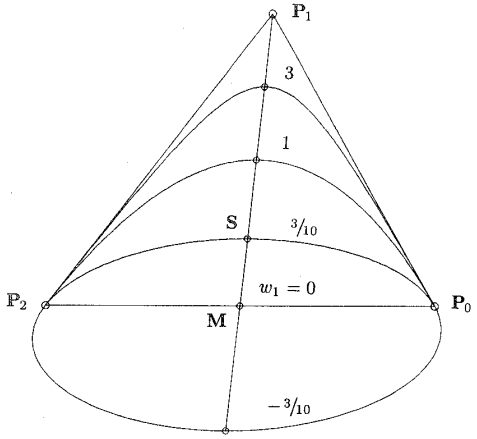
- Homogeneous coordinates에서 polynomial로 정확히 표현 가능

Quadratic Rational Polynomial Curve

- n차 polynomial curve
  - 차수를 아무리 높여도 conic curve를 근사적으로 밖에 표현 하지 못함
- Rational polynomial curve
  - 2차(quadratic)로 모든 종류의 conic curve를 정확히 표현함
  - NURB(Non-Uniform Rational B-spline)가 널리 쓰이는 이유



$$w_0 = w_2 = 1$$
$$w_1 = \cos\theta$$



$$w_0 = w_2 = 1,$$

$$\begin{cases} w_1 = 1 & \text{parabola} \\ 0 \le w_1 \le 1 & \text{ellipse} \\ w_1 > 1 & \text{hyperbola} \end{cases}$$

NURBS Curve

- Non-Uniform Rational B-spline Curve
- 가장 일반적 형태의 B-spline curve
- NURB curve data (in IGES)

p : degree

n : highest index of control points $(= number - 1)$

$\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n$: Euclidean control points

$w_0, w_1, \cdots, w_n$: weights

$t_0, t_1, \cdots, t_m$: knot vector $(m = n + p + 1)$

s_0, s_1 : start and end parameter values $(t_0 \le s_0 < s_1 \le t_m)$

- 참고사항
  - planar or nonplanar
  - open or closed
  - rational or nonrational
  - nonperiodic(clamped) or periodic(unclamped)

Quadratic Rational B-Spline

$k = 3 \ (\text{order})$

$n = 2 \ (\#\text{ of control points}(= n+1) = 3)$

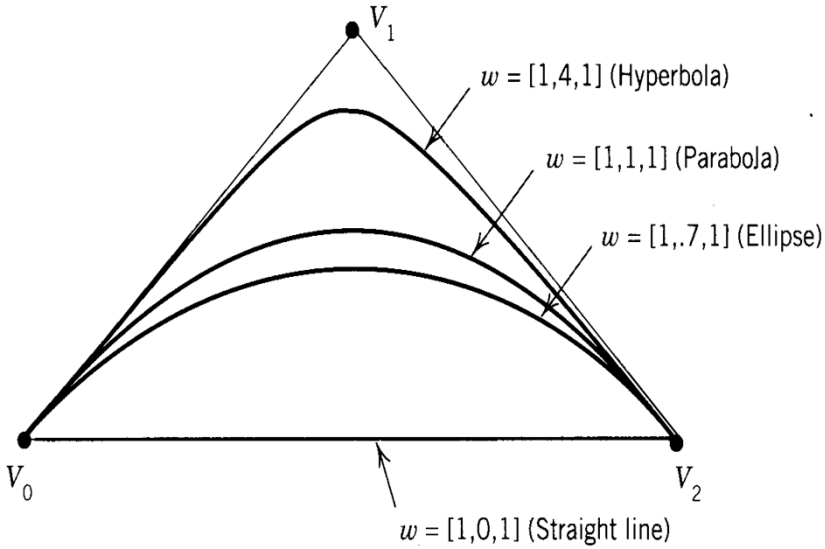
$m = n + k = 5$

knot vector : $\{0,0,0,1,1,1\}$

weight : $\{1, w_1, 1\}$

$$\mathbf{P}(u) = \frac{w_0 \mathbf{V}_0 N_{0,3}(u) + w_1 \mathbf{V}_1 N_{1,3}(u) + w_2 \mathbf{V}_2 N_{2,3}(u)}{w_0 N_{0,3}(u) + w_1 N_{1,3}(u) + w_2 N_{2,3}(u)}$$

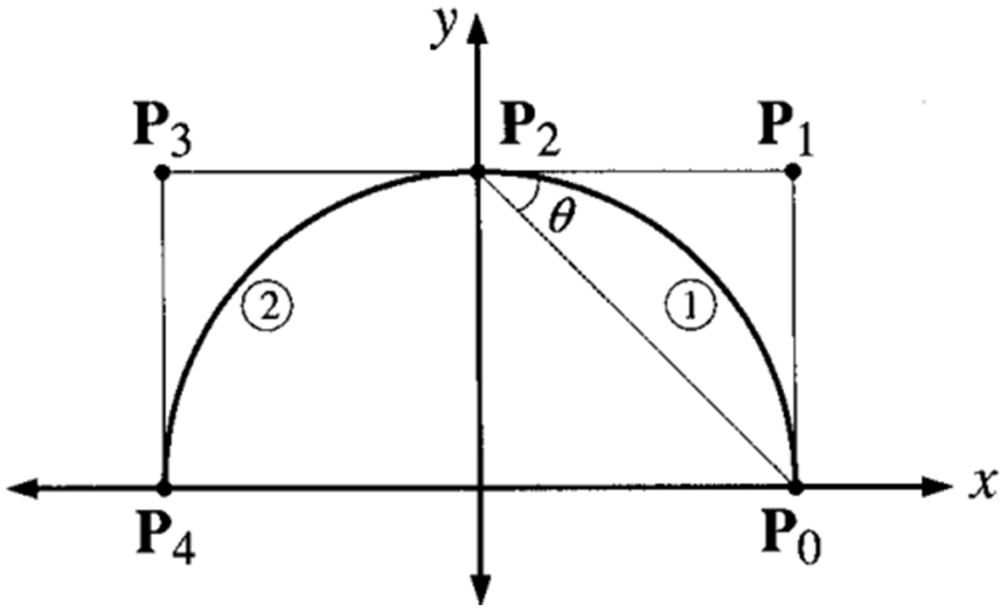
$$= \frac{\mathbf{V}_0 N_{0,3}(u) + w_1 \mathbf{V}_1 N_{1,3}(u) + \mathbf{V}_2 N_{2,3}(u)}{N_{0,3}(u) + w_1 N_{1,3}(u) + N_{2,3}(u)}$$



$$\begin{cases} w_1 = 0 & \text{straight line} \\ 0 \le w_1 \le 1 & \text{ellipse segment} \\ w_1 = \cos\theta & \text{circle} \\ w_1 = 1 & \text{parabola segment} \\ w_1 > 1 & \text{hyperbola segment} \end{cases}$$

Example (1)

- Derive a NURB representation of a half circle of radius 1 in the xy plane.
- Expand the NURBS equation of arc 1 and show that it represents the circular arc exactly.



Example (2)

degree $= 2$, order $k = 3$

half circle $\rightarrow$ two circular arcs $\left(\text{less than } 180°\right)$

$$\text{arc 1} \begin{cases} \mathbf{P}_0 = (1,0),\ \mathbf{P}_1 = (1,1),\ \mathbf{P}_0 = (0,1) \\ w_0 = 1,\ w_1 = \cos 45° = \dfrac{1}{\sqrt{2}},\ w_2 = 1 \\ knot: 0,0,0,1,1,1 \end{cases}$$

$$\text{arc 2} \begin{cases} \mathbf{P}_2 = (0,1),\ \mathbf{P}_3 = (-1,1),\ \mathbf{P}_4 = (-1,0) \\ w_2 = 1,\ w_3 = \cos 45° = \dfrac{1}{\sqrt{2}},\ w_4 = 1 \\ knot: 1,1,1,2,2,2 \end{cases}$$

$$\rightarrow \begin{cases} \mathbf{P}_0 = (1,0),\ \mathbf{P}_1 = (1,1),\ \mathbf{P}_2 = (0,1),\ \mathbf{P}_3 = (-1,1),\ \mathbf{P}_4 = (-1,0) \\ w_0 = 1,\ w_1 = \dfrac{1}{\sqrt{2}},\ w_2 = 1,\ w_3 = \dfrac{1}{\sqrt{2}},\ w_4 = 1 \\ knot: 0,0,0,1,1,2,2,2 \end{cases}$$

$$\mathbf{P}(u) = \frac{\sum_{i=0}^{n} w_i \mathbf{P}_i N_{i,k}(u)}{\sum_{i=0}^{n} w_i N_{i,k}(u)} = \frac{w_0 \mathbf{P}_0 N_{0,3}(u) + w_1 \mathbf{P}_1 N_{1,3}(u) + w_2 \mathbf{P}_2 N_{2,3}(u)}{w_0 N_{0,3}(u) + w_1 N_{1,3}(u) + w_2 N_{2,3}(u)}$$

How to Choose a Spline

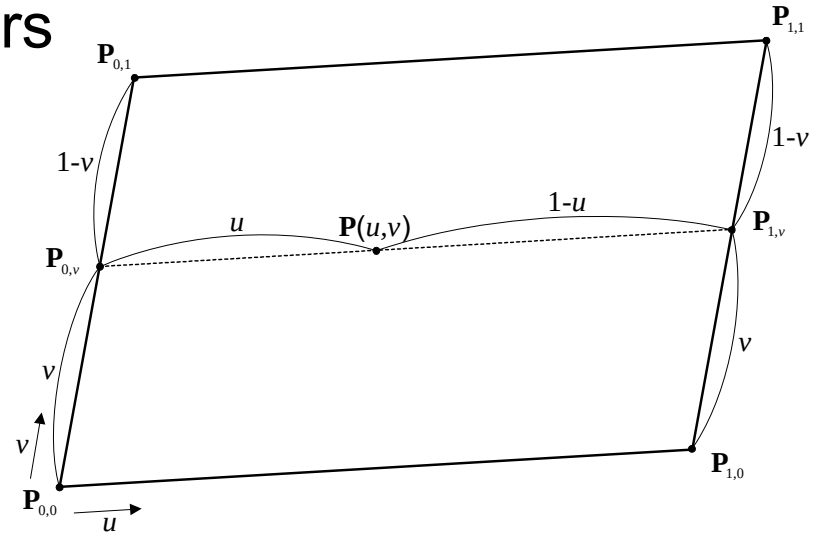
- Hermite curves are good for single segments where you know the parametric derivative or want easy control of it
- Bezier curves are good for single segments or patches where a user controls the points
- B-splines are good for large continuous curves and surfaces
- NURBS are the most general, and are good when that generality is useful, or when conic sections must be accurately represented (CAD)

Surfaces

- Bilinear Surface
- Coon’s Patch
- Bicubic Patch
- Bezier Surface
- B-Spline Surface
- NURBS Surface

Bilinear Surface

- A bilinear surface is derived by interpolating the four data points with the linear equations in the parameters *u* and *v* such that the resulting surface has the four points at its corners



$$\mathbf{P}_{0,v} = (1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0,1}$$

$$\mathbf{P}_{1,v} = (1-v)\mathbf{P}_{1,0} + v\mathbf{P}_{1,1}$$

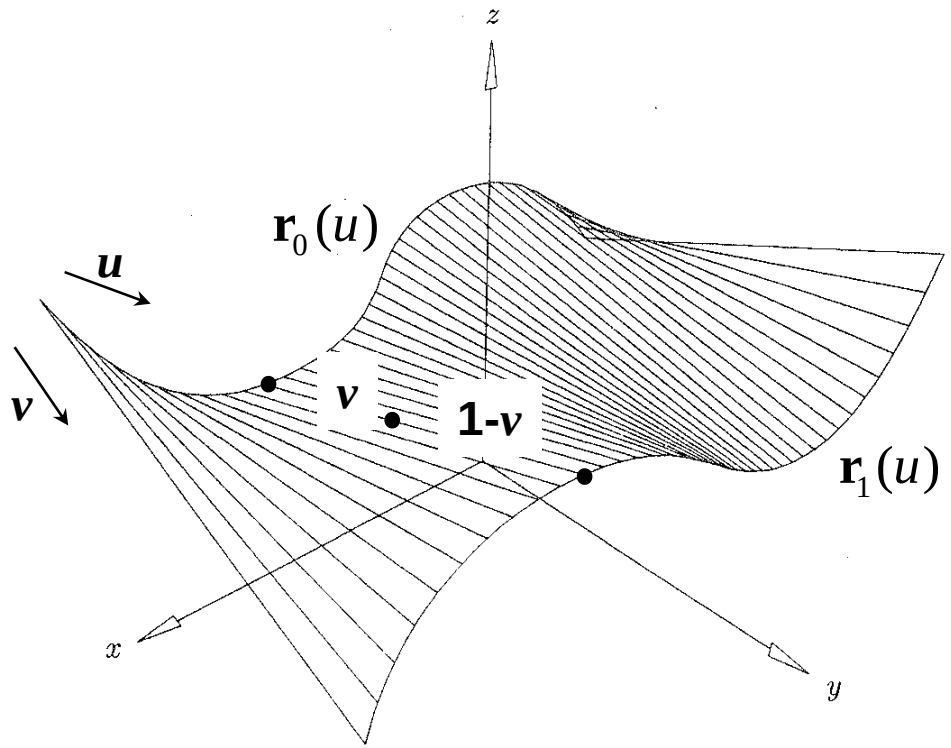
$$\mathbf{P}(u,v) = (1-u)\mathbf{P}_{0,v} + u\mathbf{P}_{1,v}$$

$$\mathbf{P}(u,v) = (1-u)\left[(1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0,1}\right] + u\left[(1-v)\mathbf{P}_{1,0} + v\mathbf{P}_{1,1}\right]$$

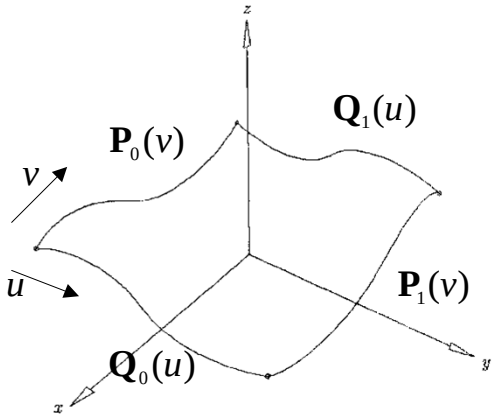
$$= \begin{bmatrix}(1-u)(1-v) & u(1-v) & (1-u)v & uv\end{bmatrix} \begin{bmatrix}\mathbf{P}_{0,0}\\ \mathbf{P}_{1,0}\\ \mathbf{P}_{0,1}\\ \mathbf{P}_{1,1}\end{bmatrix} \qquad \begin{pmatrix}0 \le u \le 1\\ 0 \le v \le 1\end{pmatrix}$$

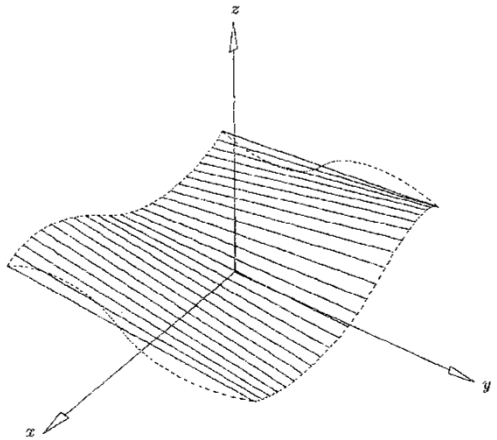
Ruled Surface

$$\mathbf{r}(u,v) = (1-v)\mathbf{r}_0(u) + v\mathbf{r}_1(u), \qquad 0 \le v \le 1$$

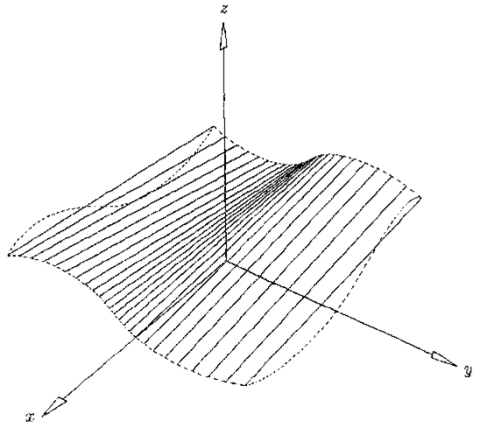


Coon's Patch: 4 boundary curves





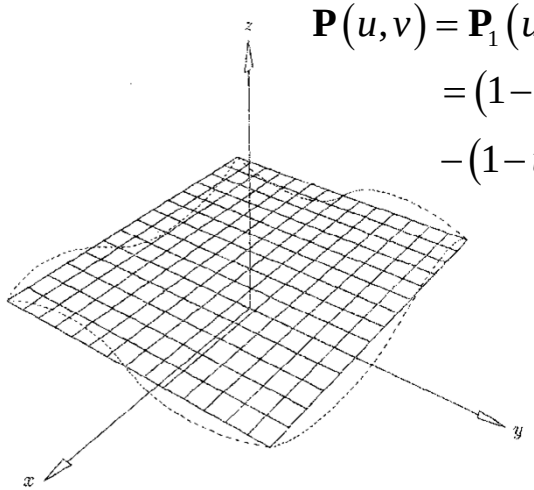
$$\mathbf{P}_1(u,v) = (1-u)\mathbf{P}_0(v) + u\mathbf{P}_1(v)$$

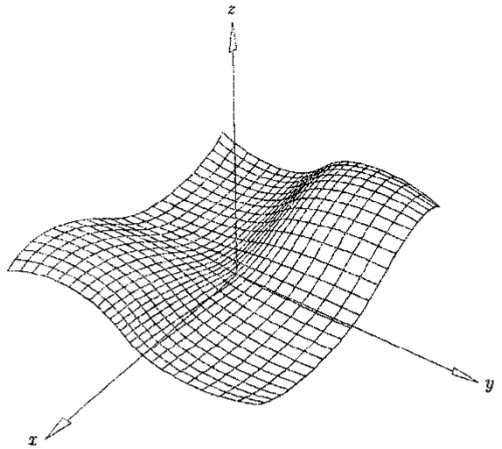


$$\mathbf{P}_2(u,v) = (1-v)\mathbf{Q}_0(u) + v\mathbf{Q}_1(u)$$

$$\begin{aligned}\mathbf{P}(u,v) &= \mathbf{P}_1(u,v) + \mathbf{P}_2(u,v) - \mathbf{P}_3(u,v)\\ &= (1-u)\mathbf{P}_0(v) + u\mathbf{P}_1(v) + (1-v)\mathbf{Q}_0(u) + v\mathbf{Q}_1(u)\\ &\quad -(1-u)(1-v)\mathbf{P}_{0,0} - u(1-v)\mathbf{P}_{1,0} - (1-u)v\mathbf{P}_{0,1} - uv\mathbf{P}_{1,1}\end{aligned}$$

$$0 \le u \le 1,\ 0 \le v \le 1$$





$$\boldsymbol{P}_3(u,v) = (1-u)(1-v)\boldsymbol{P}_{0,0} + u(1-v)\boldsymbol{P}_{1,0} + (1-u)v\boldsymbol{P}_{0,1} + uv\boldsymbol{P}_{1,1}$$

Bicubic Patch (1)

- Bicubic Patch
  - Extension of the parametric cubic curve formulation
  - Boundary curves are parametric cubics or Hermites
  - The Interior is defined by blending functions
- Algebraic form

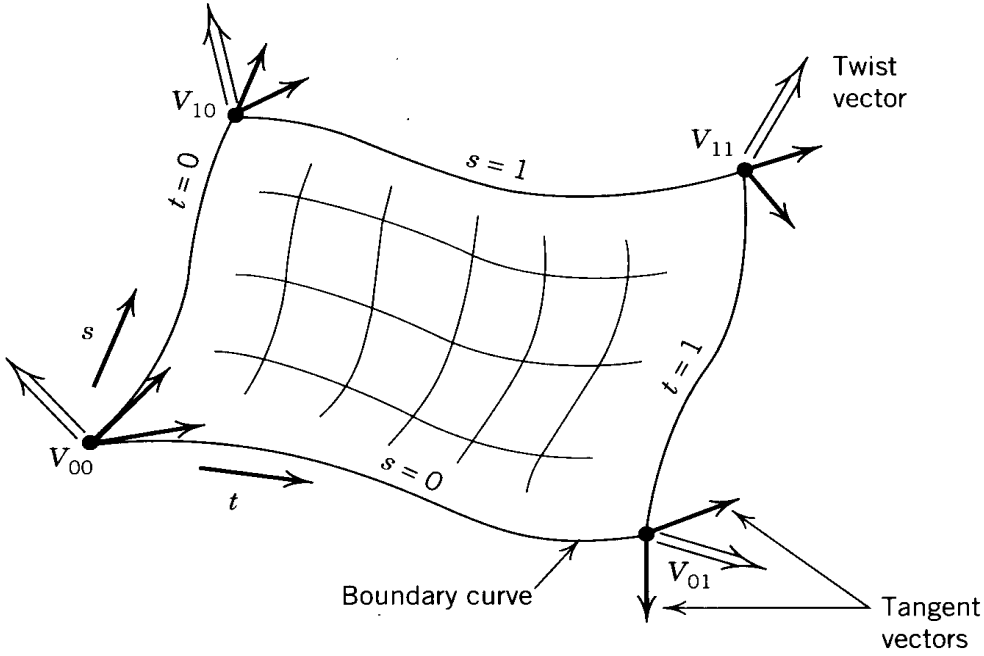
$$\mathbf{P}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3}\mathbf{a}_{ij}u^i v^j \qquad 0 \le u \le 1, 0 \le v \le 1$$

- Matric form

$$\mathbf{P}(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} & \mathbf{a}_{02} & \mathbf{a}_{03} \\ \mathbf{a}_{10} & \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{30} & \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

Bicubic Patch (2)

- Geometric Form
  - 16 vector equations are required for 16 algebraic coefficient vectors
  - 16 Boundary conditions
    - 4 corner points: P(0,0), P(0,1), P(1,0), P(1,1)
    - 8 tangent vectors at corner points: $P_u(0,0)$, $P_u(0,1)$, $P_u(1,0)$, $P_u(1,1)$, $P_v(0,0)$, $P_v(0,1)$, $P_v(1,0)$, $P_v(1,1)$
    - 4 twist vectors at corner points: $P_{uv}(0,0)$, $P_{uv}(0,1)$, $P_{uv}(1,0)$, $P_{uv}(1,1)$



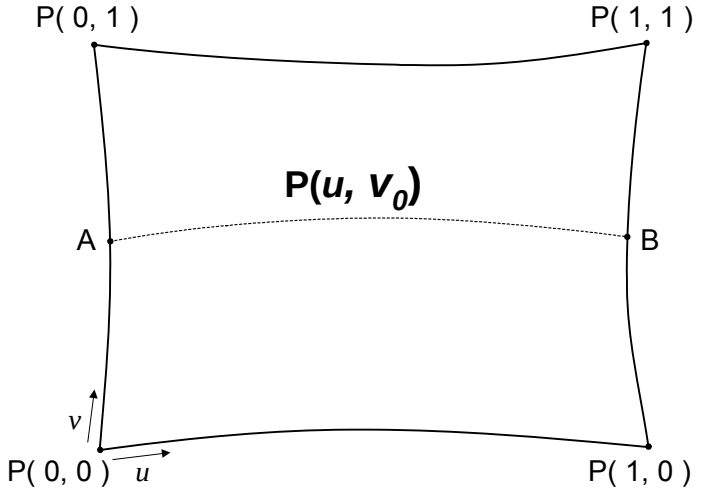
Bicubic Patch (3)

$$\mathbf{P}(u,v) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} \mathbf{P}(0,0) & \mathbf{P}(0,1) & \mathbf{P}_v(0,0) & \mathbf{P}_v(0,1) \\ \mathbf{P}(1,0) & \mathbf{P}(1,1) & \mathbf{P}_v(1,0) & \mathbf{P}_v(1,1) \\ \mathbf{P}_u(0,0) & \mathbf{P}_u(0,1) & \mathbf{P}_{uv}(0,0) & \mathbf{P}_{uv}(0,1) \\ \mathbf{P}_u(1,0) & \mathbf{P}_u(1,1) & \mathbf{P}_{uv}(1,0) & \mathbf{P}_{uv}(1,1) \end{bmatrix} \begin{bmatrix} F_1(v) \\ F_2(v) \\ F_3(v) \\ F_4(v) \end{bmatrix}$$

$$0 \le u \le 1,\ 0 \le v \le 1$$

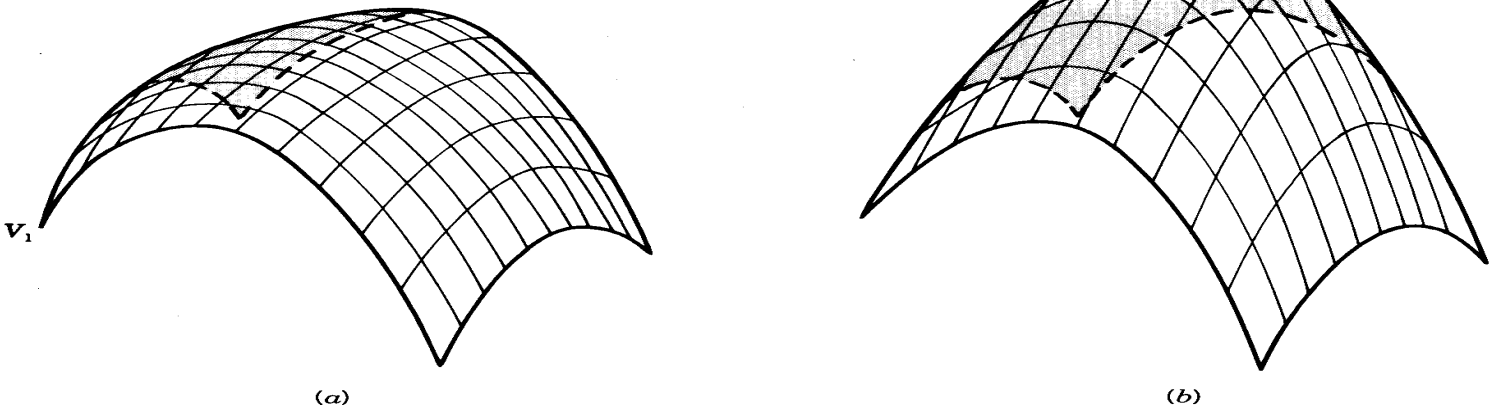
- Blending Functions (Hermite Curve Eqns)

$$F_1(u) = 1 - 3u^2 + 2u^3$$
$$F_2(u) = 3u^2 - 2u^3$$
$$F_3(u) = u - 2u^2 + u^3$$
$$F_4(u) = -u^2 + u^3$$

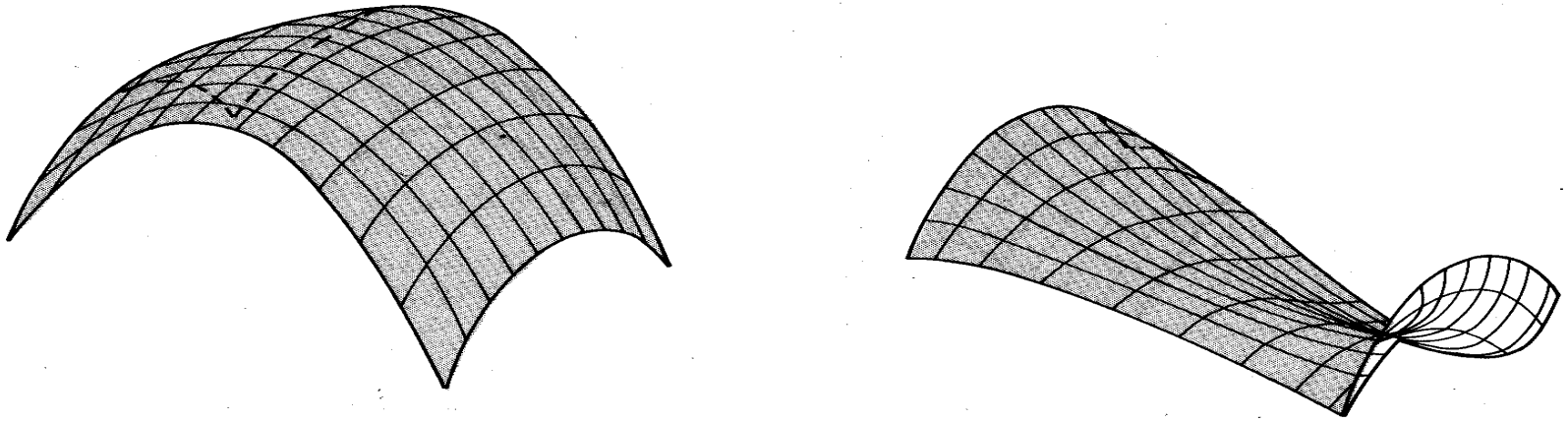


Bicubic Patch (4)

- Effect of Variation in Twist Vector



- Effect of Variation in Tangent Vector



Ferguson Patch (F-Patch)

- Definition
  - Setting all twist vector to zero
    - $P_{uv}(0,0) = P_{uv}(0,1) = P_{uv}(1,0) = P_{uv}(1,1) = 0$
  - Not commonly used in practice because they force the surface to flatten at the corners

- Disadvantage
  - No intuitive feel for the values of the tangent and twist vectors is available to the user

Bezier Surface

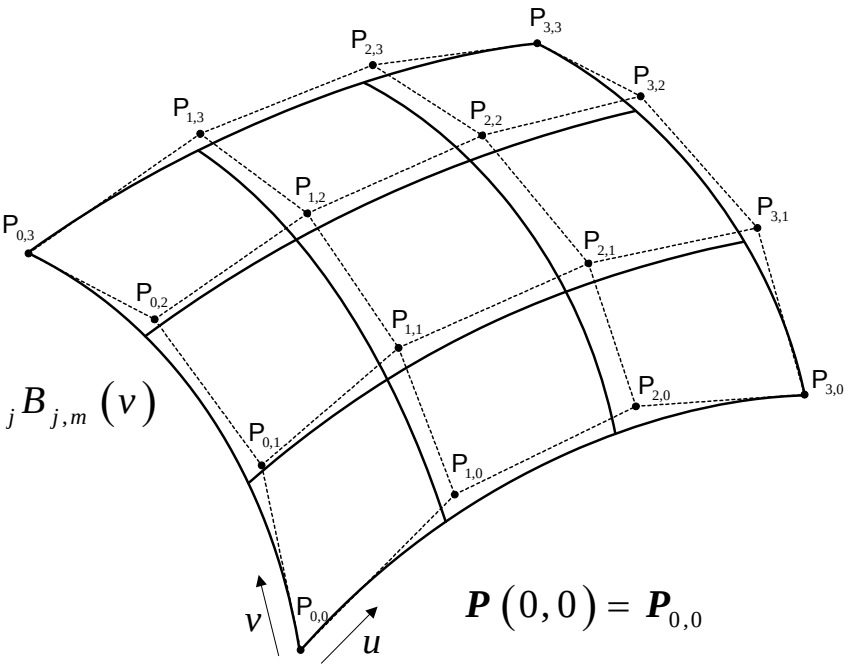
$$\boldsymbol{P}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \boldsymbol{P}_{i,j} B_{i,n}(u) B_{j,m}(v) \quad 0 \le u \le 1, \quad 0 \le v \le 1$$

$$= \sum_{i=0}^{n} \left[\boldsymbol{P}_{i,0} B_{0,m}(v) + \boldsymbol{P}_{i,1} B_{1,m}(v) + \cdots + \boldsymbol{P}_{i,m} B_{m,m}(v) \right] B_{i,n}(u)$$

$\boldsymbol{P}_{i,j}$: control points

$B_{i,n}(u), B_{j,m}(v)$: Bernstein blending functions in the u and v directions

Note : n does not have to be the same as m



$$\boldsymbol{P}(0,v) = \sum_{j=0}^{m} \boldsymbol{P}_{0,j} B_{j,m}(v)$$

$$\boldsymbol{P}(0,0) = \boldsymbol{P}_{0,0}$$

$\boldsymbol{P}_{1,0}, \boldsymbol{P}_{2,0}, \boldsymbol{P}_{1,3}, \boldsymbol{P}_{2,3}$

$\leftrightarrow \boldsymbol{P}_u(0,0), \boldsymbol{P}_u(1,0), \boldsymbol{P}_u(0,1), \boldsymbol{P}_u(1,1)$

$\boldsymbol{P}_{0,1}, \boldsymbol{P}_{0,2}, \boldsymbol{P}_{3,1}, \boldsymbol{P}_{3,2}$

$\leftrightarrow \boldsymbol{P}_v(0,0), \boldsymbol{P}_v(0,1), \boldsymbol{P}_v(1,0), \boldsymbol{P}_v(1,1)$

$\boldsymbol{P}_{1,1}, \boldsymbol{P}_{2,1}, \boldsymbol{P}_{1,2}, \boldsymbol{P}_{2,2}$

$\leftrightarrow$ twisting vectors (internal shape)

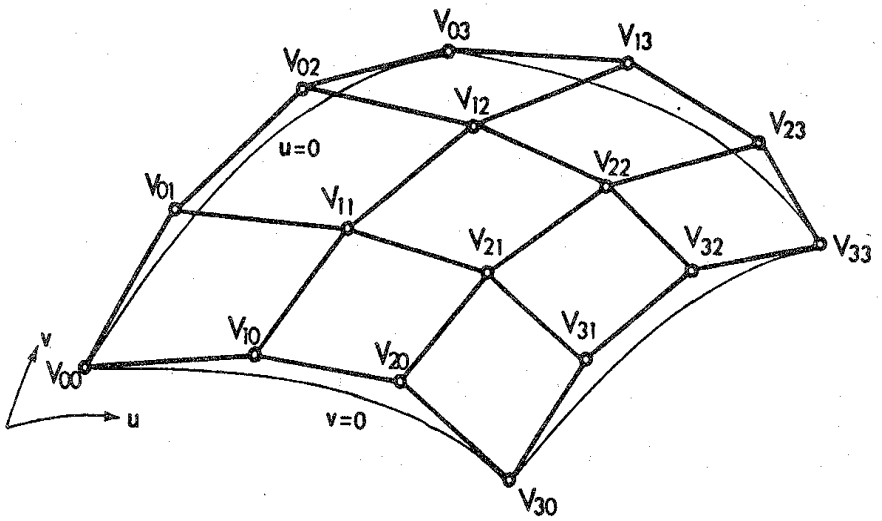
Bicubic Bezier Surface

$$\mathbf{P}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} B_i^3(u)B_j^3(v)\mathbf{V}_{ij}$$

$$= \sum_{i=0}^{3} B_i^3(u)\left(\sum_{j=0}^{3} B_j^3(v)\mathbf{V}_{ij}\right)$$

$$= \sum_{i=0}^{3} B_i^3(u)\mathbf{b}_i(v)$$

$$= \mathbf{UMBM}^T\mathbf{V}^T$$

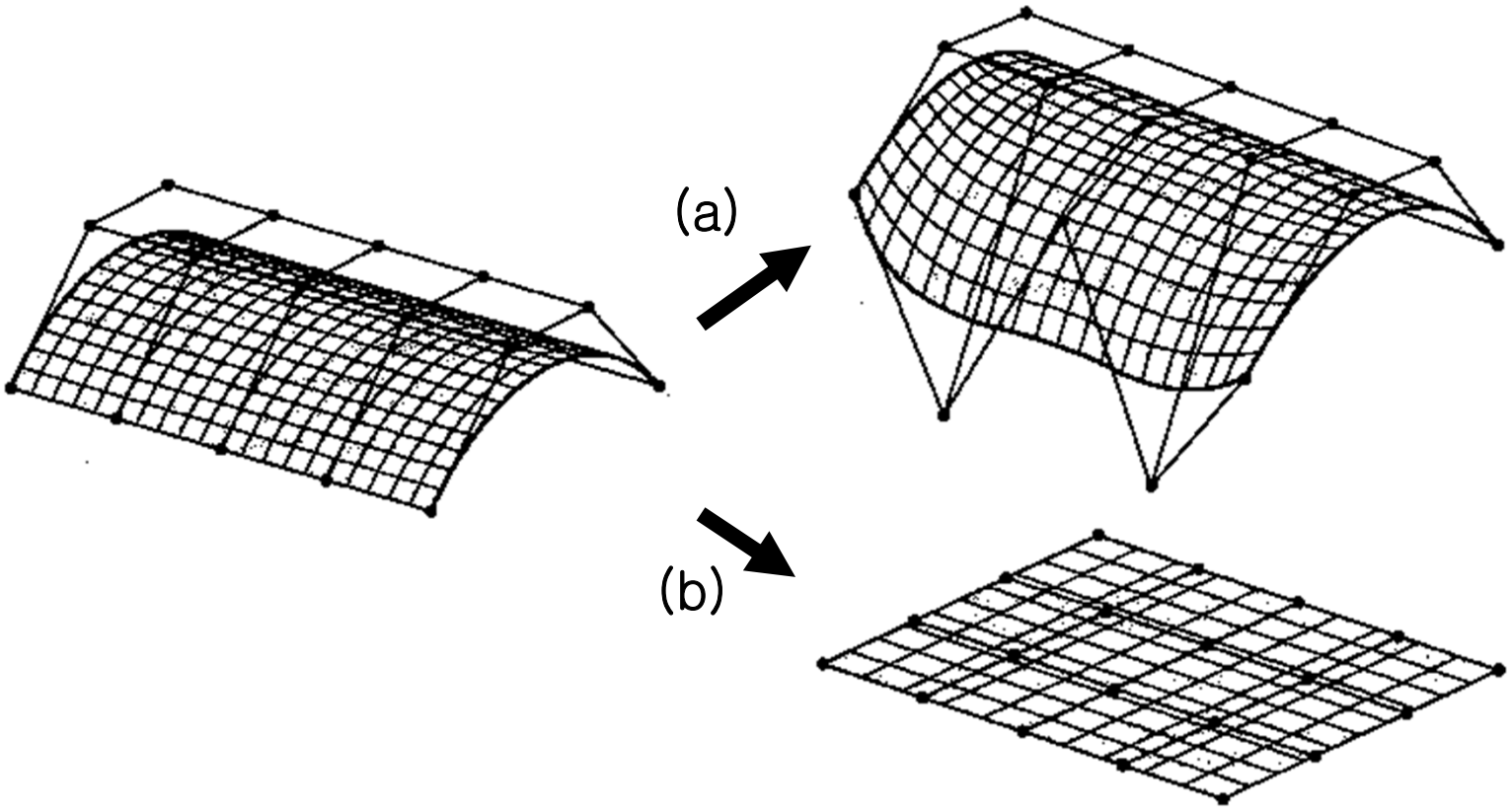


$$\mathbf{U} = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1 & v & v^2 & v^3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{V}_{00} & \mathbf{V}_{01} & \mathbf{V}_{02} & \mathbf{V}_{03} \\ \mathbf{V}_{10} & \mathbf{V}_{11} & \mathbf{V}_{12} & \mathbf{V}_{13} \\ \mathbf{V}_{20} & \mathbf{V}_{21} & \mathbf{V}_{22} & \mathbf{V}_{23} \\ \mathbf{V}_{30} & \mathbf{V}_{31} & \mathbf{V}_{32} & \mathbf{V}_{33} \end{bmatrix}$$

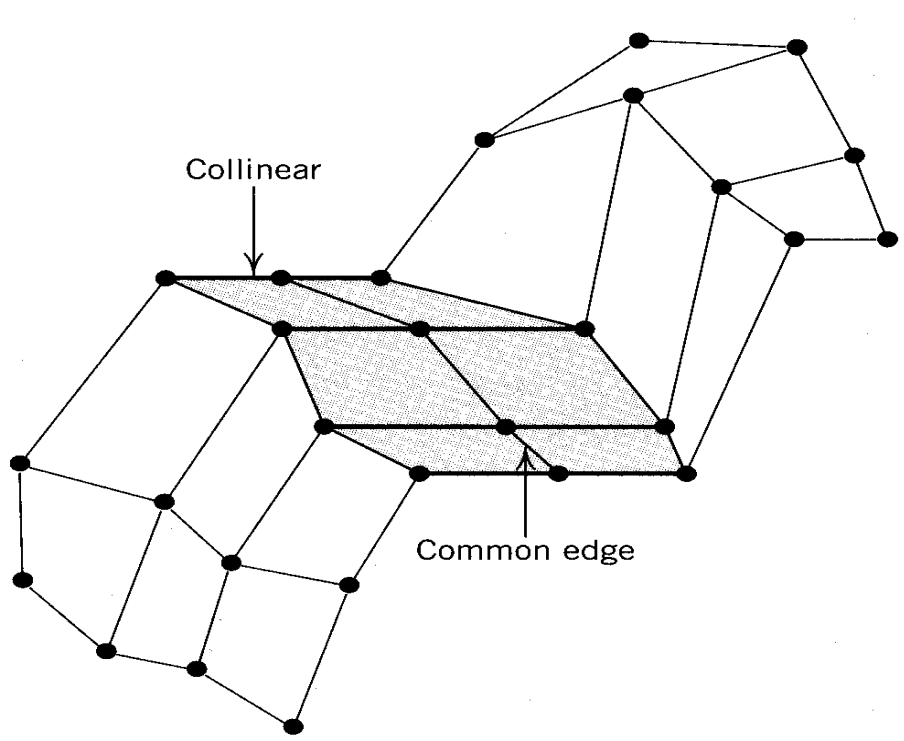
Effect of Moving Control Points

- (a) At the boundary curves
- (b) On the interior part of the surface



Continuity of Bicubic Bezier Surfaces

– First degree parametric continuity is enforced along the common edge between two patches



B-Spline Surface

$$\mathbf{P}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{P}_{i,j} N_{i,k}(u) N_{j,l}(v) \qquad \left(s_{k-1} \le u \le s_{n+1},\ t_{l-1} \le v \le t_{m+1}\right)$$

$\mathbf{P}_{i,j}$: control points

$N_{i,k}(u), N_{j,l}(v)$: B – spline blending functions in the u and v directions

Note : k does not have to be the same as l

$$\mathbf{P}(0,v) = \sum_{j=0}^{m}\left[\sum_{i=0}^{n} \mathbf{P}_{i,j} N_{i,k}(u)\right]_{u=0} N_{j,l}(v) = \sum_{j=0}^{m} \mathbf{P}_{0,j} N_{j,l}(v) \leftarrow \left[\sum_{i=0}^{n} \mathbf{P}_{i} N_{i,k}(u)\right]_{u=0} = \mathbf{P}_0$$

$$\mathbf{P}(u_0,v_0) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{P}_{i,j} N_{i,k}(u_0) N_{j,l}(v_0)$$

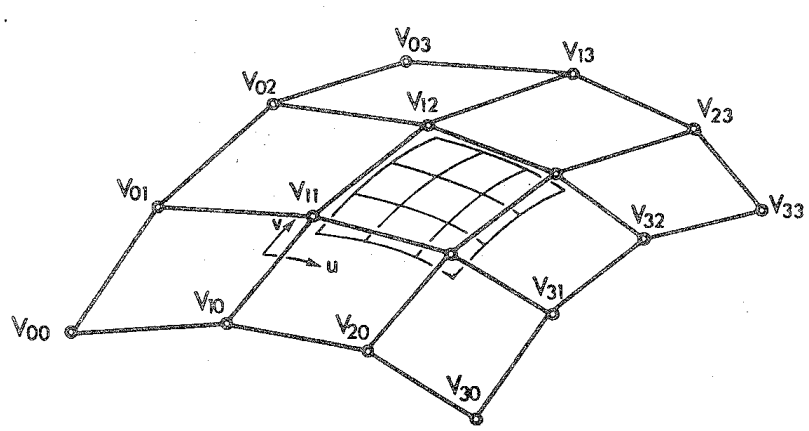
$$= \left[\sum_{j=0}^{m} \mathbf{P}_{0,j} N_{j,l}(v_0)\right] N_{0,k}(u_0) + \cdots + \left[\sum_{j=0}^{m} \mathbf{P}_{n,j} N_{j,l}(v_0)\right] N_{n,k}(u_0)$$

$$= \mathbf{C}_0 N_{0,k}(u_0) + \mathbf{C}_1 N_{1,k}(u_0) + \cdots + \mathbf{C}_n N_{n,k}(u_0)$$

Bicubic B-spline Surface (Uniform)

$$\mathbf{P}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} N_{i,3}(u)\, N_{j,3}(v)\, \mathbf{V}_{ij}$$

$$= \mathbf{UNBN}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \qquad (0 \le u,v \le 1)$$

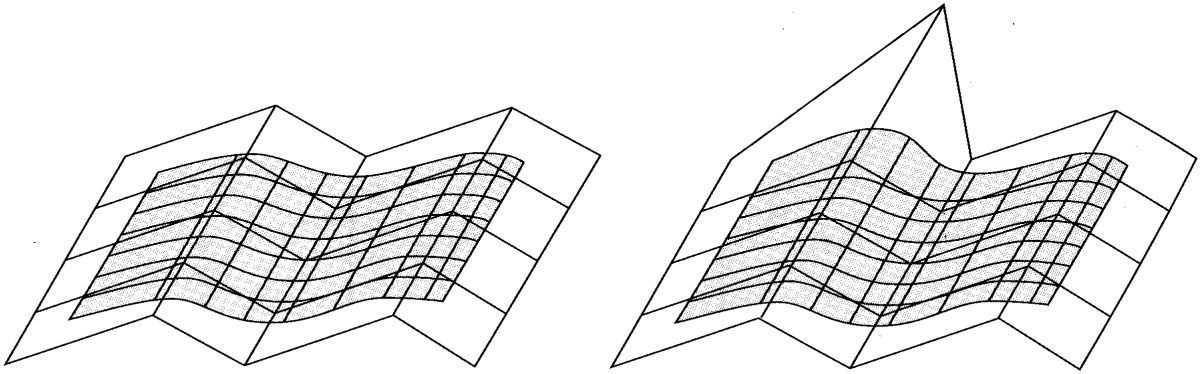


$$\boldsymbol{U} = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix},\ \boldsymbol{V} = \begin{bmatrix} 1 & v & v^2 & v^3 \end{bmatrix}$$

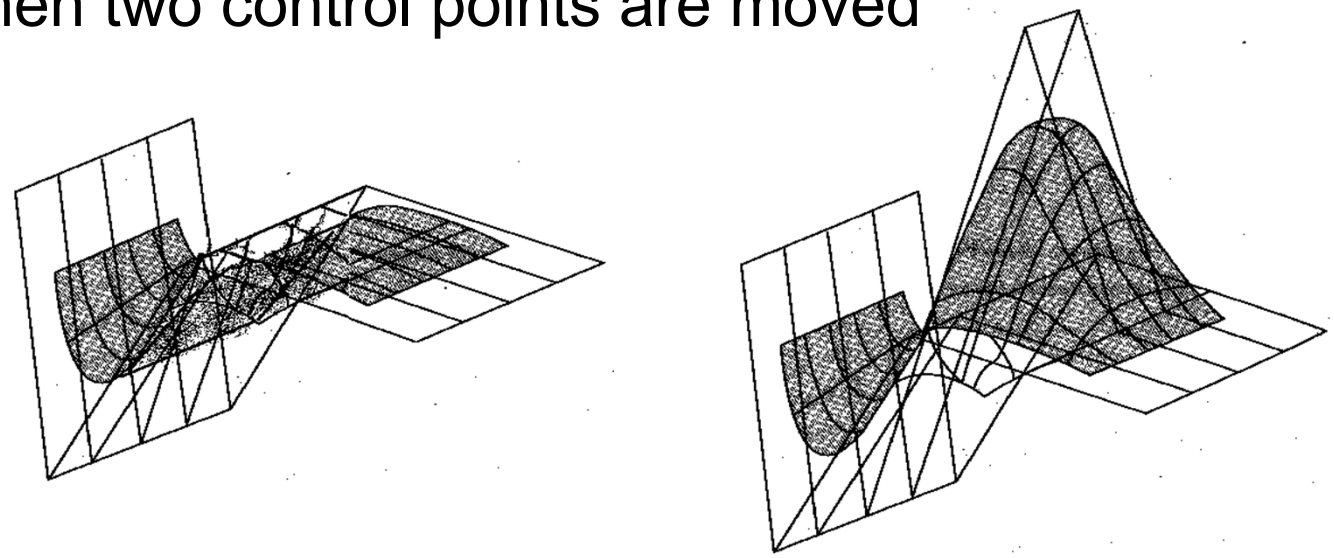
$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{V}_{00} & \boldsymbol{V}_{01} & \boldsymbol{V}_{02} & \boldsymbol{V}_{03} \\ \boldsymbol{V}_{10} & \boldsymbol{V}_{11} & \boldsymbol{V}_{12} & \boldsymbol{V}_{13} \\ \boldsymbol{V}_{20} & \boldsymbol{V}_{21} & \boldsymbol{V}_{22} & \boldsymbol{V}_{23} \\ \boldsymbol{V}_{30} & \boldsymbol{V}_{31} & \boldsymbol{V}_{32} & \boldsymbol{V}_{33} \end{bmatrix},\ \boldsymbol{N} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

B-Spline Surface

- When one control point is moved
  - only a small portion of the B-spline surface is affected



- When two control points are moved

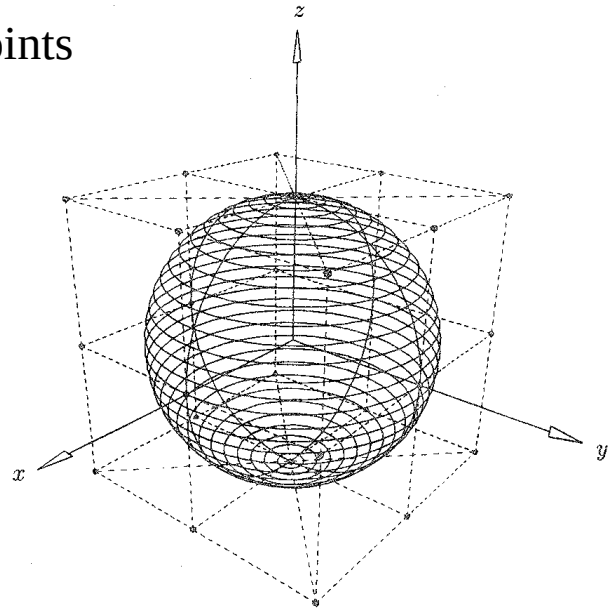


NURBS Surface (1)

$$\mathbf{P}(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} h_{i,j}\ \mathbf{P}_{i,j}\ N_{i,k}(u)\ N_{j,l}(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m} h_{i,j}\ N_{i,k}(u)\ N_{j,l}(v)} \qquad \begin{pmatrix} s_{k-1} \le u \le s_{n+1} \\ t_{l-1} \le v \le t_{m+1} \end{pmatrix}$$

$\mathbf{P}_{i,j}$: x, y and z coordinates of the control points

$h_{i,j}$: homogeneous coordinates of the control points

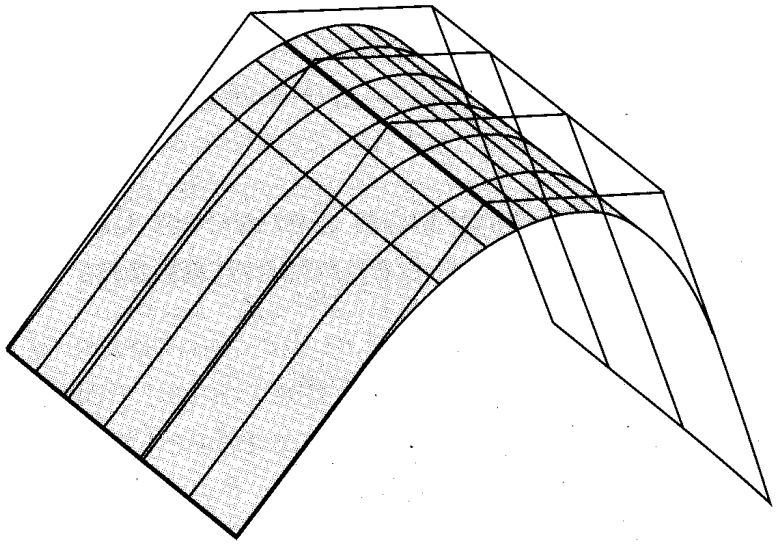
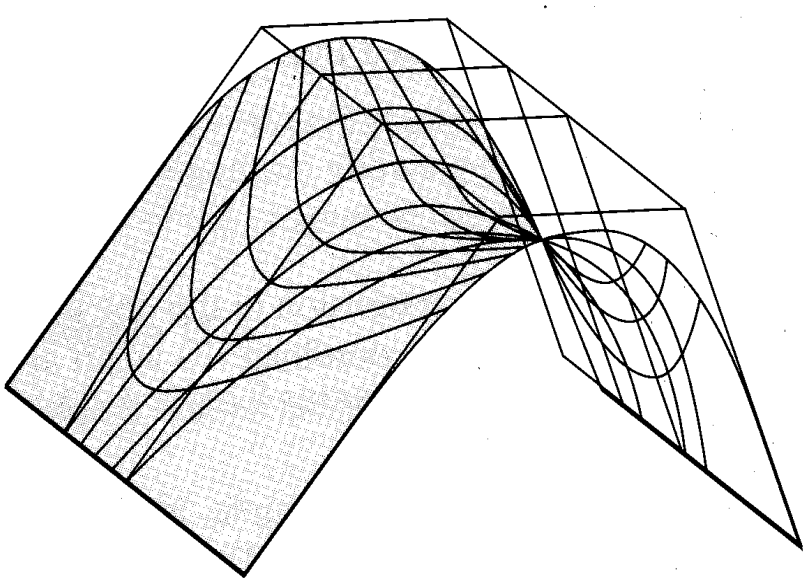


– Quadric (Quadratic) NURBS Surface로 Cylinder, Cone, Sphere, Paraboloid, Hyperboloid를 정확히 나타낼 수 있다.

NURBS Surface (2)

- Effect of Weights
  - The weights provide an additional degree of freedom for the shape of surface

– Larger values of weights at the interior control points



– Lower values of weights at the top interior control points