# Contents

# Direct Stiffness Method (DSM)

- Importance: DSM is used for all major commercial FEM codes
- A democratic method works the same no matter what the element:

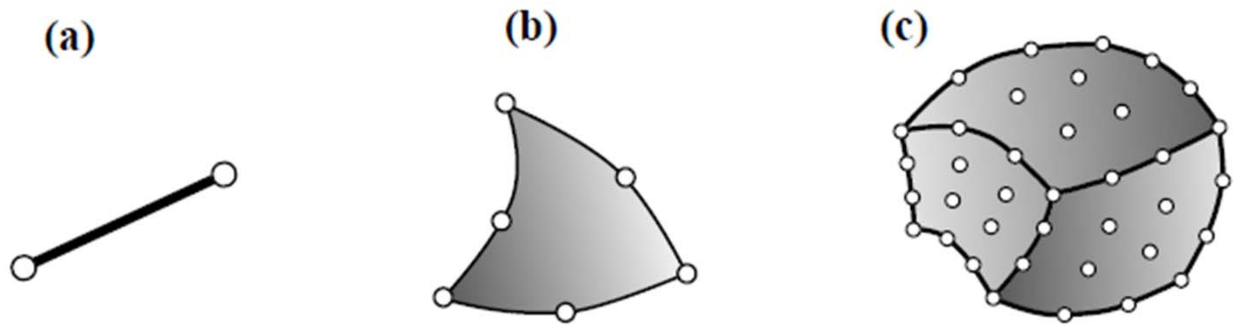
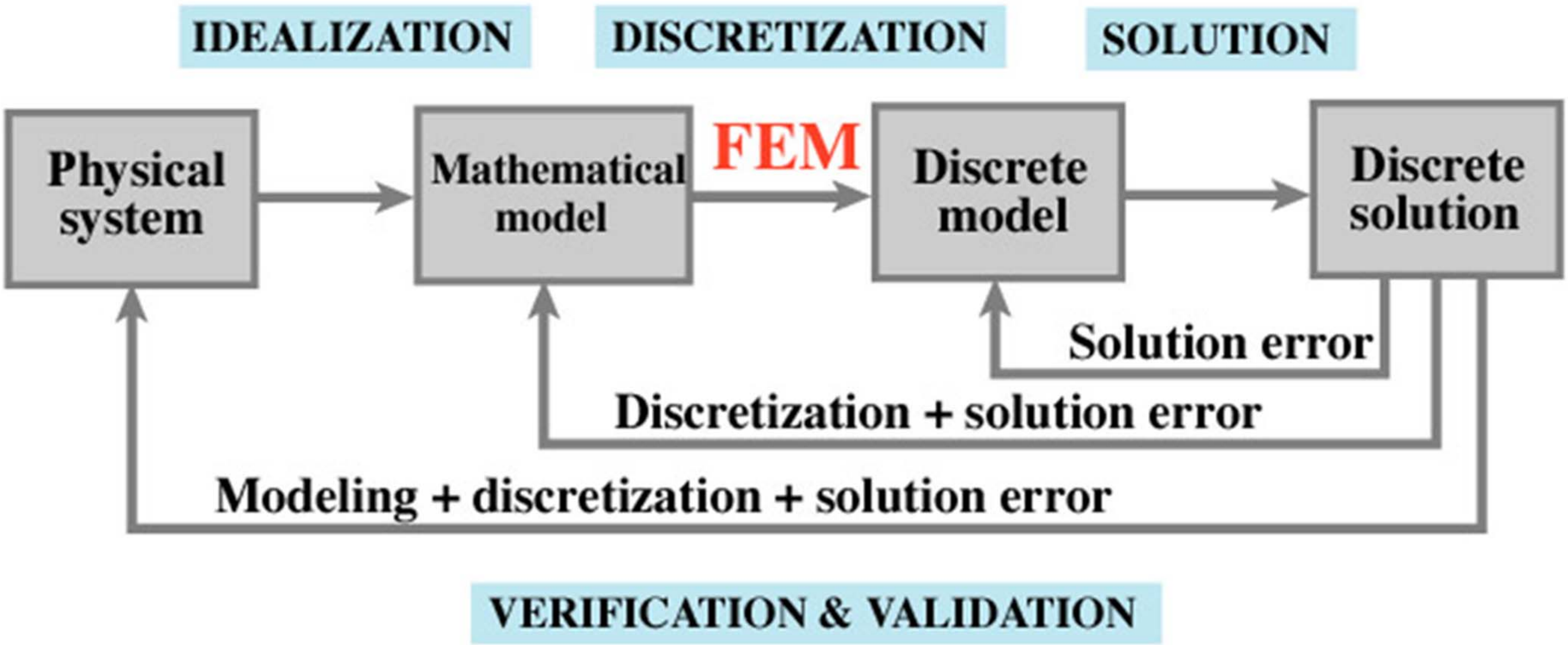



FIGURE 2.1. From the simplest through progressively more complex structural finite elements: (a) two-node bar element for trusses, (b) six-node triangle for thin plates, (b) 64-node tricubic, "brick" element for three-dimensional solid analysis.

- Obvious decision: use the truss to teach the DSM

# Model Based Simulation



# A Physical Plane Truss

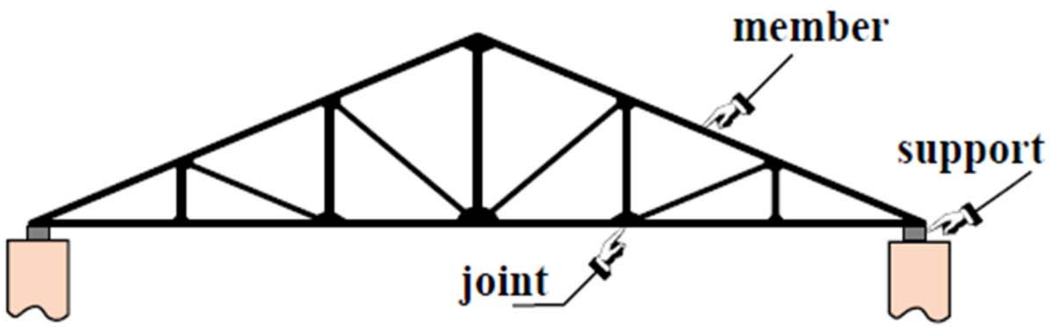
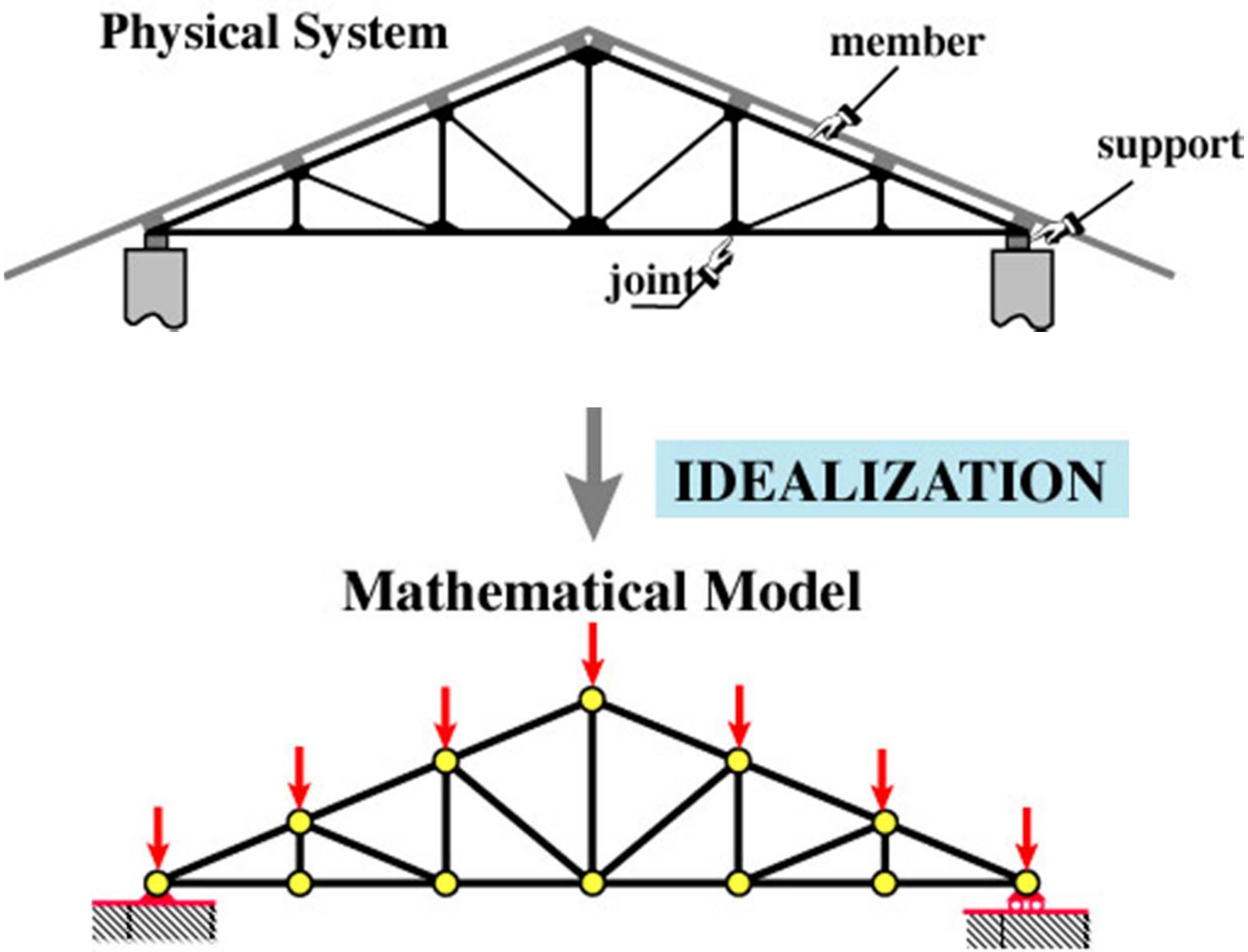


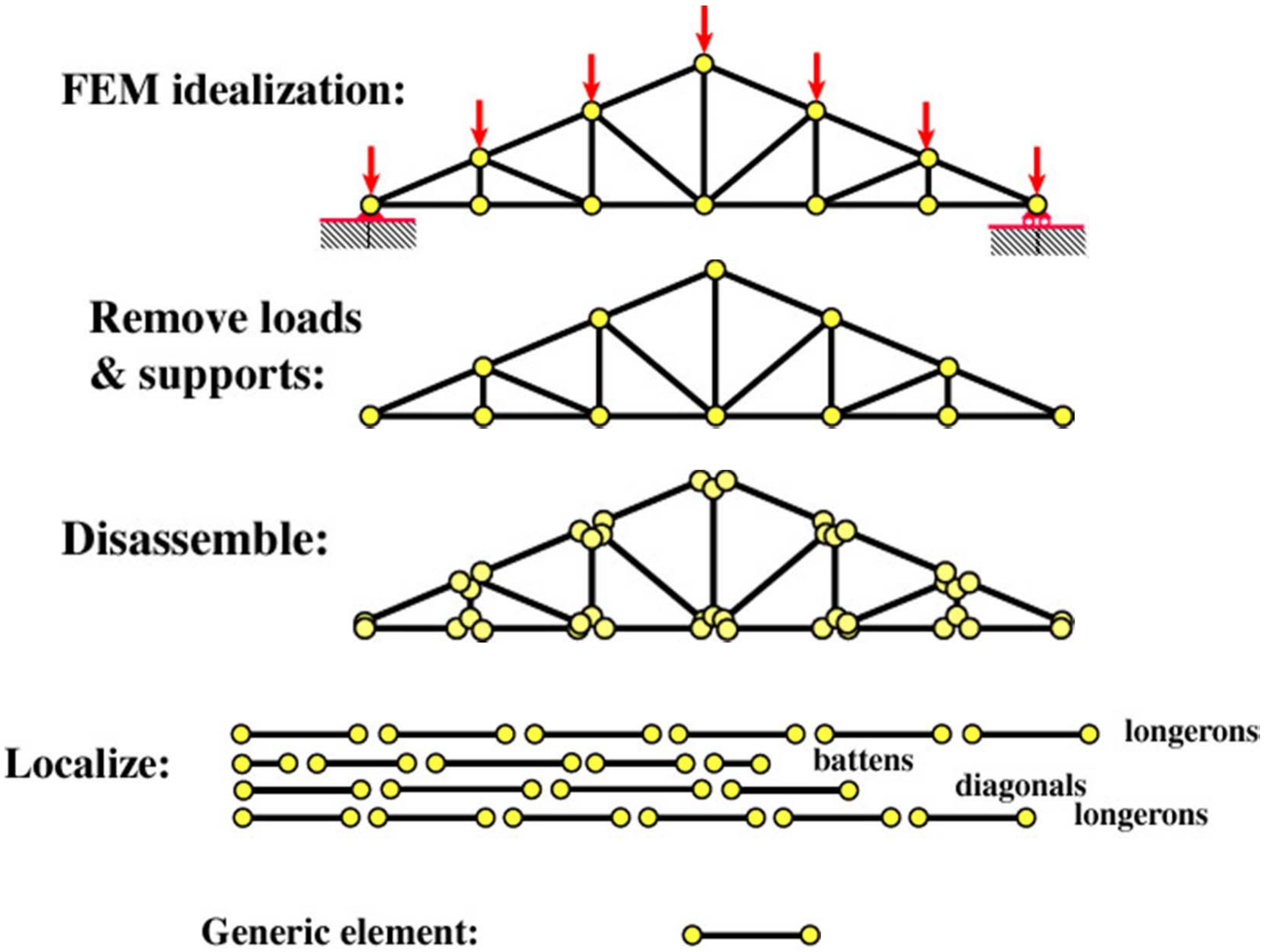

FIGURE 2.2. An actual plane truss structure. That shown is typical of a roof truss used in building construction for rather wide spans, say, over 10 meters. For shorter spans, as in residential buildings, trusses are simpler, with fewer bays.

Too complicated to do by hand.
We will use a simpler one to illustrate DSM steps.

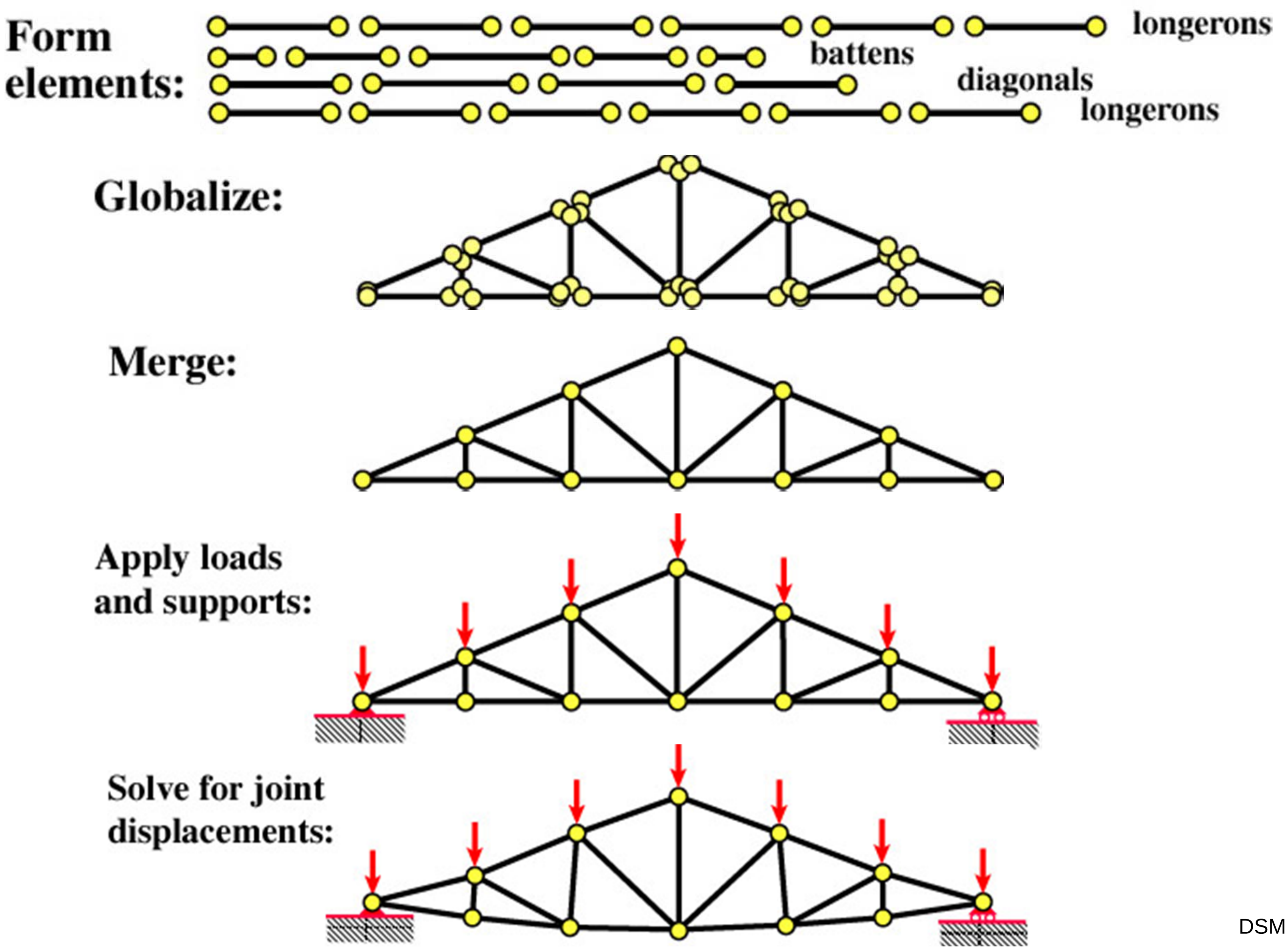
# Idealization Process



# DSM: Breakdown Steps



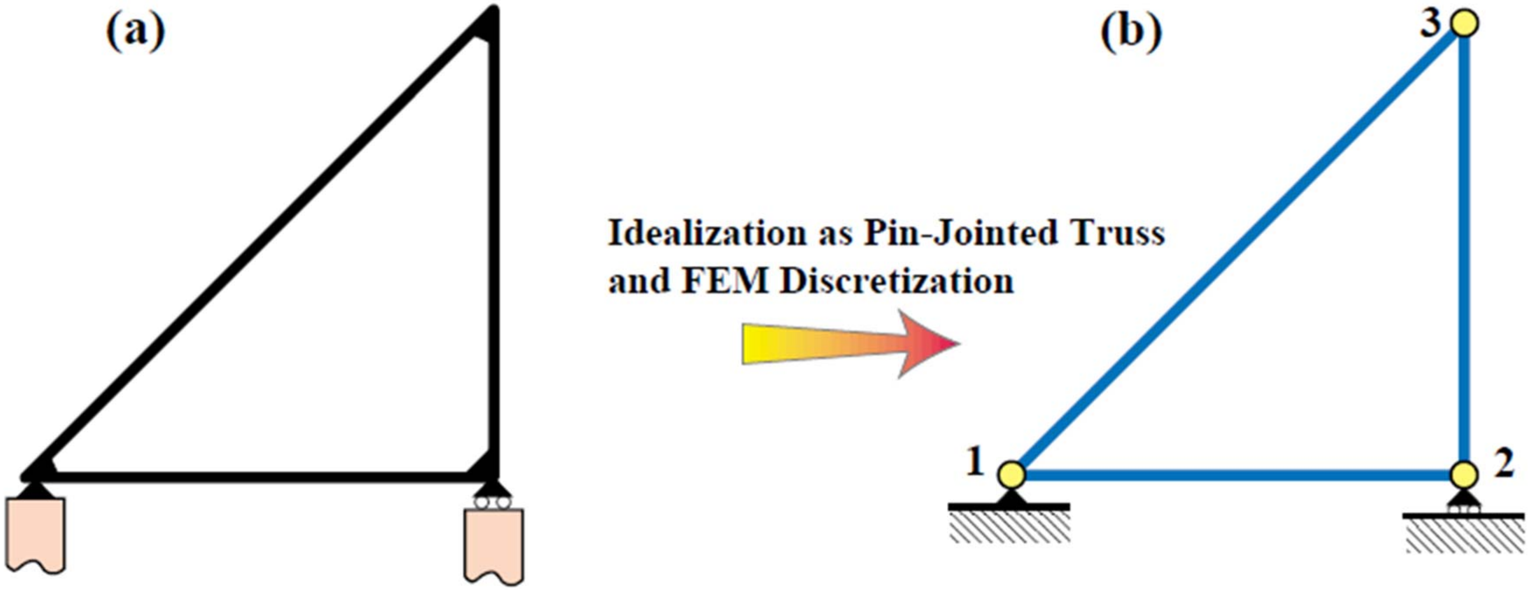
# DSM: Assembly & Solution Steps



# DSM Steps

| | | |
|---|---|---|
| Idealization | | |
| Breakdown | Disconnection | Conceptual steps |
| | Localization | |
| | Member (element) formulation | Processing steps |
| Assembly and Solution | Globalization | |
| | Merge | |
| | Application of BCs | |
| | Solution | |
| | Recovery of derived quantities | Post-processing steps |

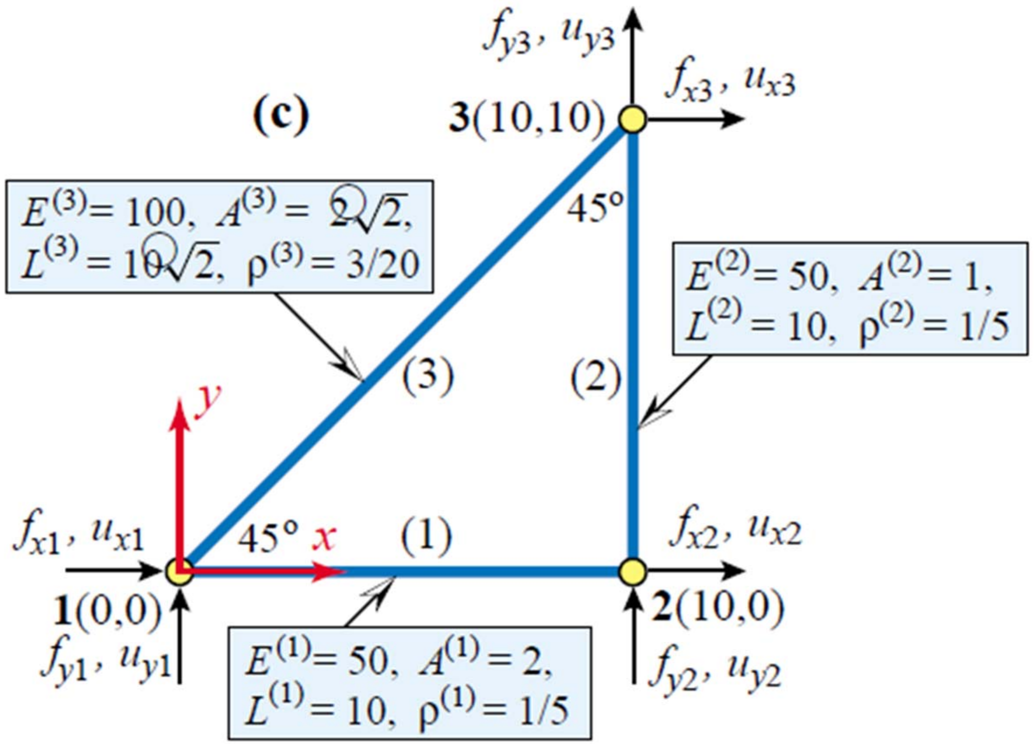
# Example Truss: Idealization




Physical structure

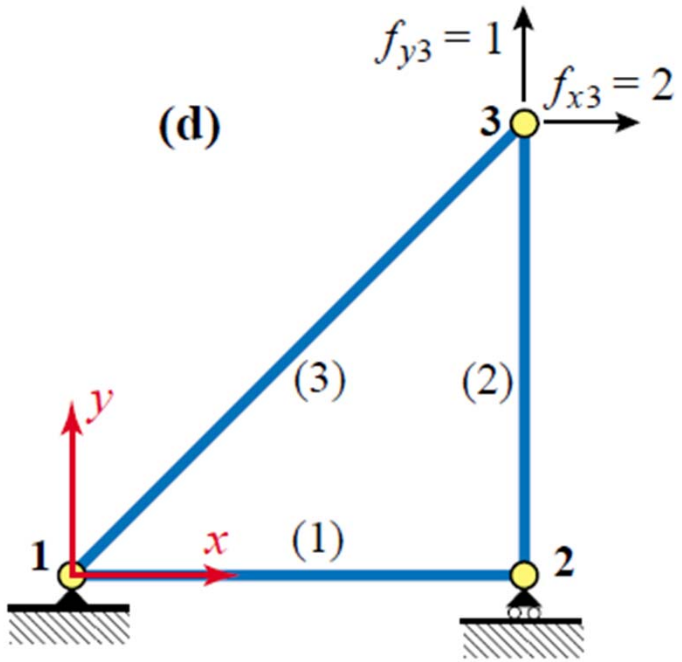
Idealization as a pin-jointed bar assemblage

# FEM Model: Nodes, Elements and DOFs




Geometric, material and fabrication properties

# FEM Model: BCs




Support conditions and applied loads

# Master (Global) Stiffness Equations

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Linear structure:

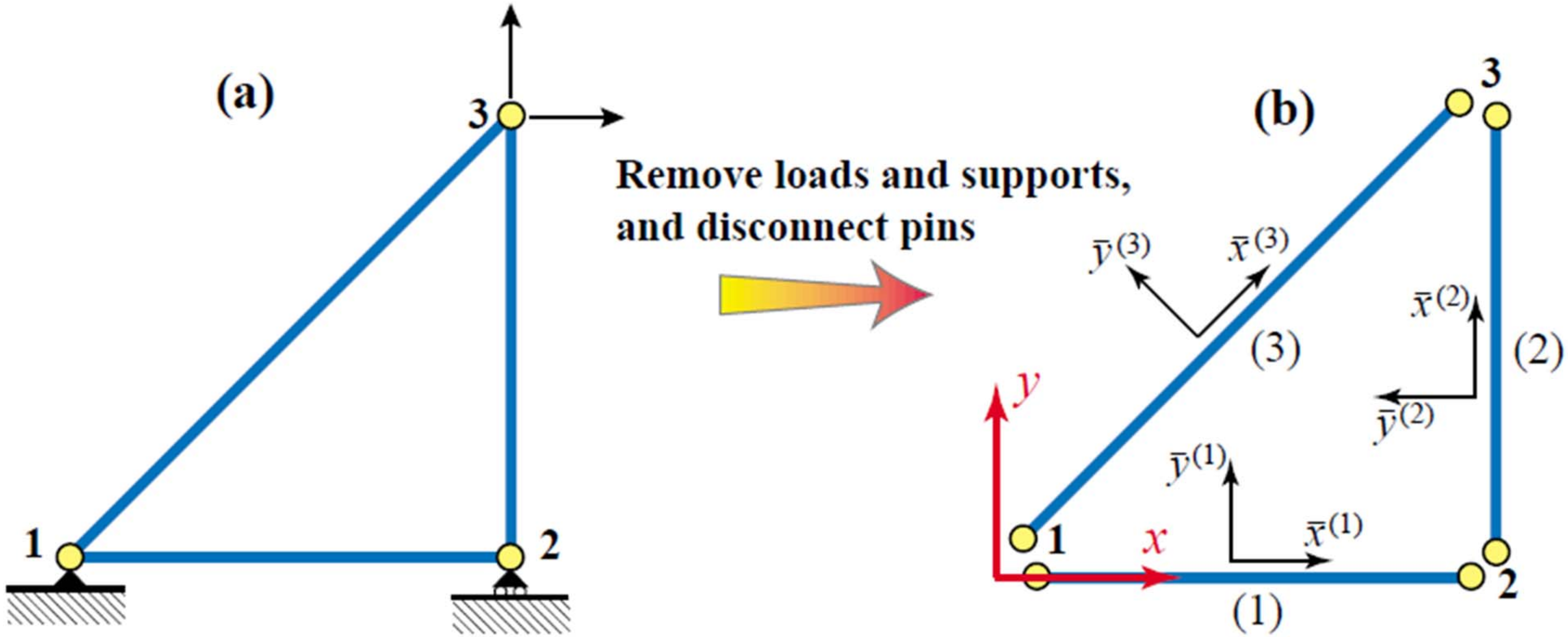
$$\underbrace{\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}}_{\text{Nodal force}} = \underbrace{\begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x2} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix}}_{\text{Master stiffness matrix}} \underbrace{\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}}_{\text{Nodal displacements}} \Leftrightarrow \mathbf{f} = \mathbf{K}\mathbf{u}$$

# Member (Element) Stiffness Equations

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\bar{\mathbf{u}}$$

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxi} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

# Breakdown: Disconnection and Localization




These steps are conceptual (not actually programmed as part of the DSM)

# 2-Node Truss (Bar) Element

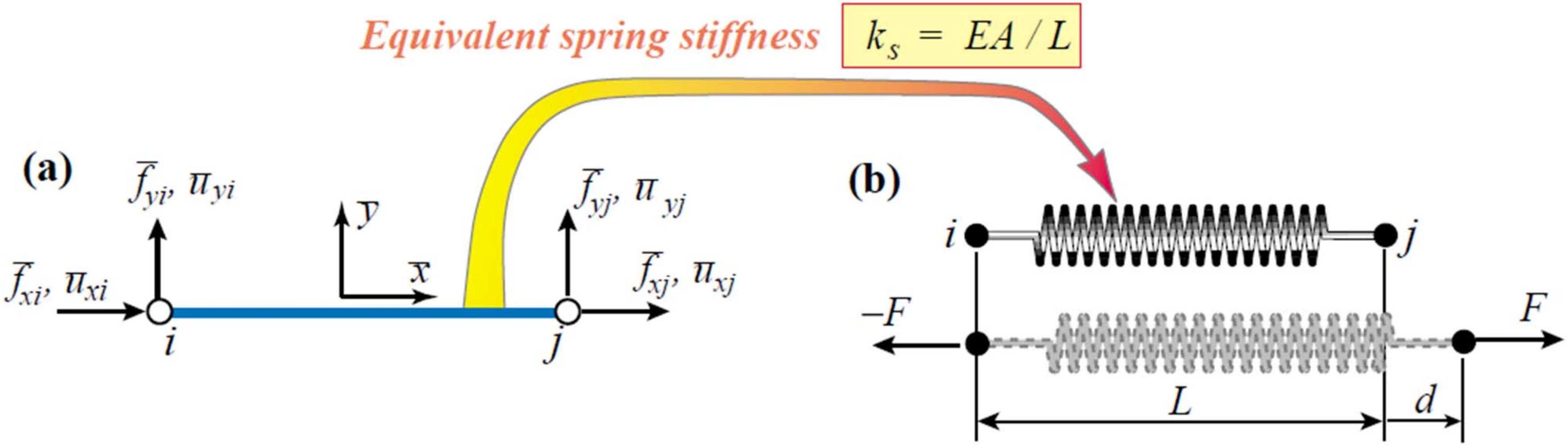



FIGURE 2.9. Generic truss member referred to its local coordinate system $\{\bar{x}, \bar{y}\}$: (a) idealization as 2-node bar element, (b) interpretation as equivalent spring. Element identification number $e$ dropped to reduce clutter.

$$F = \bar{f}_{xj} = -\bar{f}_{xi}$$

$$d = \bar{u}_{xj} - \bar{u}_{xi}$$

$$F = k_s d = \frac{EA}{L} d$$

# Truss (Bar) Element Formulation

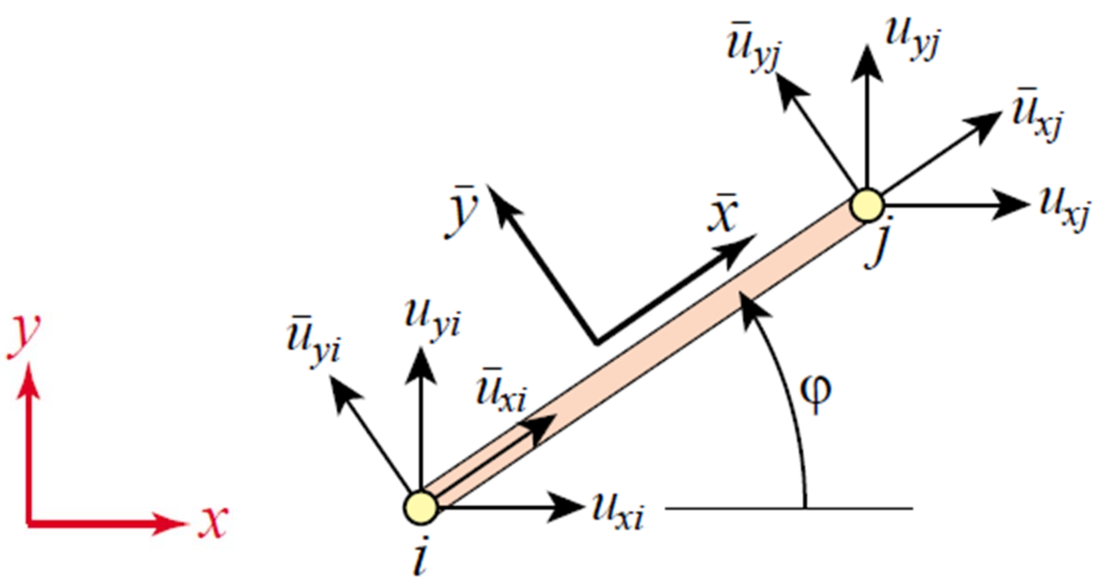
- Mechanics of Materials (MoM)

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} F = \frac{EA}{L} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} d \leftarrow d = \bar{u}_{xj} - \bar{u}_{xi} = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} \qquad \leftarrow \begin{cases} \text{Element stiffness equations} \\ \text{in local coordinates} \end{cases}$$

$$\bar{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \leftarrow \begin{cases} \text{Element stiffness matrix} \\ \text{in local coordinates} \end{cases}$$

# Globalization: Displacement Transformation



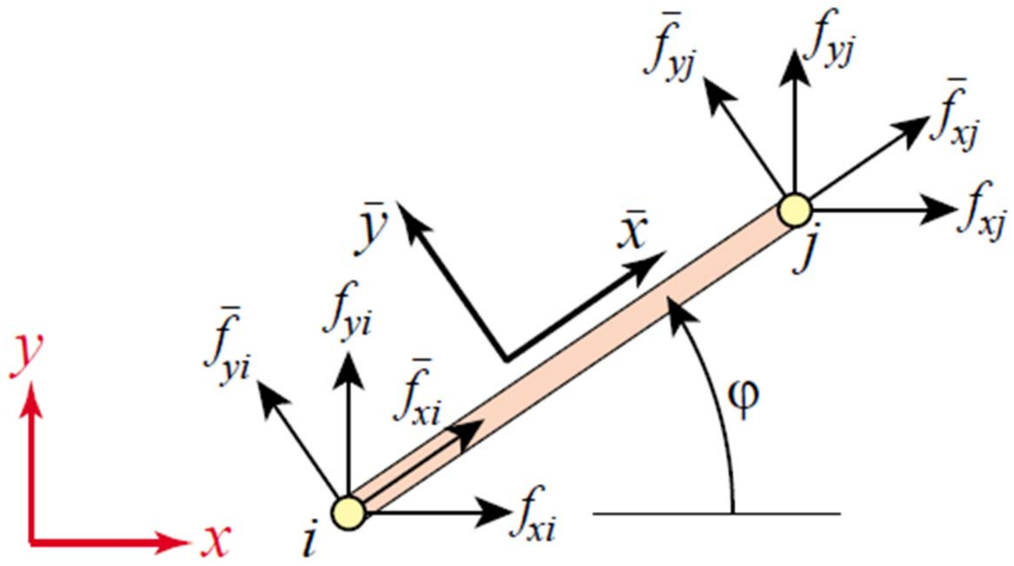

Node displacemens transform as

$$\bar{u}_{xi} = u_{xi}c + u_{yi}s, \quad \bar{u}_{yi} = -u_{xi}s + u_{yi}c$$
$$\bar{u}_{xj} = u_{xj}c + u_{yj}s, \quad \bar{u}_{yj} = -u_{xj}s + u_{yj}c$$
$$c = \cos\varphi, \quad s = \sin\varphi$$

$\rightarrow$ In matrix form

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} \Leftrightarrow \bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e$$

# Globalization: Force Transformation




Node force transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} \Leftrightarrow \mathbf{f}^e = \left(\mathbf{T}^e\right)^T \bar{\mathbf{f}}^e$$

Note: global on LHS, local on RHS

# Globalization: Transformation of Element Stiffness Matrices

$$\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$$

$$\left(\mathbf{T}^e\right)^T \bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \left(\mathbf{T}^e\right)^T \bar{\mathbf{f}}^e \leftarrow \begin{cases} \bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e \\ \mathbf{f}^e = \left(\mathbf{T}^e\right)^T \bar{\mathbf{f}}^e \end{cases}$$

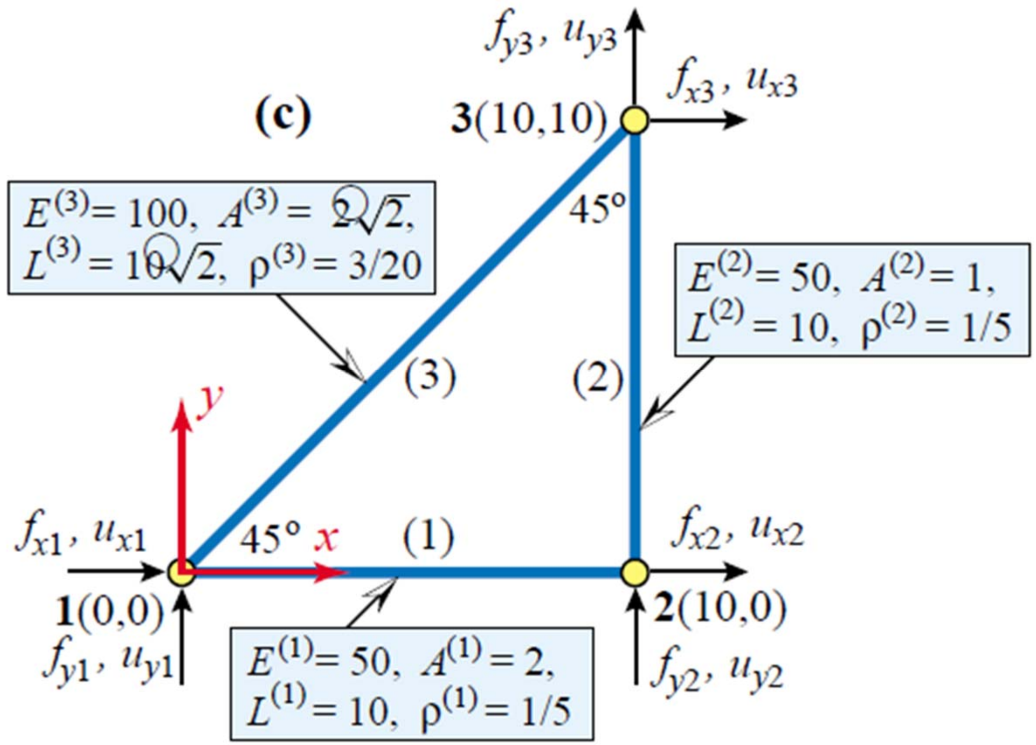
$$\left(\mathbf{T}^e\right)^T \bar{\mathbf{K}}^e \mathbf{T}^e \mathbf{u}^e = \mathbf{f}^e$$

$$\mathbf{K}_e \mathbf{u}^e = \mathbf{f}^e \quad \text{where} \quad \mathbf{K}_e = \left(\mathbf{T}^e\right)^T \bar{\mathbf{K}}^e \mathbf{T}^e$$

$$\mathbf{K}_e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

# Example Truss

Insert the geometric & physical properties of this model into the globalized member stiffness equations



# Globalized Element Stiffness Equations

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$

# Assembly Rules

- Compatibility
  - The joint displacements of all members meeting at a joint must be the same
- Equilibrium
  - The sum of forces exerted by all members that meet at a joint must balance the external force applied to that joint

# Expanded Element Stiffness Equations

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x3}^{(1)} \\ u_{y3}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x2}^{(3)} \\ u_{y2}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$

# Compatibility Rule

To apply compatibility, drop the member index from the nodal displacements

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \Rightarrow \mathbf{f}^{(1)} = \mathbf{K}^{(1)}\mathbf{u},$$

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \Rightarrow \mathbf{f}^{(2)} = \mathbf{K}^{(2)}\mathbf{u}$$

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \Rightarrow \mathbf{f}^{(3)} = \mathbf{K}^{(3)}\mathbf{u}$$

# Equilibrium Rule

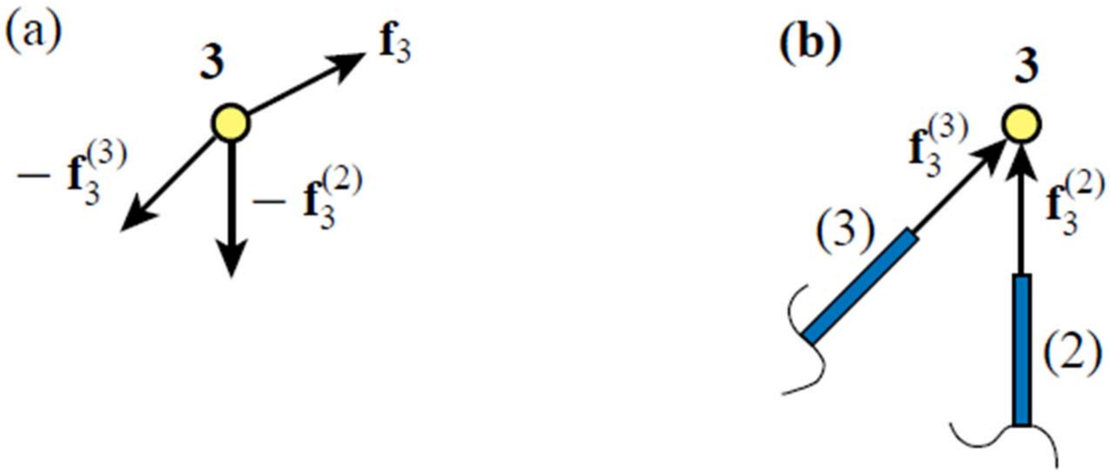



FIGURE 3.2. The force equilibrium of joint 3 of the example truss, depicted as an FBD in (a). Here $\mathbf{f}_3$ is the known external joint force applied on the joint. Internal forces $\mathbf{f}_3^{(2)}$ and $\mathbf{f}_3^{(3)}$ are applied *by the joint on the members*, as illustrated in (b). Thus the forces applied *by the members on the joint* are $-\mathbf{f}_3^{(2)}$ and $-\mathbf{f}_3^{(3)}$. These forces would act in the directions shown in (a) if members (2) and (3) were in tension. The free-body equilibrium statement is $\mathbf{f}_3 - \mathbf{f}_3^{(2)} - \mathbf{f}_3^{(3)} = \mathbf{0}$ or $\mathbf{f}_3 = \mathbf{f}_3^{(2)} + \mathbf{f}_3^{(3)}$. This translates into the two component equations: $f_{x3} = f_{x3}^{(2)} + f_{x3}^{(3)}$ and $f_{y3} = f_{y3}^{(2)} + f_{y3}^{(3)}$, of (3.2).

Applying this to all joints:

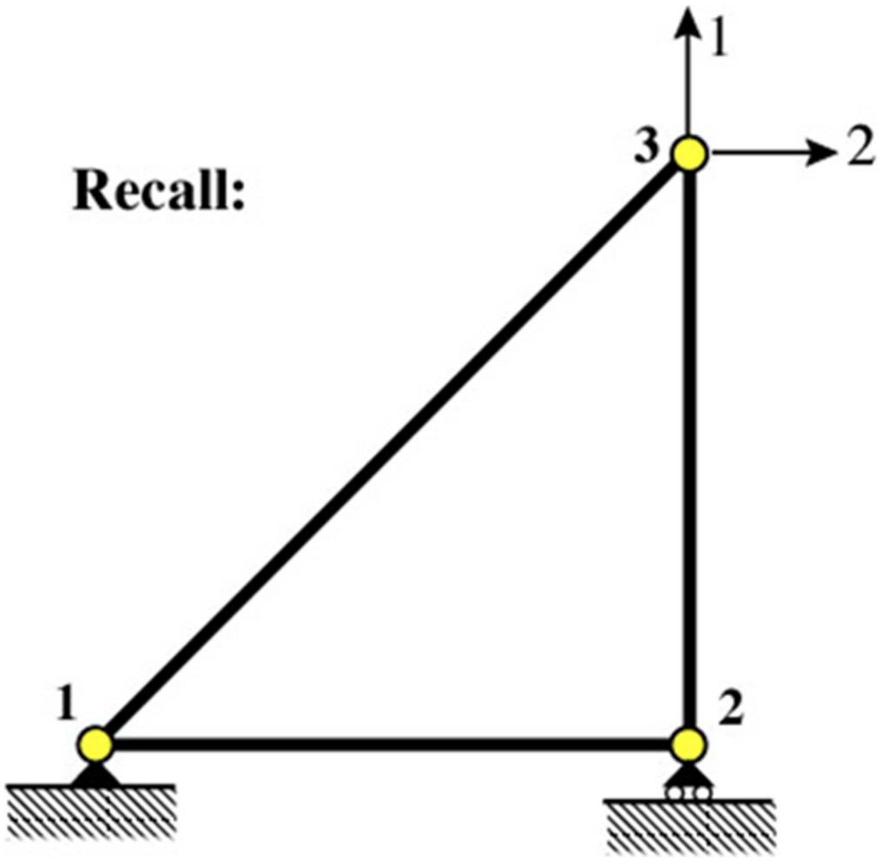
$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)}$$

# Master Stiffness Equations

$$\underbrace{\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = \left(\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\right)\mathbf{u} = \mathbf{K}\mathbf{u}}_{\Downarrow}$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

# BCs: Support and Loading




Displacement BCs:

$$u_{x1} = u_{y1} = u_{y2} = 0$$

Force BCs:

$$f_{x2} = 0, \ \ f_{x3} = 2, \ \ f_{y3} = 1$$

# Where Do BCs Go? Reduced Master Stiffness Equations

$$\begin{cases}\text{Displacement BCs: } u_{x1} = u_{y1} = u_{y2} = 0 \\ \text{Force BCs: } f_{x2} = 0,\ \ f_{x3} = 2,\ \ f_{y3} = 1\end{cases}$$

$$\rightarrow \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}$$

Strike out rows and columns pertaining to known displacements:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Leftrightarrow \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

Solve by Gauss elimination for unknown node displacements

# Solve for Unknown Node Displacements

Strike out rows and columns pertaining to known displacements:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Leftrightarrow \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

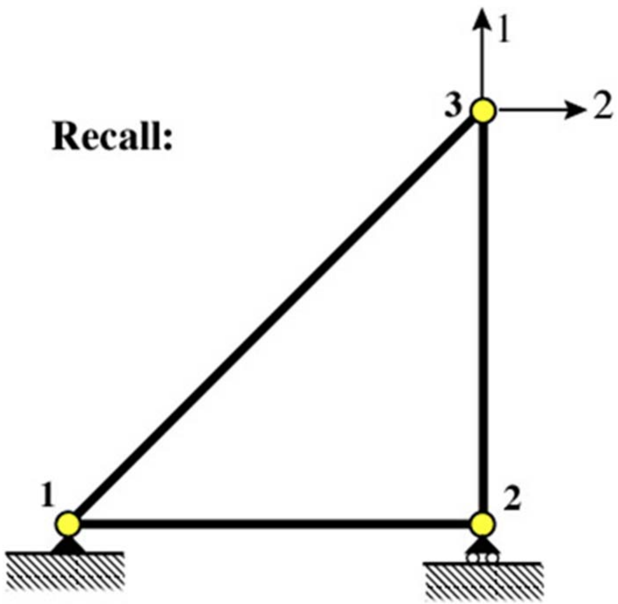
Solve by Gauss elimination for unknown node displacements

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix} \xrightarrow{\text{expand with known displacement BCs}} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ \color{red}{0} \\ 0 \\ \color{red}{0.4} \\ \color{red}{-0.2} \end{bmatrix}$$

# Recovery of Node Forces

$$\mathbf{f} = \mathbf{K}\mathbf{u} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{Reaction Forces}}$$

**Recall:**



# Recovery of Internal Forces

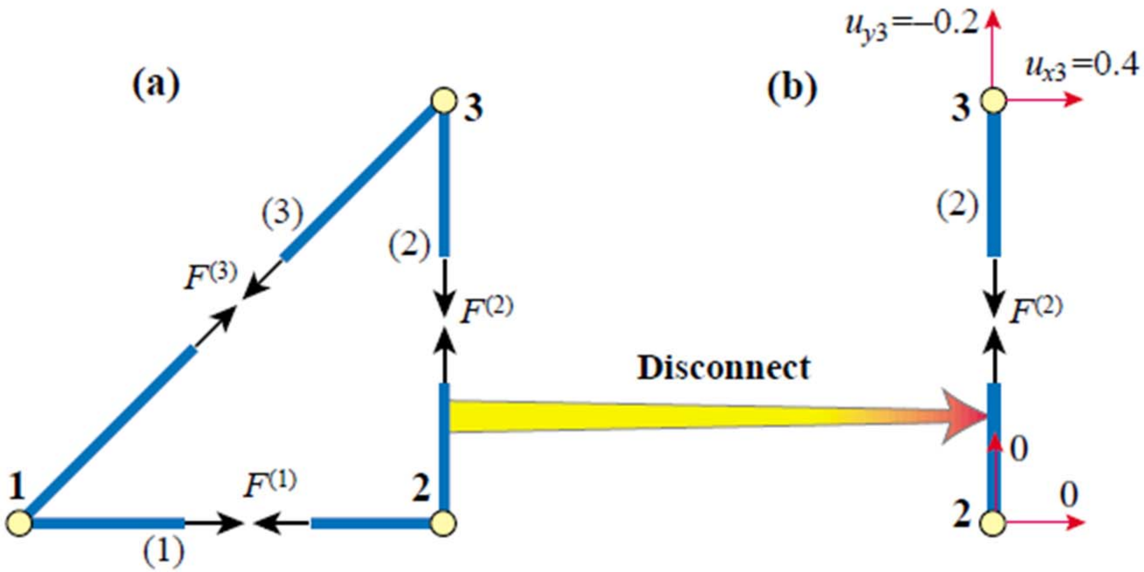



FIGURE 3.3. Internal force recovery for example truss: (a) member axial forces $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$, with arrow directions pertaining to tension; (b) details of computation for member (2).

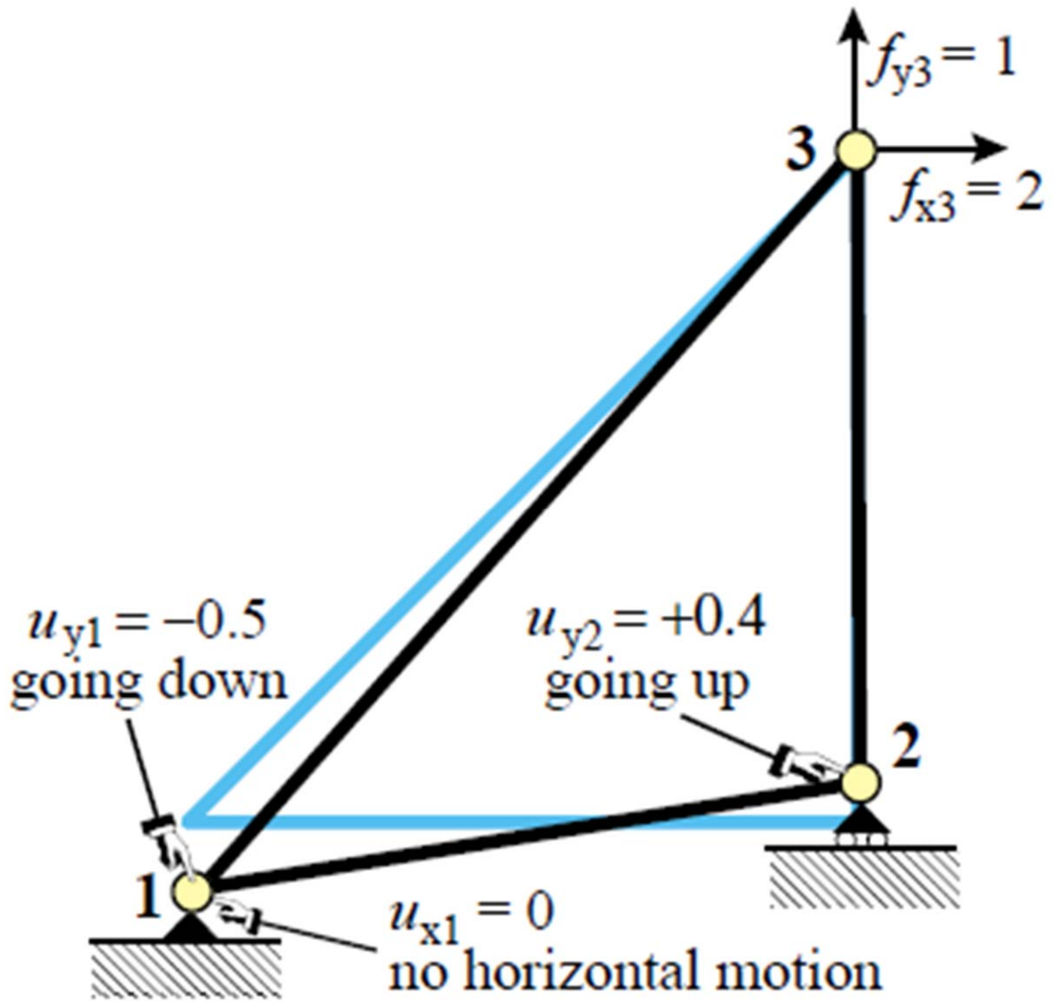
(1) extract $\mathbf{u}^e$ from $\mathbf{u}$

(2) transform to local (element) displacements: $\mathbf{u}^e = \mathbf{T}^e \mathbf{u}^e$

(3) compute elongation: $d^e = \bar{u}^e_{xj} - \bar{u}^e_{xi}$

(4) compute axial force: $F^e = \dfrac{E^e A^e}{L^e} d^e$

# Example: Prescribed Nonzero Displacements



Recall the master stiffness equations

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}$$

The displacement BCs are now

$$u_{x1} = 0, \quad u_{y1} = -0.5, \quad u_{y2} = 0.4$$

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 0 \\ f_{y2} \\ 2 \\ 1 \end{bmatrix}$$

Remove rows 1, 2, 4 but (for now) keep columns

$$\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Transfer effect of known displacements to RHS, and delete columns:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} (-10)\times 0 + 0\times(-0.5) + 0\times 0.4 \\ (-10)\times 0 + (-10)\times(-0.5) + 0\times 0.4 \\ (-10)\times 0 + (-10)\times(-0.5) + (-5)\times 0.4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$$

Solving gives

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix} \xrightarrow{\text{Complete the displacement vector with known values}} \mathbf{u} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.4 \\ -0.5 \\ 0.2 \end{bmatrix}$$

In summary, the only changes to the SDM is in the application of displacement boundary conditions before solve