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Introduction

PDE	order	linearity
$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1$	2	*linear*
$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y$	3	*linear*
$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x$	3	*nonlinear*
$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = y$	2	*nonlinear*

linear second-order PDEs

with two independent variable:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$$\begin{cases} A, B, C: \text{ functions of } x \text{ and } y \\ D: \text{ functions of } x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \end{cases}$$

depending on the values of the coefficients of the second-derivative terms

$$\rightarrow B^2 - 4AC \begin{cases} < 0: \text{ elliptic} \\ = 0: \text{ parabolic} \\ > 0: \text{ hyperbolic} \end{cases}$$

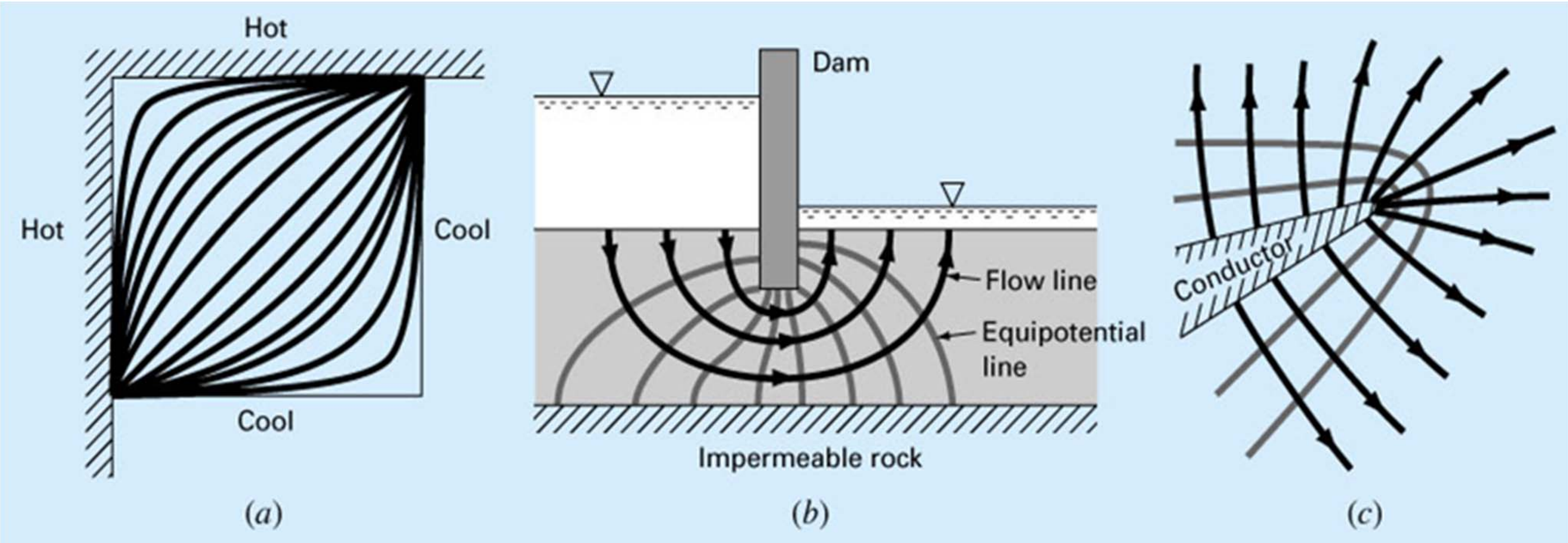
Classification of PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$B^2 - 4AC$	category	example
< 0	elliptic	$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \begin{cases} 0 & (\text{Laplace equation}) \\ f(x, y) & (\text{Poisson equation}) \end{cases}$
$= 0$	parabolic	$\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$ (heat conduction equation)
> 0	hyperbolic	$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ (wave equation)

Elliptic Equations

- Characterize steady-state systems
  - Steady-state distribution of an unknown in two spatial dimensions (no time derivative)
    - (a) temperature distribution on a heated plate
    - (b) seepage of water under a dam
    - (c) electric field near the point of a conductor



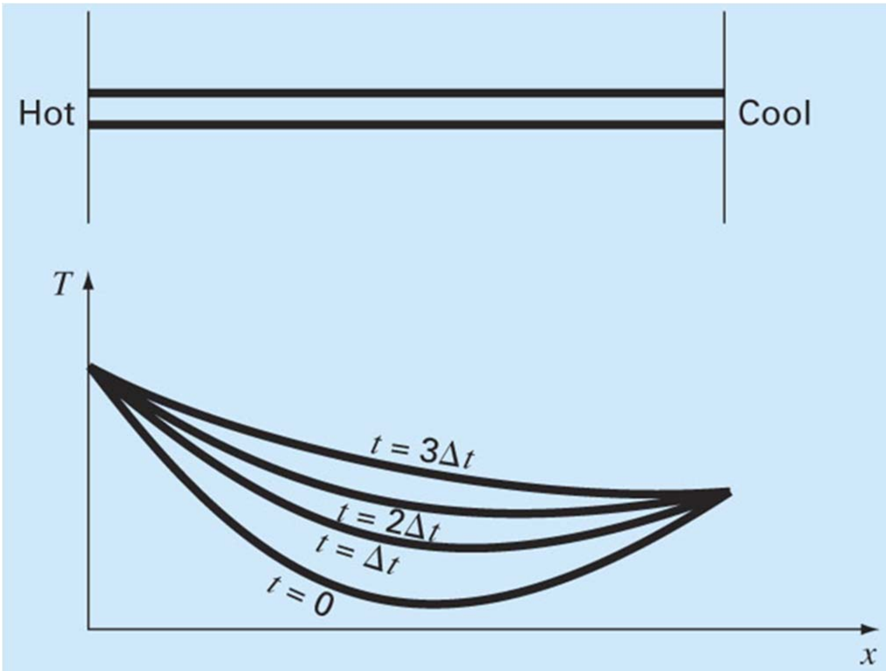
Comparison of Analogous Properties

Generic Term	Discipline		
	Elasticity	Heat Conduction	Magnetostatics
Potential function	$\{u\}$	u	$\{A\}$
"Strain"	$\{\varepsilon\}$	$-\{\nabla u\}$	$\{B\}$
"Stress"	$\{\sigma\}$	$\{q\}$	$\{H\}$
Load Density	$\{p\}$	$\{Q\}$	$\{J\}$
Boundary Load	$\{t\}$	$\{q^b\}$	$H \times n$
Definition of Strain	$\{\varepsilon\} = [L]\{u\}$	$-\{\nabla u\}$	$\{B\} = \nabla \times \{A\}$
Constitutive Relationship	$\{\sigma\} = [D]\{\varepsilon\}$	$\{q\} = -[K]\{\nabla u\}$	$\{H\} = [\mu]^{-1}[B]$
Potential Energy	$\frac{1}{2}\{\sigma\}^T\{\varepsilon\}$		$\frac{1}{2}\{H\}^T\{B\}$
Equilibrium	$\Sigma\, \sigma_{ij,j} = -P_i$	$\nabla \cdot \{q\} = Q$	$\nabla \times \{H\} = \{J\}$
Inertia Loading	$\{p^i\} = -\rho\{u_{,tt}\}$	$Q^s = -\rho c u_{,t}$	

R. H. MacNeal, Finite Elements: Their Design and Performance, Marcel Dekker, 1994

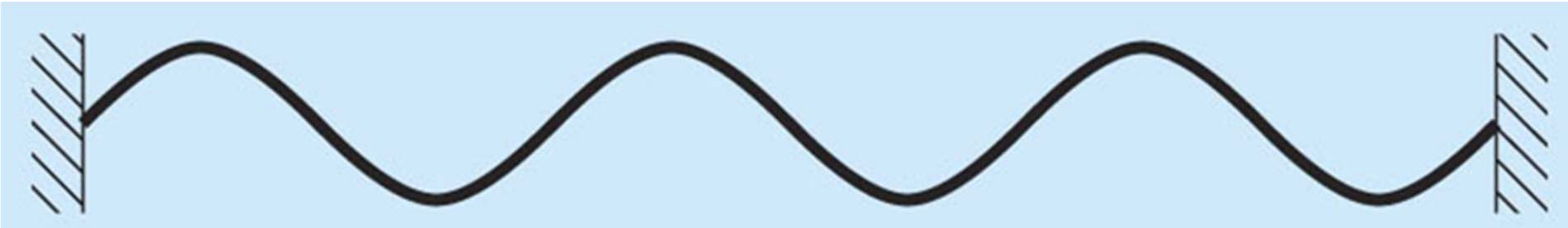
Parabolic Equations

- Determine how an unknown varies in both space and time
- Propagation problems: solution changes in time
  - Long, thin rod insulated everywhere but at its end

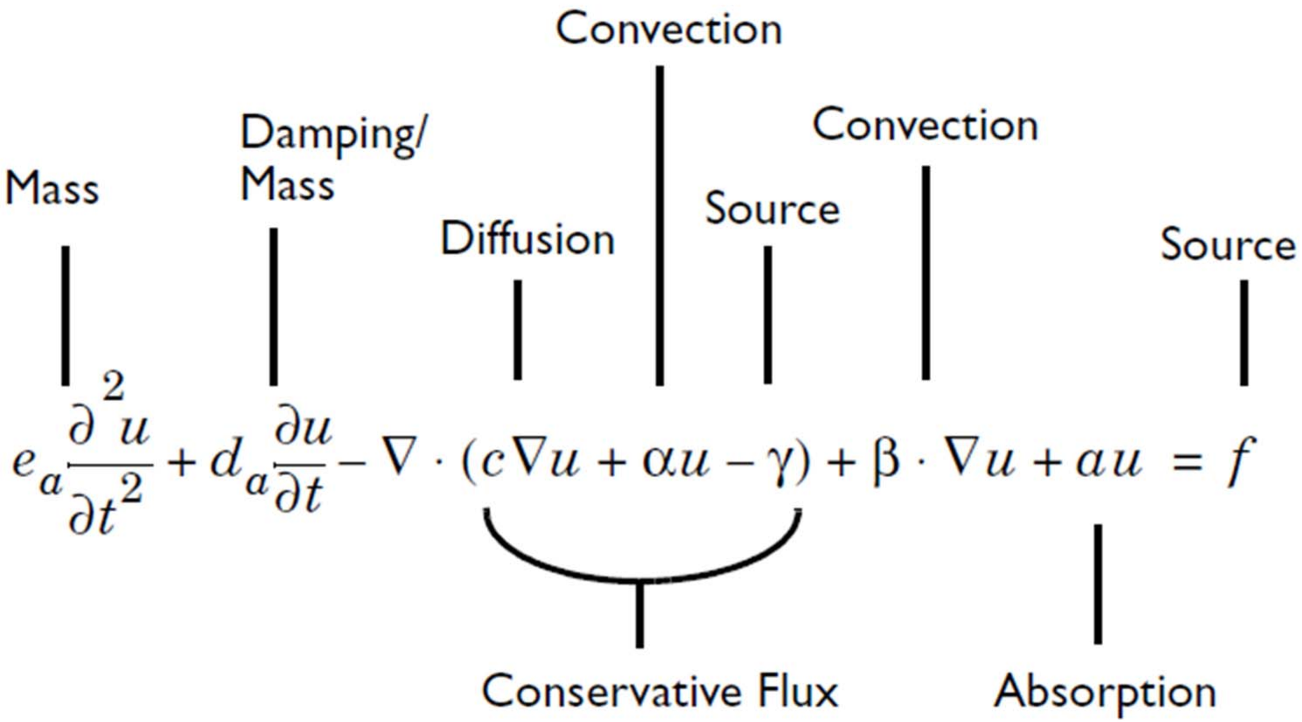


Hyperbolic Equations

- Propagation problems
  - Unknown is characterized by a second derivative w.r.t. time
  - Solution oscillates
    - Taut string vibrating at a low amplitude



COMSOL: Coefficient Form PDE



$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} - \nabla \cdot (c \nabla u + \alpha u - \gamma) + \beta \cdot \nabla u + a u = f$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \quad (\text{vector differential operator, gradient})$$

$$\nabla u = grad(u)$$

$$\nabla \cdot \mathbf{u} = \mathrm{div}(\mathbf{u})$$

$$\nabla \times \mathbf{u} = \mathrm{curl}(\mathbf{u})$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad (\text{Laplace operator})$$

$$\nabla \cdot (c \nabla u) = \frac{\partial}{\partial x_1} \left(c \frac{\partial u}{\partial x_1} \right) + \cdots + \frac{\partial}{\partial x_n} \left(c \frac{\partial u}{\partial x_n} \right)$$

$$\beta \cdot \nabla u = \beta_1 \frac{\partial u}{\partial x_1} + \cdots + \beta_n \frac{\partial u}{\partial x_n}$$

COMSOL: Classical PDEs

EQUATION	COMPACT NOTATION	STANDARD NOTATION (2D)
Laplace's equation	$-\nabla\cdot(\nabla u) = 0$	$-\frac{\partial}{\partial x}\frac{\partial u}{\partial x}-\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = 0$
Poisson's equation	$-\nabla\cdot(c\nabla u) = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Helmholtz equation	$-\nabla\cdot(c\nabla u)+au = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)+au = f$
Heat equation	$d_a\frac{\partial u}{\partial t}-\nabla\cdot(c\nabla u) = f$	$d_a\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Wave equation	$e_a\frac{\partial^2 u}{\partial t^2}-\nabla\cdot(c\nabla u) = f$	$e_a\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Convection-diffusion equation	$d_a\frac{\partial u}{\partial t}-\nabla\cdot(c\nabla u)+\beta\cdot\nabla u = f$	$d_a\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)+\beta_x\frac{\partial u}{\partial x}+\beta_y\frac{\partial u}{\partial y} = f$

Elliptic: Laplace Equation

- Model a variety of problems involving the potential of an unknown variable
  - Heated thin rectangular plate
    - Insulated everywhere but at its edges
    - Heat balance at steady state: (In) - (Out) = 0

$$q(x)(\Delta y\Delta z)\Delta t + q(y)(\Delta x\Delta z)\Delta t = q(x+\Delta x)(\Delta y\Delta z)\Delta t + q(y+\Delta y)(\Delta x\Delta z)\Delta t$$

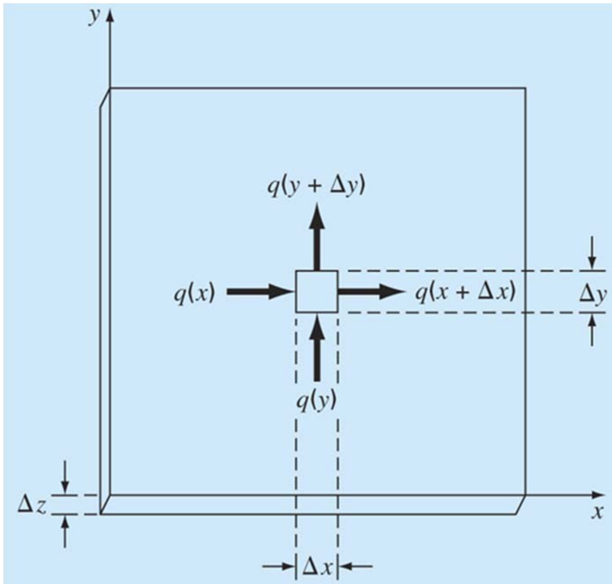
$$\xrightarrow{\div\Delta z\Delta t}\left[q(x)-q(x+\Delta x)\right]\Delta y + \left[q(y)-q(y+\Delta y)\right]\Delta x = 0$$

$$\rightarrow \frac{q(x)-q(x+\Delta x)}{\Delta x}\Delta x\Delta y + \frac{q(y)-q(y+\Delta y)}{\Delta y}\Delta x\Delta y = 0$$

$$\xrightarrow[\text{taking the limit}]{\div\Delta x\Delta y} -\frac{\partial q}{\partial x} - \frac{\partial q}{\partial y} = 0 \left(\leftarrow \text{need heat fluxes at the plate's edges}\right)$$

$$\left(\underbrace{\text{heat flux} \rightarrow \text{temperature}}_{\text{Fourier's law of heat conduction}}\right) q_i = -k'\frac{\partial T}{\partial i}\begin{cases}\xrightarrow{k'=k\rho C} q_i = -k\rho C\dfrac{\partial T}{\partial i} \\ \xrightarrow{H=\rho CVT} q_i = -\dfrac{k'}{\rho CV}\dfrac{\partial H}{\partial i}\end{cases}$$

$$-\frac{\partial q}{\partial x} - \frac{\partial q}{\partial y} = 0 \rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



$$\begin{cases} \sinh x = \dfrac{e^{x}-e^{-x}}{2} \\ \sin^{2} x = \dfrac{e^{ix}-e^{-ix}}{2i} \end{cases}$$

Analytic Solution

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \text{ and BC } \begin{cases} \text{Dirichlet: } T(0,y)=T(a,y)=0, T(x,0)=0, T(x,b)=f(x) \\ \text{Neumann: } \partial T/\partial n \\ \text{Mixed: } T \text{ and } \partial T/\partial n \end{cases}$$

$$T(x,y)=F(x)G(y)\rightarrow \frac{d^2F}{dx^2}G+F\frac{d^2G}{dy^2}=0 \xrightarrow{\div FG} \frac{1}{F}\frac{d^2F}{dx^2}=-\frac{1}{G}\frac{d^2G}{dy^2}=-k$$

$$\left.\begin{matrix} T(0,y)=F(0)G(y)=0 \\ T(a,y)=F(a)G(y)=0 \end{matrix}\right\} \rightarrow G(y)\neq 0, F(0)=0 \text{ and } F(a)=0$$

$$T(x,0)=F(x)G(0)=0\rightarrow G(0)=0$$

$$\begin{cases} \dfrac{d^2F}{dx^2}+kF=0\rightarrow F=A\cos\sqrt{k}x+B\sin\sqrt{k}x\rightarrow \begin{cases} F(0)=0\rightarrow A=0 \\ F(a)=0=B\sin\sqrt{k}a\rightarrow B\neq 0, \sqrt{k}a=n\pi\rightarrow k=\left(\dfrac{n\pi}{a}\right)^2 \end{cases} \\ \dfrac{d^2G}{dy^2}-kG=0\rightarrow \dfrac{d^2G}{dy^2}-\left(\dfrac{n\pi}{a}\right)^2 G=0\rightarrow G_n(y)=A_n e^{\frac{n\pi y}{a}}+B_n e^{-\frac{n\pi y}{a}}\rightarrow \{G(0)=0\rightarrow A_n+B_n=0 \end{cases}$$

$$T_n(x,y)=F_n(x)G_n(y)=\left(B\sin\frac{n\pi}{a}x\right)\left[A_n\left(e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}\right)\right]=\tilde{A}_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}\rightarrow T(x,y)=\sum_{n=1}^{\infty}T_n(x,y)$$

$$T(x,b)=f(x)=\sum_{n=1}^{\infty}\tilde{A}_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}=\sum_{n=1}^{\infty}\left(\tilde{A}_n\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

$$\left(\tilde{A}_n\sinh\frac{n\pi b}{a}\right)=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx\rightarrow \tilde{A}_n=2\int_0^a f(x)\sin\frac{n\pi x}{a}dx\Big/ a\sinh\frac{n\pi b}{a}$$

Solution Technique (1)

– PDE → algebraic difference equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \leftarrow \begin{cases} \dfrac{\partial^2 T}{\partial x^2}=\dfrac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2} \\ \dfrac{\partial^2 T}{\partial y^2}=\dfrac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2} \end{cases}$$

$$\rightarrow \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$$

$$\xrightarrow{\Delta x=\Delta y} T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0$$

apply boundary conditions (fixed/Dirihlet)

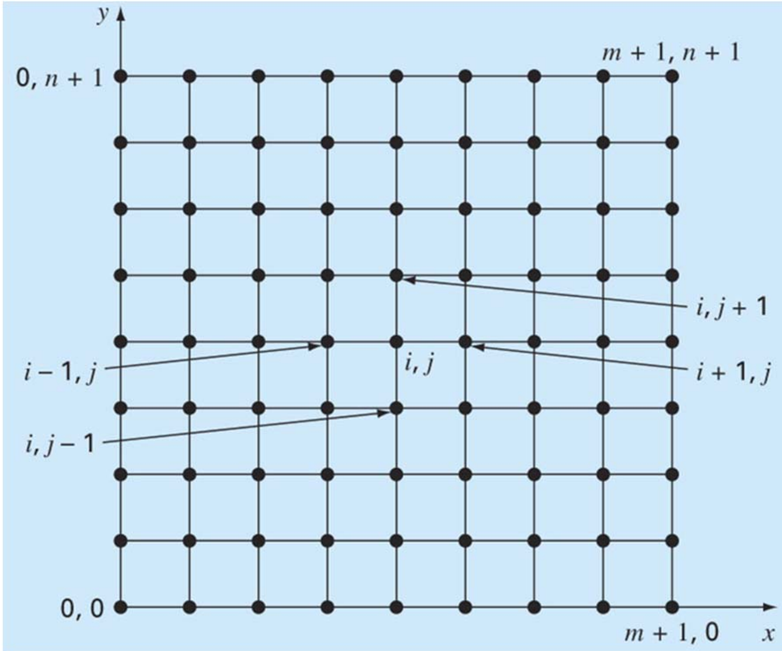
$$@(1,1): T_{21}+\underbrace{T_{01}}_{75°C}+T_{12}+\underbrace{T_{10}}_{0°C}-4T_{11}=0 \rightarrow 4T_{11}-T_{21}-T_{12}=75$$

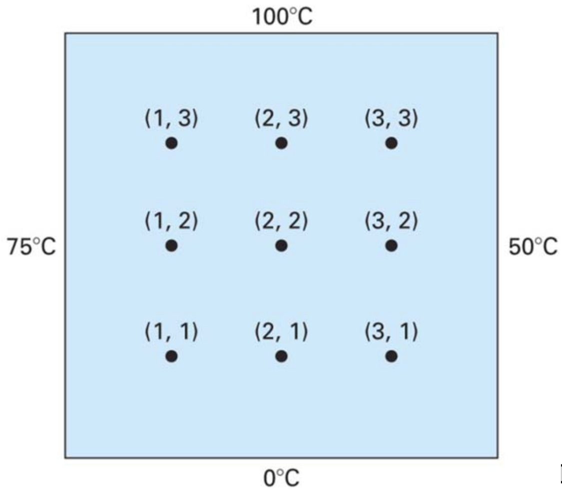
$$@(2,1): T_{31}+T_{11}+T_{22}+\underbrace{T_{20}}_{0°C}-4T_{21}=0 \rightarrow -T_{11}+4T_{21}-T_{31}-T_{22}=0$$

$$@(3,1): \underbrace{T_{41}}_{50°C}+T_{21}+T_{32}+\underbrace{T_{30}}_{0°C}-4T_{31}=0 \rightarrow -T_{21}+4T_{31}-T_{32}=50$$

⋮

9 equations





Solution Technique (2)

- How to solve linear algebraic equations (T_{ij})?
  - Gauss-Seidel (Liebmann in PDE) method

$$\begin{cases} x_1 - 0.25x_2 - 0.25x_3 = 50 \\ -0.25x_1 + x_2 - 0.25x_4 = 50 \\ -0.25x_1 + x_3 - 0.25x_4 = 25 \\ -0.25x_2 - 0.25x_3 \quad x_4 = 25 \end{cases}$$

$$\rightarrow \begin{cases} x_1^{(1)} = 0.25x_2^{(0)} + 0.25x_3^{(0)} + 50 \\ x_2^{(1)} = 0.25x_1^{(1)} + 0.25x_4^{(0)} + 50 \\ x_3^{(1)} = 0.25x_1^{(1)} + 0.25x_4^{(0)} + 25 \\ x_4^{(1)} = 0.25x_2^{(1)} + 0.25x_3^{(1)} + 25 \end{cases}$$

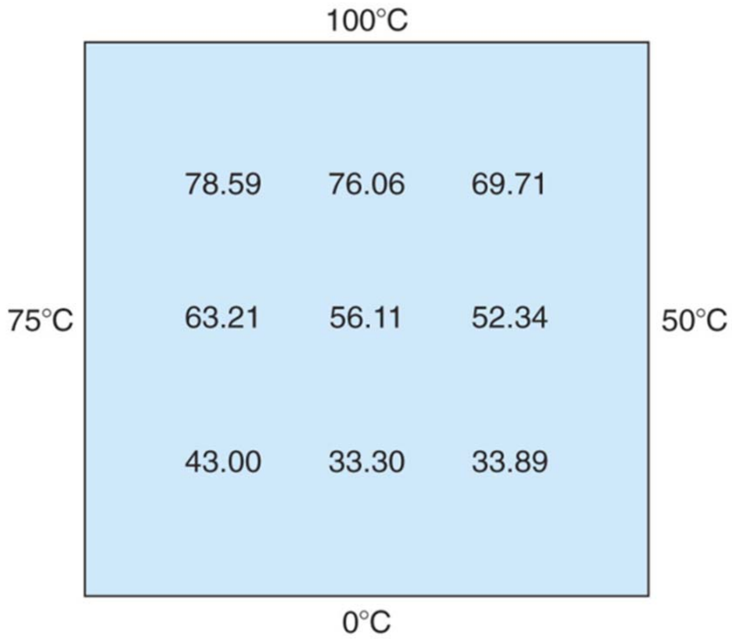
$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4} \xrightarrow{\text{overrelaxation}} T_{i,j}^{new} = \lambda T_{i,j}^{new} + (1-\lambda) T_{i,j}^{old}$$

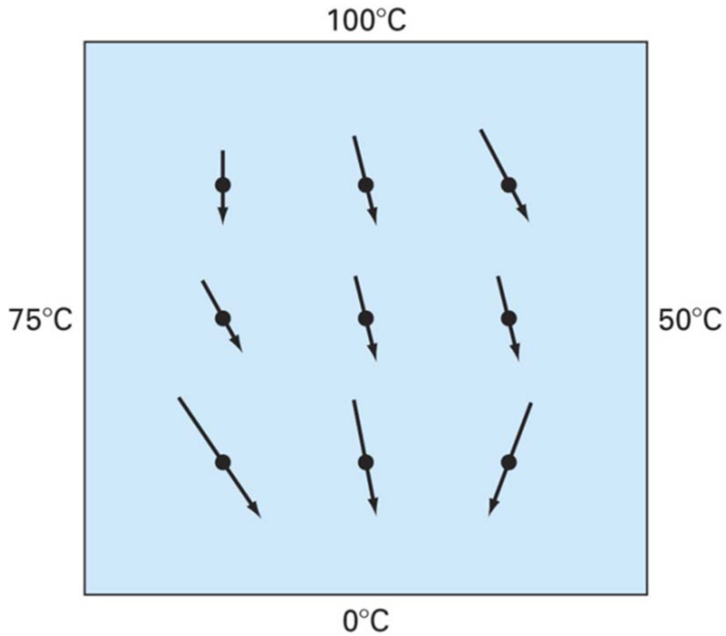
$$\xrightarrow{\text{stopping criterion}} \left| (\varepsilon_a)_{i,j} \right| = \left| \frac{T_{i,j}^{new} - T_{i,j}^{old}}{T_{i,j}^{new}} \right| 100\%$$

Example

- Primary variable (T_{ij}) → secondary variable $(q_{i,j})_l$

$$q_l = -k' \frac{\partial T}{\partial l} \xrightarrow{\text{central difference}} \left\{ \begin{aligned} (q_{i,j})_x &= -k' \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} \\ (q_{i,j})_y &= -k' \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta y} \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} (q_{i,j})_n &= \sqrt{(q_{i,j})_x^{\,2} + (q_{i,j})_y^{\,2}} \\ \theta_{i,j} &= \begin{cases} \tan^{-1}\left((q_{i,j})_y / (q_{i,j})_x\right) & \text{for } q_x > 0 \\ \tan^{-1}\left((q_{i,j})_y / (q_{i,j})_x\right) + \pi & \text{for } q_x < 0 \end{cases} \end{aligned} \right.$$





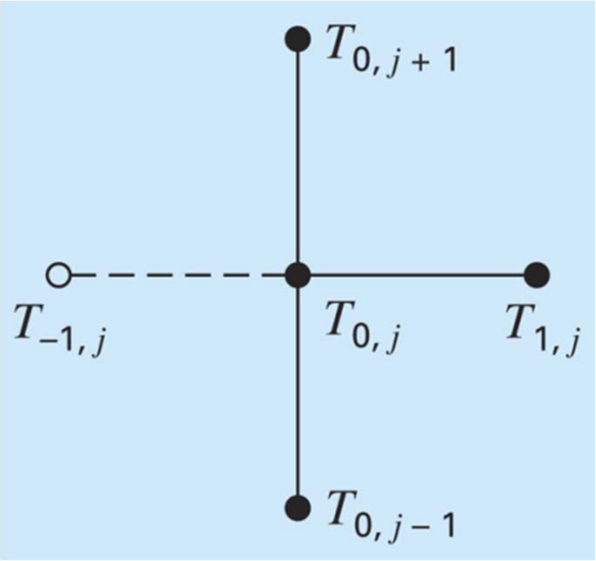
Derivative Boundary Conditions

- Neumann boundary condition
- For the heated plate problem
  - heat flux is specified at the boundary
  - If the edge is insulated, this derivative becomes zero

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

$$\frac{\partial T_{0,j}}{\partial x} \cong \frac{T_{1,j} - T_{-1,j}}{2\Delta x} \rightarrow T_{-1,j} = T_{1,j} - 2\Delta x \frac{\partial T_{0,j}}{\partial x}$$

$$2T_{1,j} - 2\Delta x \frac{\partial T_{0,j}}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$



Example

$$@(1,0): T_{2,0} + \underbrace{T_{0,0}}_{75°C} + T_{1,1} + T_{1,-1} - 4T_{1,0} = 0 \leftarrow \left(\frac{\partial T_{1,0}}{\partial y} \cong \frac{T_{1,1} - T_{1,-1}}{2\Delta y} \rightarrow T_{1,-1} = T_{1,1} - 2\Delta y \underbrace{\frac{\partial T_{1,0}}{\partial y}}_{0} \right)$$

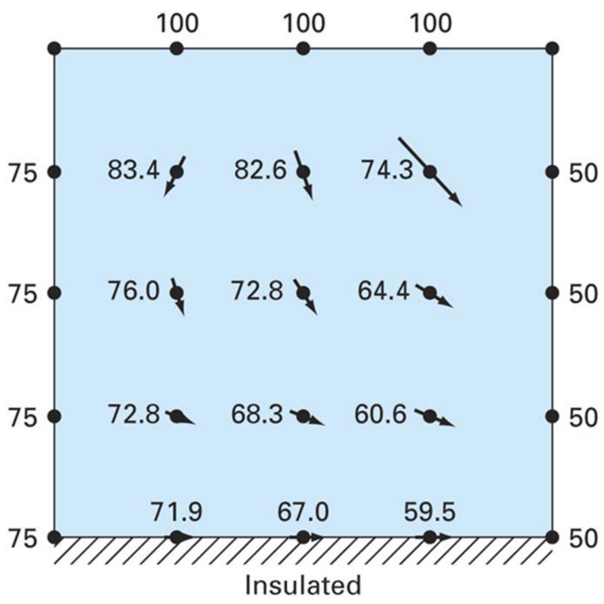
$$T_{2,0} + \underbrace{T_{0,0}}_{75°C} + T_{1,1} + \left(T_{1,1} - 2\Delta y \underbrace{\frac{\partial T_{1,0}}{\partial y}}_{0} \right) - 4T_{1,0} = 0 \rightarrow 4T_{1,0} - T_{2,0} - 2T_{1,1} = 75$$

$\vdots$

$$@(1,1): T_{2,1} + \underbrace{T_{0,1}}_{75°C} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0 \rightarrow -T_{1,0} + 4T_{1,1} - T_{2,1} - T_{1,2} = 0$$

$\vdots$

12 equations



Irregular Boundaries

$$\left(\frac{\partial T}{\partial x}\right)_{i-1,i} \cong \frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x}$$

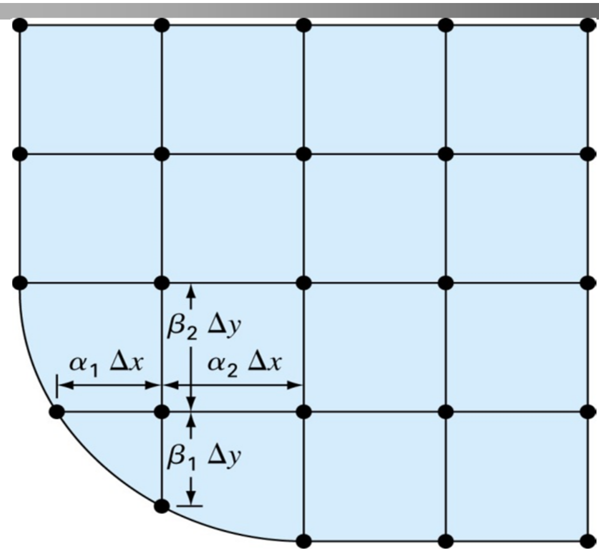
$$\left(\frac{\partial T}{\partial x}\right)_{i,i+1} \cong \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}$$

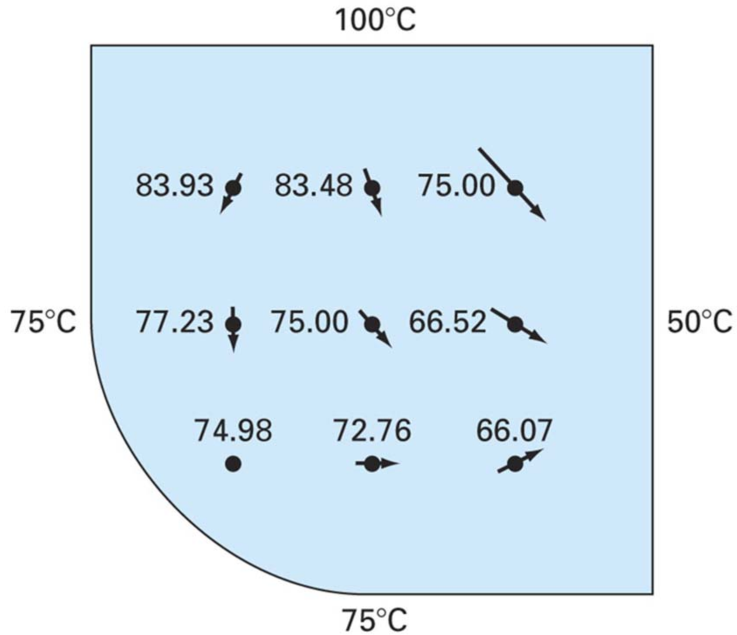
$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\left(\frac{\partial T}{\partial x}\right)_{i,i+1} - \left(\frac{\partial T}{\partial x}\right)_{i-1,i}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = 2\frac{\frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x} - \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}}{\alpha_1 \Delta x + \alpha_2 \Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{2}{\Delta x^2}\left[\frac{T_{i-1,j} - T_{i,j}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{T_{i+1,j} - T_{i,j}}{\alpha_2(\alpha_1 + \alpha_2)}\right]$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{2}{\Delta y^2}\left[\frac{T_{i,j-1} - T_{i,j}}{\beta_1(\beta_1 + \beta_2)} + \frac{T_{i,j+1} - T_{i,j}}{\beta_2(\beta_1 + \beta_2)}\right]$$





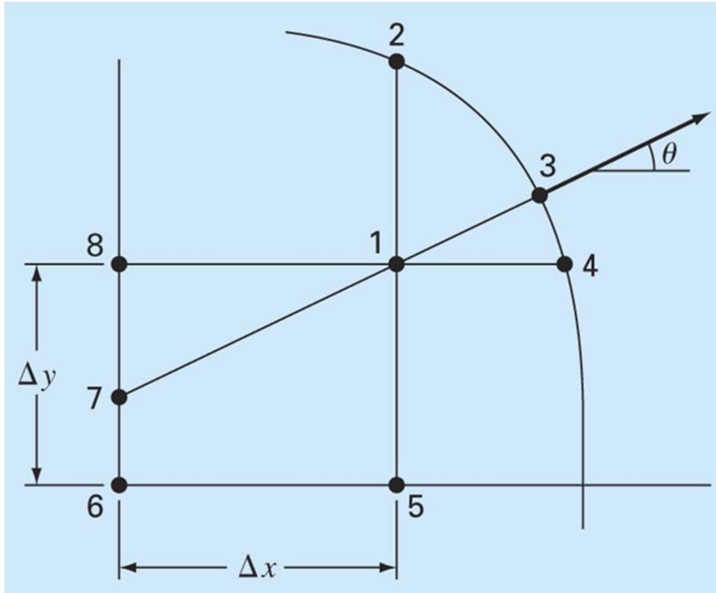
Derivative for Irregular Boundaries

$$\left(\frac{\partial T}{\partial \eta}\right)_3 = \frac{T_1 - T_7}{L_{17}} \rightarrow T_1 = \frac{\Delta x}{\cos\theta}\left(\frac{\partial T}{\partial \eta}\right)_3 + T_6\frac{\Delta x \tan\theta}{\Delta y} + T_8\left(1 - \frac{\Delta x \tan\theta}{\Delta y}\right)$$

$$L_{17} = \frac{\Delta x}{\cos\theta}$$

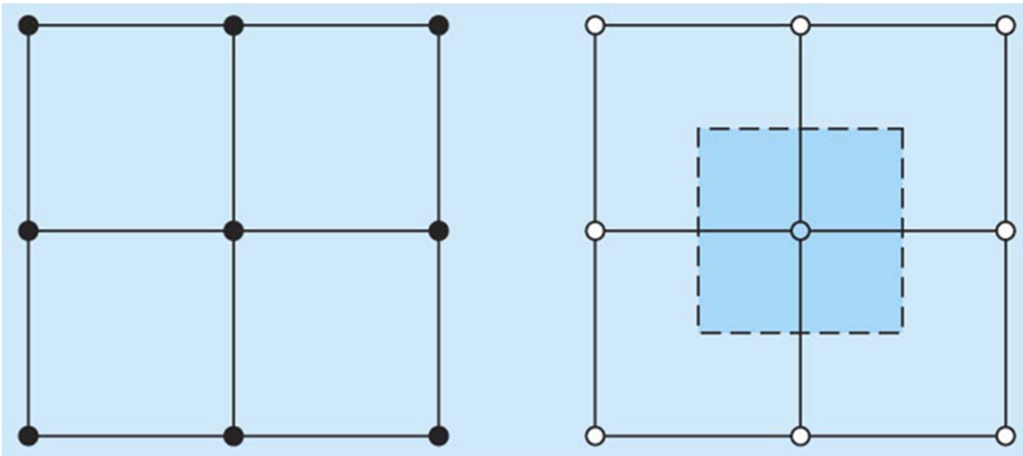
$$(T_7 - T_8) : (T_6 - T_8) = \Delta x \tan\theta : \Delta y$$

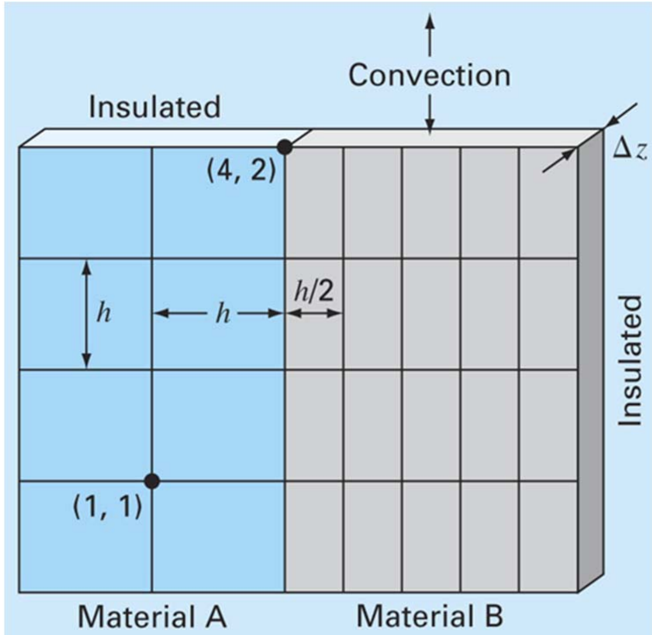
$$\rightarrow T_7 = T_8 + (T_6 - T_8)\frac{\Delta x \tan\theta}{\Delta y} \quad (\text{when } \theta < \pi/4)$$



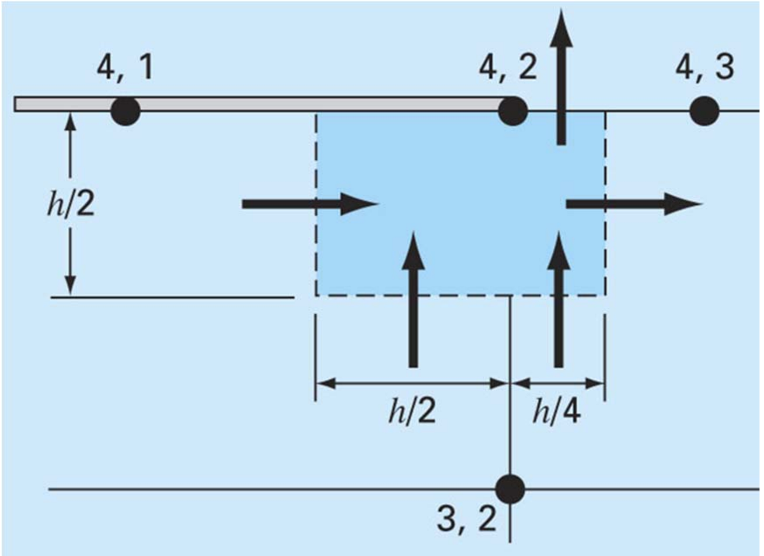
Control-Volume Approach

- Volume-integral approach
- points are determined across the domain
- approximation is applied to a volume surrounding the point rather than approximating the PDE at a point
  - Heated plate with unequal grid spacing, two materials, and mixed boundary conditions





Example



steady-state heat balance for the volume: $0 = (\text{In}) - (\text{Out})$

$$\begin{pmatrix}\text{left-side}\\\text{conduction}\end{pmatrix} + \begin{pmatrix}\text{lower}\\\text{conduction}\\\text{Material A}\end{pmatrix} + \begin{pmatrix}\text{lower}\\\text{conduction}\\\text{Material B}\end{pmatrix} = \begin{pmatrix}\text{right-side}\\\text{conduction}\end{pmatrix} + \begin{pmatrix}\text{upper}\\\text{convection}\end{pmatrix}$$

$$\left(-k'_A \frac{T_{42} - T_{41}}{h}\right)\left(\frac{h}{2}\Delta z\right) + \left(-k'_A \frac{T_{42} - T_{32}}{h}\right)\left(\frac{h}{2}\Delta z\right) + \left(-k'_B \frac{T_{42} - T_{32}}{h}\right)\left(\frac{h}{4}\Delta z\right)$$

$$= \left(-k'_B \frac{T_{43} - T_{42}}{h/2}\right)\left(\frac{h}{2}\Delta z\right) - h_c\left(\frac{h}{4}\Delta z\right)\left(T_a - T_{42}\right)$$

Parabolic: Heat Conduction Equation

- Conservation of energy can be used to develop an unsteady-state energy balance for the differential element in a long, thin insulated rod

$$\text{heat stored in the element over a unit time period } \Delta t = (In) - (Out)$$

$$q(x)(\Delta y \Delta z)\Delta t - q(x+\Delta x)(\Delta y \Delta z)\Delta t = \rho C(\Delta x \Delta y \Delta z)\Delta T$$

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} = \rho C \frac{\Delta T}{\Delta t} \xrightarrow{\text{taking the limit}} -\frac{\partial q}{\partial x} = \rho C \frac{\partial T}{\partial t} \xrightarrow{q=-k\rho C \frac{\partial T}{\partial x}} k \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Analytic Solution (1)

$$k\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}\quad \text{with}\quad \begin{cases}\text{BC: } T(0,t)=T_1 \text{ and } T(L,t)=T_2 \ \ (t>0)\\ \text{IC: } T(x,0)=f(x)=0\end{cases}$$

$$T(x,t)=T_S(x)+T_T(x,t)\rightarrow\begin{cases}T_S(x)=T_1+(T_2-T_1)\dfrac{x}{L}\\ T_T(x,\infty)=0\\ T_T(0,t)=T_T(L,t)=0 \ \ (t>0) \text{ since } T_S(x) \text{ satisfies BCs}\end{cases}$$

$$T_T(x,t)=F(x)G(t)\rightarrow kF''G=F\dot{G}\xrightarrow{\div kFG}\frac{F''}{F}=\frac{\dot{G}}{kG}=(\text{constant})=-p^2\ (\text{why not 0 or positive?})$$

$$\begin{cases}F''+p^2F=0\rightarrow F(x)=A\cos px+B\sin px\rightarrow\begin{cases}F(0)=0\rightarrow A=0\\ F(L)=0=B\sin pL\rightarrow B\neq 0,\ pL=n\pi\rightarrow p=\dfrac{n\pi}{L}\end{cases}\\ \dot{G}+kp^2G=0\rightarrow G(t)=\exp\left(-kp^2t\right)=\exp\left(-\dfrac{kn^2\pi^2t}{L^2}\right)\\ \left.\begin{matrix}F(0)G(t)=0\\ F(L)G(t)=0\end{matrix}\right\}\rightarrow G(t)\neq 0, F(0)=0 \text{ and } F(L)=0\end{cases}$$

Analytic Solution (2)

$$T_{T,n}\left(x,t\right)=F_n\left(x\right)G_n\left(t\right)=\left(B\sin\frac{n\pi}{L}x\right)\left[a_n\exp\left(-\frac{kn^2\pi^2t}{L^2}\right)\right]$$

$$T\left(x,t\right)=T_S\left(x\right)+\sum_{n=1}^{\infty}T_{T,n}\left(x,t\right)=T_1+\left(T_2-T_1\right)\frac{x}{L}+\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}\exp\left(-\frac{kn^2\pi^2t}{L^2}\right)$$

$$T\left(x,0\right)=f\left(x\right)=T_1+\left(T_2-T_1\right)\frac{x}{L}+\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}$$

$$\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}=f\left(x\right)-T_1-\left(T_2-T_1\right)\frac{x}{L}$$

$$a_n=\frac{2}{L}\int_0^L\left[f\left(x\right)-T_1-\left(T_2-T_1\right)\frac{x}{L}\right]\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L f\left(x\right)\sin\frac{n\pi x}{L}dx+\frac{2}{n\pi}\left(T_2\cos n\pi-T_1\right)$$

$$T_1=0,T_2=\overline{T},\ f\left(x\right)=0$$

$$T\left(x,t\right)=\overline{T}\left[\frac{x}{L}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}\sin\frac{n\pi x}{L}\exp\left(-\frac{kn^2\pi^2t}{L^2}\right)\right]$$

Explicit Methods

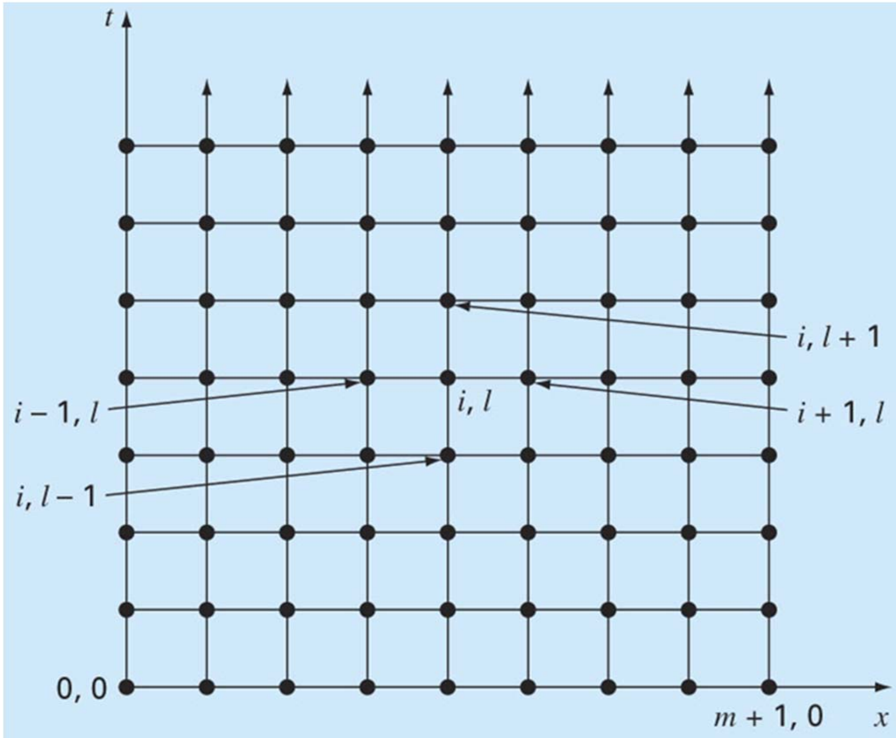
- approximations for the second derivative in space and the first derivative in time
- Time-variable: open-ended, stability problem

$$\left.\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2} \to O\left((\Delta x)^2\right)\\ \frac{\partial T}{\partial t} &= \frac{T_i^{l+1} - T_i^{l}}{\Delta t} \to O(\Delta t)\end{aligned}\right\}$$

$$\to k\,\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$T_i^{l+1} = T_i^{l} + \lambda\left(T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}\right) \text{ where } \lambda = \frac{k\Delta t}{(\Delta x)^2}$$

for all interior nodes



Convergence and Stability

$$\lambda = \frac{k\Delta t}{(\Delta x)^2} \le \begin{cases} \frac{1}{2}\left(\text{errors do not grow, but oscillate}\right) \rightarrow \Delta t \le \frac{1}{2}\frac{(\Delta x)^2}{k} \\ \frac{1}{4}\left(\text{ensure no oscillation}\right) \\ \frac{1}{6}\left(\text{minimize trucation error}\right) \end{cases}$$

$$\left(\Delta x \rightarrow \frac{1}{2}\Delta x\right) \Rightarrow \left(\Delta t \rightarrow \frac{1}{4}\Delta t\right)$$

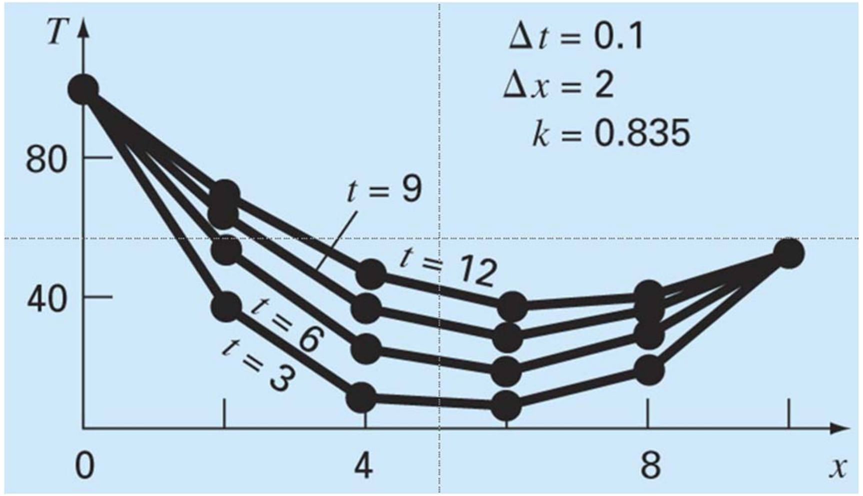
- Derivative boundary conditions

$$T_i^{l+1} = T_i^l + \lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right)$$

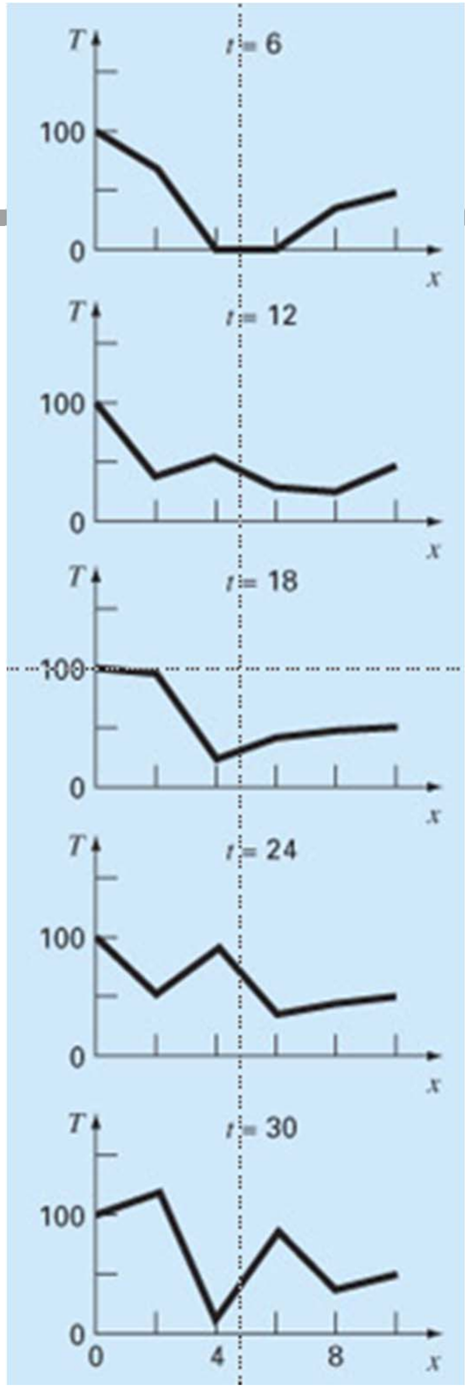
$$i = 0 : T_0^{l+1} = T_0^l + \lambda\left(T_1^l - 2T_0^l + T_{-1}^l\right)$$

Example

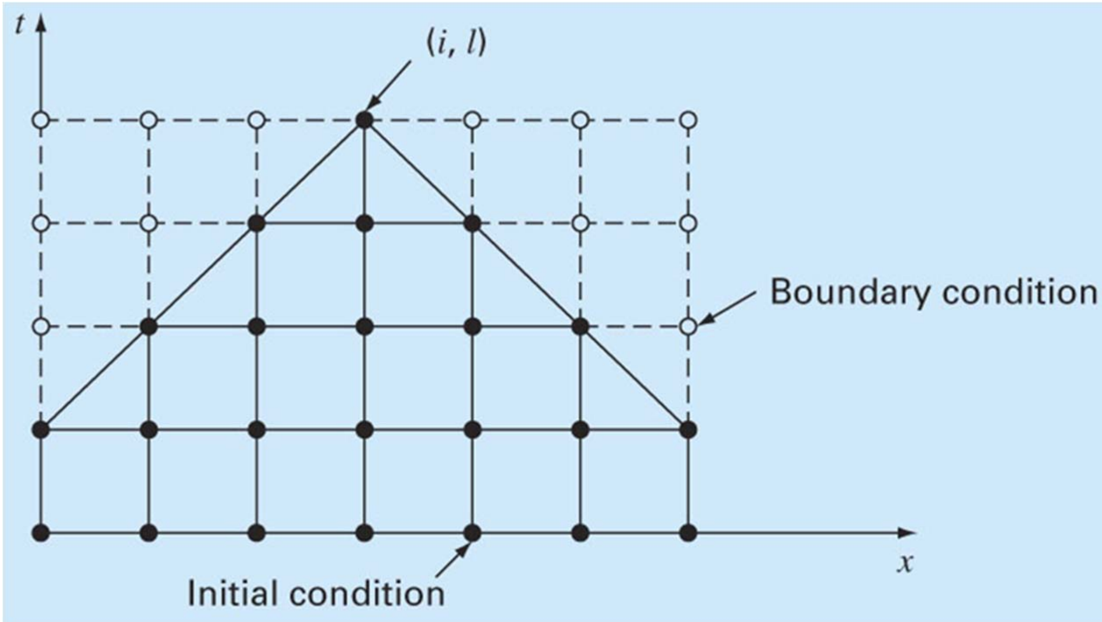
$$\lambda = \frac{k\Delta t}{(\Delta x)^2} = 0.020875$$

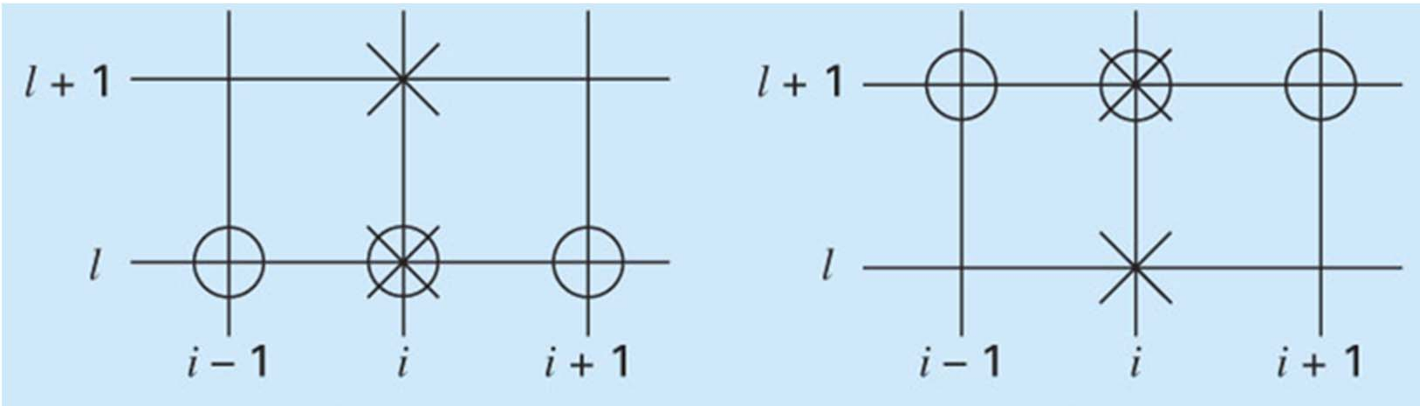


$$\lambda = 0.735$$



Explicit vs. Implicit





(a) Explicit (b) Implicit

Simple Implicit Method

- overcome difficulties associated with explicit methods at the expense of somewhat more complicated algorithms
- spatial derivative is approximated at an advanced time interval l+1
- Unconditionally stable ↔ accuracy limit
- Steady-state results in an efficient manner

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \xrightarrow{\frac{\partial^2 T}{\partial x^2} \cong \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}} k\frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\begin{cases} i = 2,\ldots,m-1: -\lambda T_{i-1}^{l+1} + (1+2\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^l \\ i = 1: -\lambda T_0^{l+1} + (1+2\lambda)T_1^{l+1} - \lambda T_2^{l+1} = T_1^l \to (1+2\lambda)T_1^{l+1} - \lambda T_2^{l+1} = T_1^l + \lambda f_0\left(t^{l+1}\right) \\ i = m: -\lambda T_{m-1}^{l+1} + (1+2\lambda)T_m^{l+1} - \lambda T_{m+1}^{l+1} = T_m^l \to -\lambda T_{m-1}^{l+1} + (1+2\lambda)T_m^{l+1} = T_m^l + \lambda f_{m+1}\left(t^{l+1}\right) \end{cases}$$

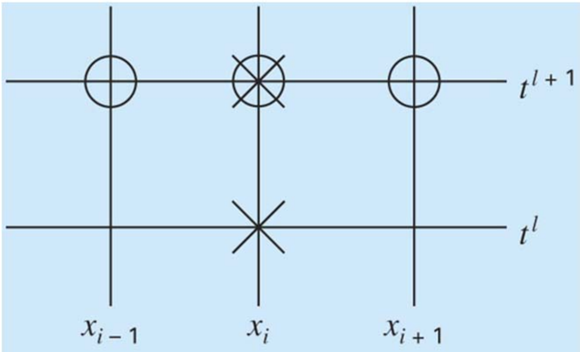
m equations / m unknowns

Crank-Nicolson Method

– alternative implicit scheme that is second order accurate both in space and time @$t^{l+1/2}$

– difference approximations are developed at the midpoint of the time increment

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \leftarrow \begin{cases} \dfrac{\partial T}{\partial t} \cong \dfrac{T_i^{l+1} - T_i^l}{\Delta t} \\ \dfrac{\partial^2 T}{\partial x^2} \cong \dfrac{1}{2}\left[\dfrac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \dfrac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right] \end{cases}$$



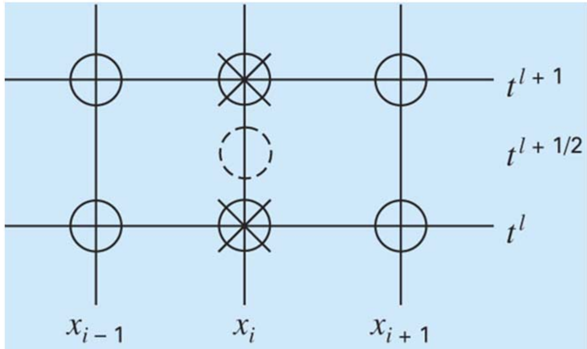
$$k\frac{1}{2}\left[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right] = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l + T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}\right) = 2\left(T_i^{l+1} - T_i^l\right)$$

$$\lambda T_{i-1}^{l+1} - 2(1+\lambda)T_i^{l+1} + \lambda T_{i+1}^{l+1} = -\lambda T_{i-1}^l - 2(1-\lambda)T_i^l - \lambda T_{i+1}^l$$

$$i = 1: \lambda T_0^{l+1} - 2(1+\lambda)T_1^{l+1} + \lambda T_2^{l+1} = -\lambda T_0^l - 2(1-\lambda)T_1^l - \lambda T_2^l$$

$$i = m: \lambda T_{m-1}^{l+1} - 2(1+\lambda)T_m^{l+1} + \lambda T_{m+1}^{l+1} = -\lambda T_{m-1}^l - 2(1-\lambda)T_m^l - \lambda T_{m+1}^l$$



Parabolic Equations in 2-D

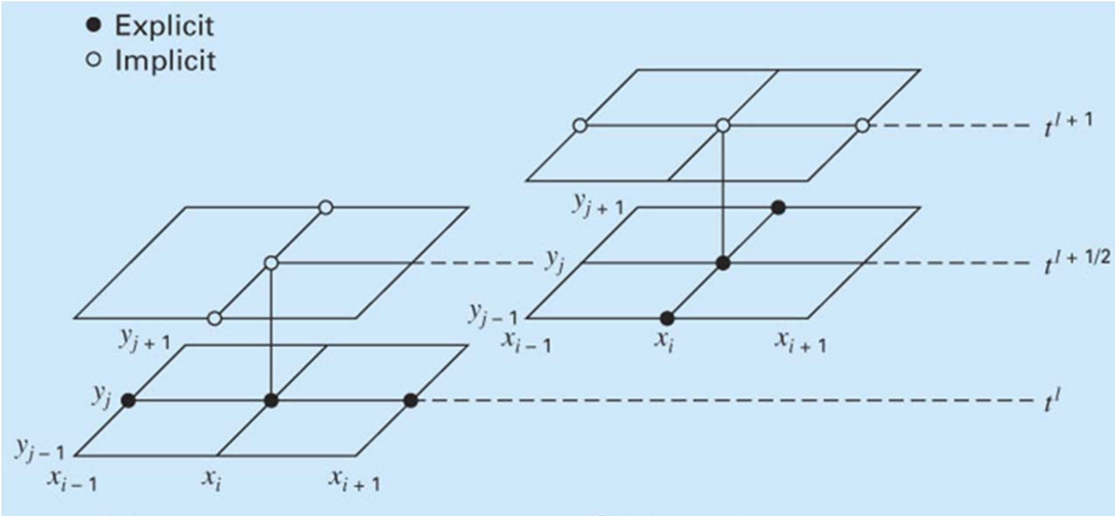
- alternating-direction implicit(ADI) scheme
  - solve parabolic equations in two spatial dimensions using tridiagonal matrices

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \rightarrow \text{stability: } \Delta t \le \frac{1}{8}\frac{(\Delta x)^2 + (\Delta y)^2}{k}$$

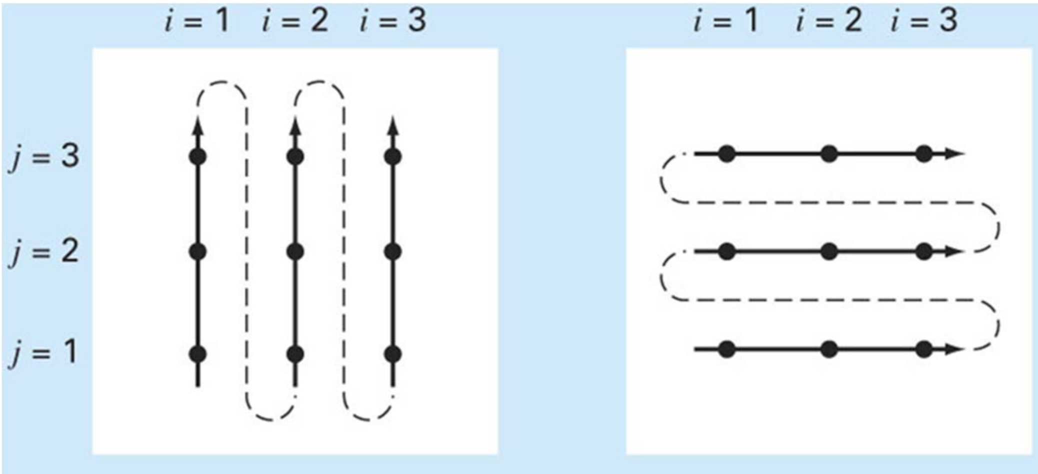
$$\text{first step: } \frac{T_{i,j}^{l+1/2} - T_{i,j}^{l}}{\Delta t/2} = k\left[\underbrace{\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{(\Delta x)^2}}_{\text{explicit}} + \underbrace{\frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{(\Delta y)^2}}_{\text{implicit}}\right]$$

$$\text{second step: } \frac{T_{i,j}^{l+1} - T_{i,j}^{l+1/2}}{\Delta t/2} = k\left[\underbrace{\frac{T_{i+1,j}^{l+1} - 2T_{i,j}^{l+1} + T_{i-1,j}^{l+1}}{(\Delta x)^2}}_{\text{implicit}} + \underbrace{\frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{(\Delta y)^2}}_{\text{explicit}}\right]$$

ADI scheme



(*a*) First half-step (*b*) Second half-step



(*a*) First direction (*b*) Second direction

FDM vs. FEM

- FDM
  - Solution domain is divided into a grid of discrete points or nodes (pointwise approximation)
  - Shortcomings: irregular geometry, unusual boundary conditions, heterogeneous composition
- FEM
  - Solution domain is divided into simply shaped regions or elements