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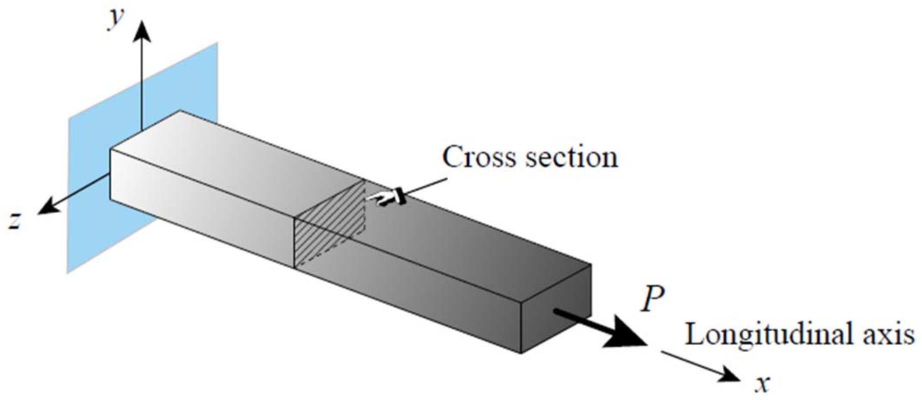
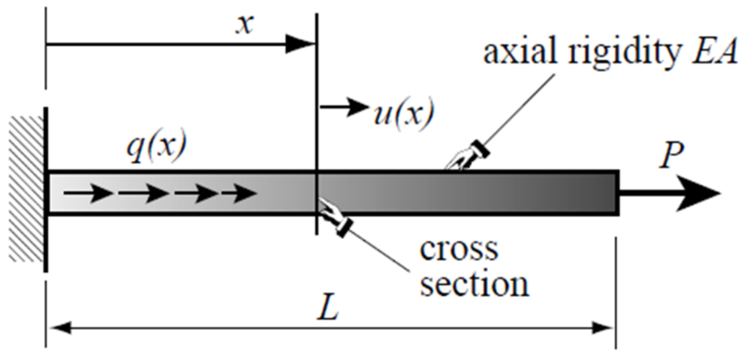
Bar Member

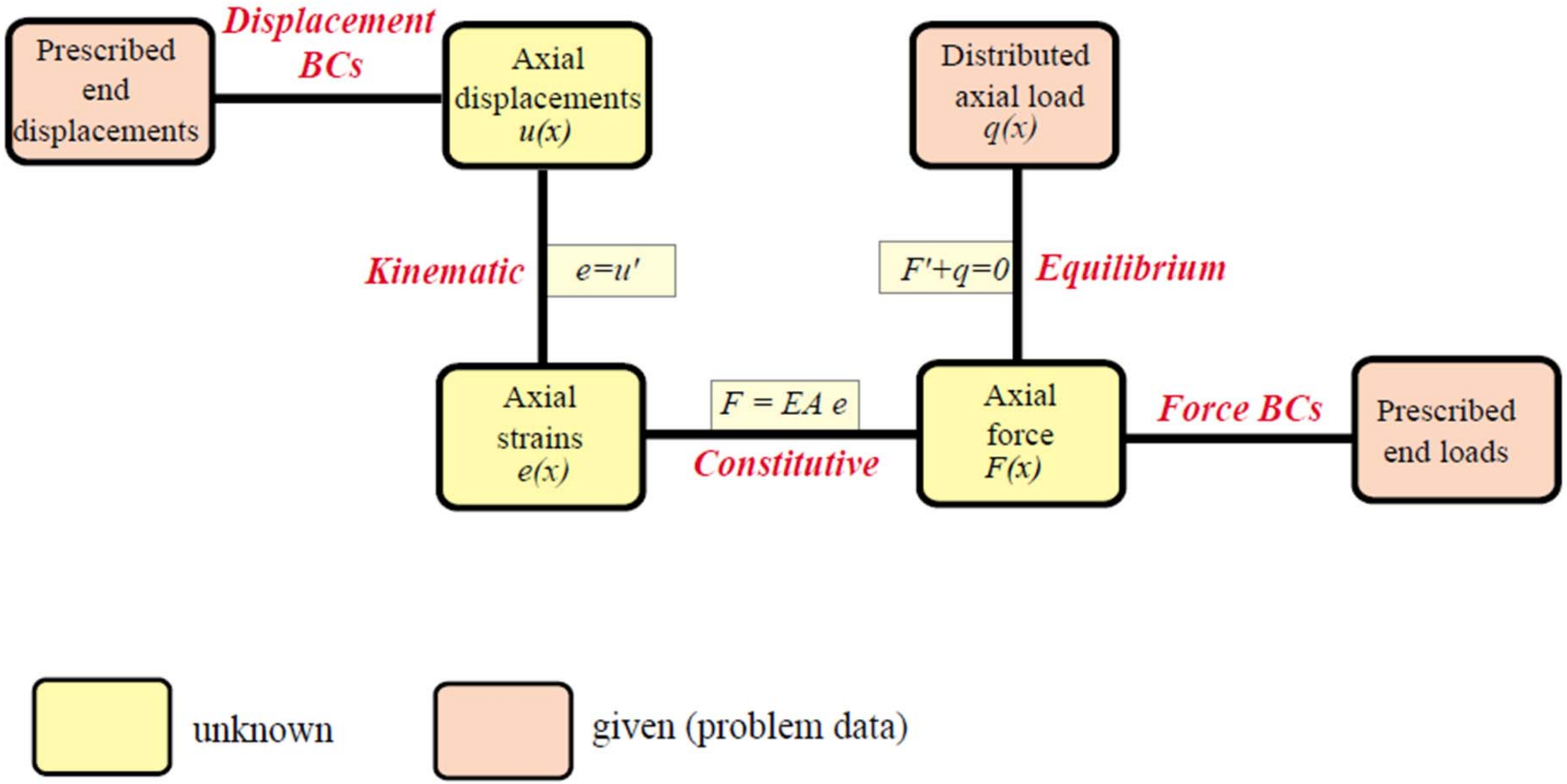
- Characteristics
  - One preferred (longitudinal dimension or axial) dimension
    - much larger than the other two (transverse) dimensions
    - cross section: intersection of a plane normal to the longitudinal dimension and the bar
  - Resist an internal axial force along its longitudinal dimension
- Modeling (truss)
  - cable, chain, rope
  - fictitious elements in penalty function method





Tonti Diagram of Governing Equations

- Straight bar: cross section may vary
- Linearly elastic material: Hooke's law
- Infinitesimal displacements and strains



Potential Energy of the Bar Member

Internal energy (=strain energy):

$$U = \frac{1}{2}\int_V \sigma e dV = \frac{1}{2}\int_0^L \sigma e\left(Adx\right)\left[=\frac{1}{2}\int_0^L Fedx\right] = \frac{1}{2}\int_0^L \left(EAu'\right)u'dx = \frac{1}{2}\int_0^L u'EAu'dx$$

External work: $W = \int_0^L qudx$

Total Potential Energy: $\Pi = U - W$

Minimum Total Potential Energy(MTPE) Principle:

actual displacement solution $u^*\left(x\right)$ that satisfies the governing equations is

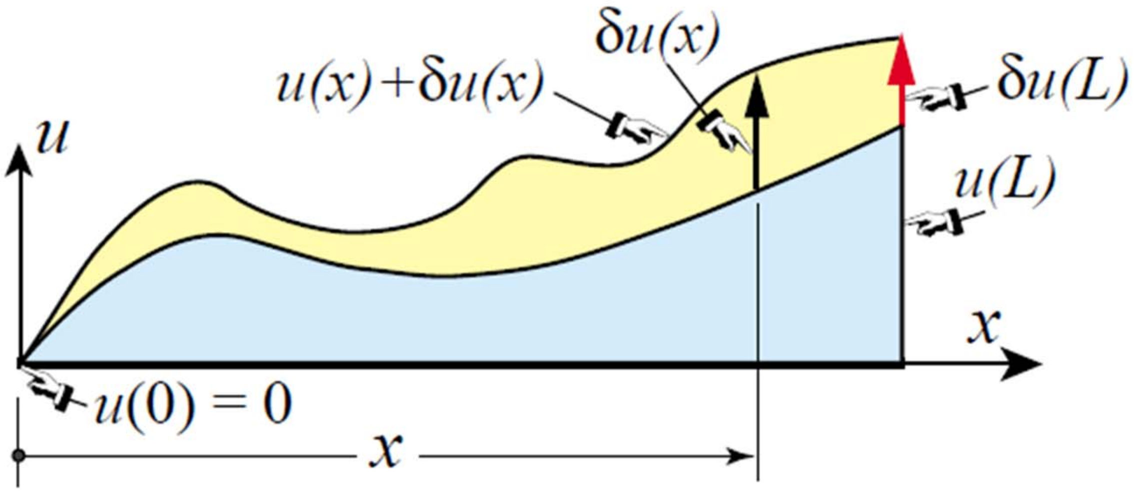
that which renders the TPE function $\Pi\left[u\right]$ stationary

$\delta\Pi = \delta U - \delta W = 0 \;\text{ iff }\; u = u^*$

with respect to *admissible* variations $u = u^* + \delta u$ of

the exact displacement solution $u^*\left(x\right)$

Concept of Kinematically Admissible Variation

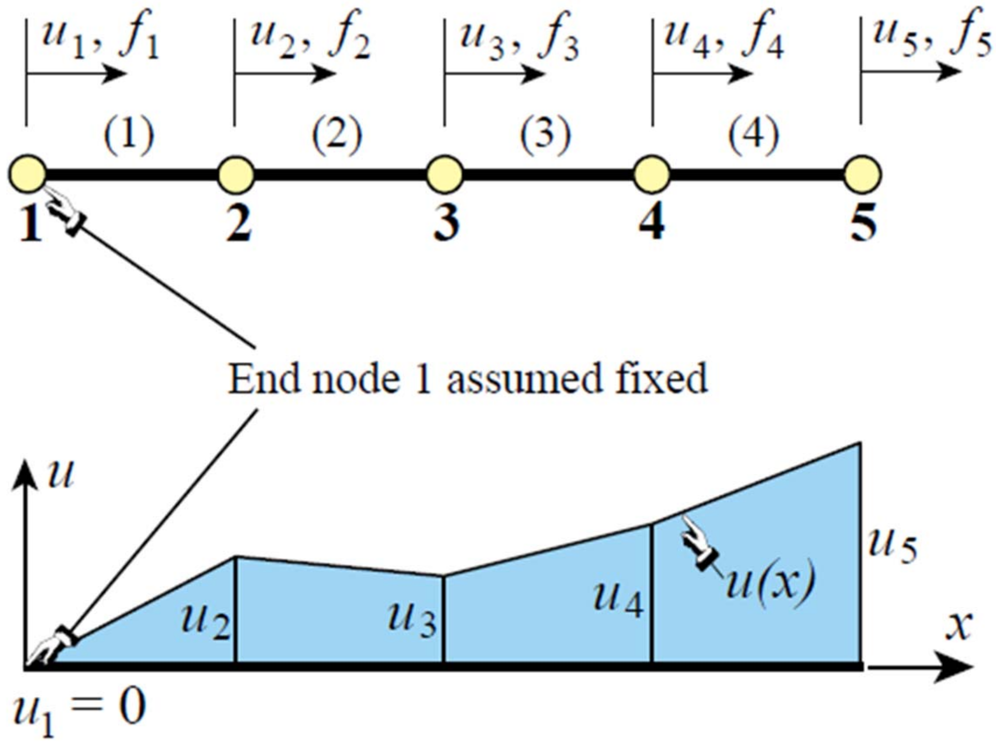


$\delta u(x)$ is kinematically admissible if $u(x)$ and $u(x)+\delta u(x)$

(i) are continuous over bar length, i.e. $u(x) \in C^0$ in $x \in [0, L]$

(ii) satisfy exactly displacement BC, in the figure, $u(0)=0$

FEM Discretization and Displacement Trial Function



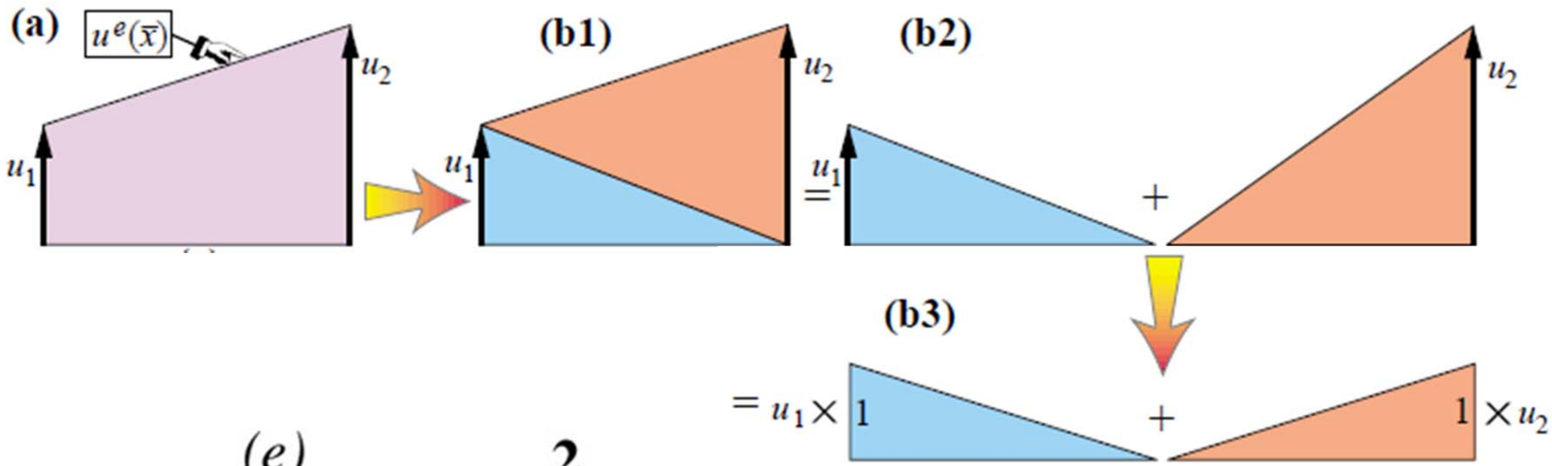
$$\delta\Pi = \delta U - \delta W = 0 \ \text{ iff } \ u = u^* \left(\text{exact solution}\right)$$

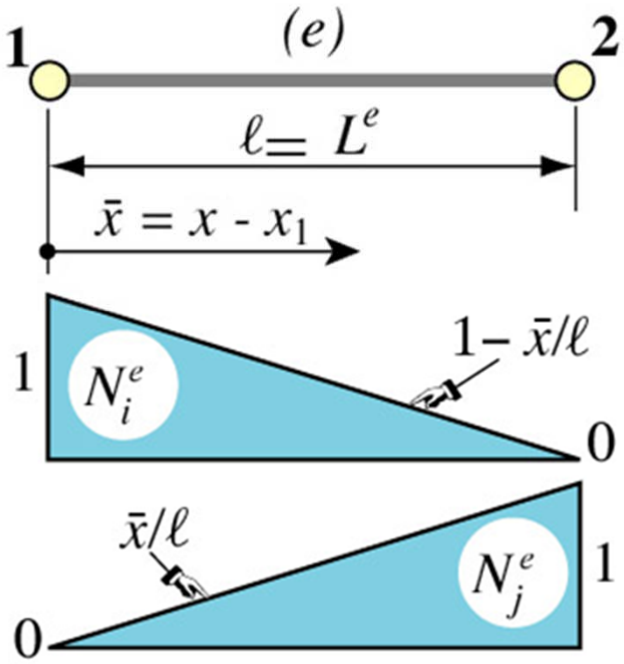
$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots + \Pi^{(N_e)}$$

$$\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \cdots + \delta\Pi^{(N_e)} = 0$$

$$\delta\Pi^e = \delta U^e - \delta W^e = 0$$

Element Shape Functions





$$u^*(x) \approx u^e(x) = N_1^e u_1^e + N_2^e u_2^e$$

$$= \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}\mathbf{u}^e$$

$$N_1^e = 1 - \frac{\bar{x}}{l} = 1 - \zeta,\; N_2^e = \frac{\bar{x}}{l} = \zeta$$

$$\zeta = \frac{x - x_1}{l} = \frac{\bar{x}}{l}$$

dimensionless (natural) coordinate

Finite Element Equation

$$\Pi^e = U^e - W^e \leftarrow \begin{cases} U^e = \dfrac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \mathbf{u}^e \\ W^e = \left(\mathbf{u}^e\right)^T \mathbf{f}^e \end{cases}$$

$$\delta\Pi^e = \delta U^e - \delta W^e = \frac{1}{2}\left(\delta\mathbf{u}^e\right)^T \mathbf{K}^e \mathbf{u}^e + \frac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \delta\mathbf{u}^e - \left(\delta\mathbf{u}^e\right)^T \mathbf{f}^e = 0$$

$$\xrightarrow{\mathbf{u}^e = \left(\mathbf{u}^e\right)^T,\ \delta\mathbf{u}^e = \left(\delta\mathbf{u}^e\right)^T} \left(\delta\mathbf{u}^e\right)^T \left[\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e\right] = 0$$

since $\delta\mathbf{u}^e$ is arbitrary, $\left[\ldots\right] = 0$

$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e$ (element stiffness equations)

Bar Element Stiffness and Nodal Force Vector

$$[\text{strain-displacement}]\, e = \frac{du^e}{dx} = \left(u^e\right)' = \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{B}\mathbf{u}^e$$

$$[\text{internal energy}]\, U^e = \frac{1}{2}\int_0^l \left(u^e\right)' EA \left(u^e\right)' dx = \frac{1}{2}\int_0^1 \left(u^e\right)' EA \left(u^e\right)' l d\zeta$$

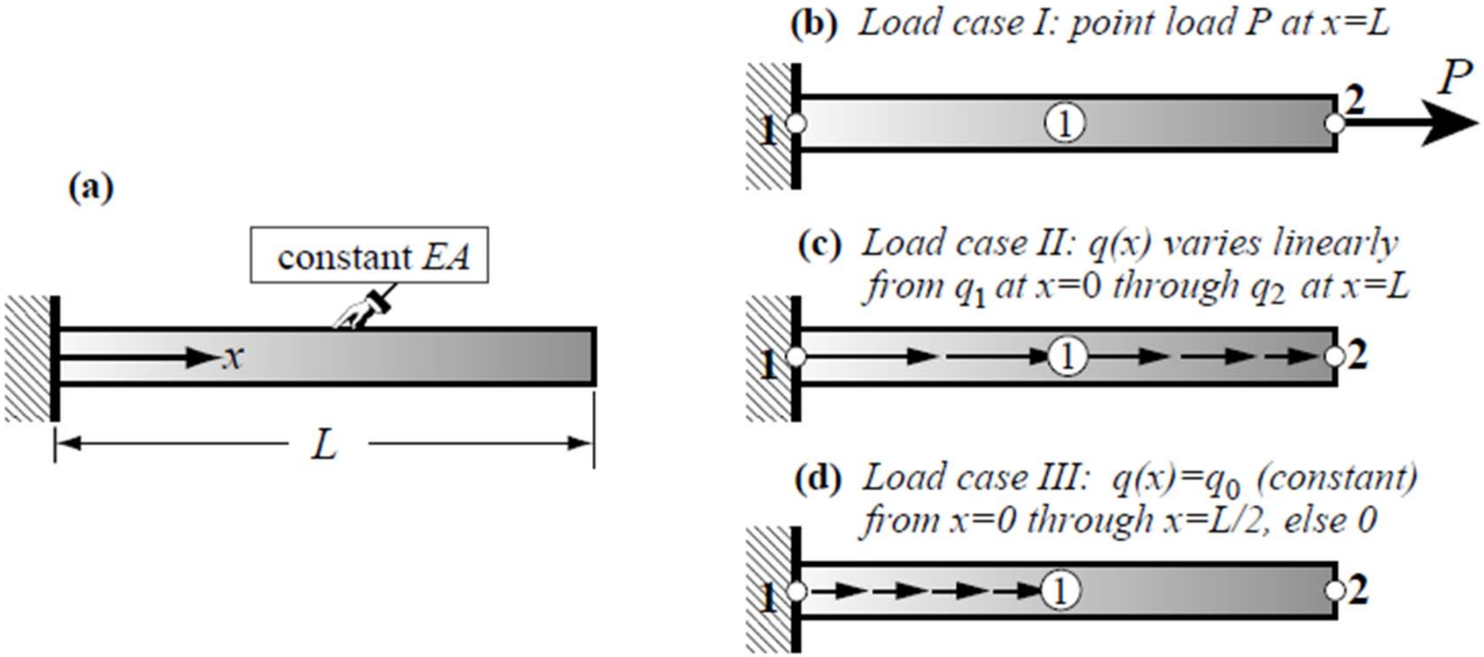
$$= \frac{1}{2}\int_0^1 \left(\mathbf{u}^e\right)^T \mathbf{B}^T EA\mathbf{B}\mathbf{u}^e l d\zeta = \frac{1}{2}\left(\mathbf{u}^e\right)^T \underbrace{\left[\int_0^1 EA\mathbf{B}^T\mathbf{B} l d\zeta\right]}_{\mathbf{K}^e} \left(\mathbf{u}^e\right) = \frac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \left(\mathbf{u}^e\right)$$

$$\mathbf{K}^e = \int_0^1 \frac{EA}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} l d\zeta \xrightarrow[\text{over the element}]{\text{if } EA \text{ is constant}} \mathbf{K}^e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\text{external work}]\, W^e = \int_0^l qu dx = \int_0^1 q\mathbf{N}^T\mathbf{u}^e l d\varsigma = \left(\mathbf{u}^e\right)^T \underbrace{\int_0^1 q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l d\zeta}_{\mathbf{f}^e} = \left(\mathbf{u}^e\right)^T \mathbf{f}^e$$

$$\mathbf{f}^e = \int_0^1 q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l d\zeta \xrightarrow[\text{along the element}]{\text{if } q \text{ is constant}} \mathbf{f}^e = q\int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} l d\varsigma = ql \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \leftarrow \text{EbE load lumping}$$

Example: Fixed-Free, Prismatic Bar (1)



$$q^{I}\left(x\right)=P\delta\left(L\right)\rightarrow f^{I}=\begin{bmatrix}0\\P\end{bmatrix}$$

$$q^{II}\left(x\right)=q_1\left(1-\zeta\right)+q_2\varsigma\rightarrow f^{II}=\int_0^1 q^{II}\begin{bmatrix}1-\zeta\\ \zeta\end{bmatrix}Ld\varsigma=\frac{L}{6}\begin{bmatrix}2q_1+q_2\\q_1+2q_2\end{bmatrix}$$

$$q^{III}\left(x\right)=q_0\left[H\left(x\right)-H\left(x-\frac{L}{2}\right)\right]\rightarrow f^{III}=\int_0^L q^{III}\begin{bmatrix}1-x/L\\x/L\end{bmatrix}dx=\frac{q_0L}{8}\begin{bmatrix}3\\1\end{bmatrix}$$

Example: Fixed-Free, Prismatic Bar (2)

$$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}f_1\\ f_2\end{bmatrix}\xrightarrow{u_1=0}\begin{cases} f^{I}=\begin{bmatrix}0\\ P\end{bmatrix}\rightarrow u_2=\dfrac{PL}{EA}\\ f^{II}=\dfrac{L}{6}\begin{bmatrix}2q_1+q_2\\ q_1+2q_2\end{bmatrix}\rightarrow u_2=\dfrac{(q_1+2q_2)L^2}{6EA}\\ f^{III}=\dfrac{q_0L}{8}\begin{bmatrix}3\\ 1\end{bmatrix}\rightarrow u_2=\dfrac{q_0L^2}{8EA}\end{cases}$$

[analytical solution]

$(EAu')'+q=0$ with $u(0)=0$ and

$$\begin{cases} F^{I}(L)=EAu'(L)=P,\ q=0\rightarrow u(x)=\dfrac{Px}{EA}\\ F^{II}(L)=EAu'(L)=0,\ q=q_1\left(1-\dfrac{x}{L}\right)+q_2\dfrac{x}{L}\rightarrow u(x)=\dfrac{x\left[3(q_1+q_2)L-3q_1x+(q_1-q_2)x^2/L\right]}{6EA}\\ F^{III}(L)=EAu'(L)=0,\ q=q_0\left[\langle x\rangle^0-\left\langle x-\dfrac{1}{2}L\right\rangle^0\right]\rightarrow u(x)=\dfrac{q_0}{2EA}\left(Lx-x^2+\left\langle x-\dfrac{1}{2}L\right\rangle^2\right)\end{cases}$$

Weak Forms

$$[\text{Strong Form}]\begin{cases}\left(EAu'(x)\right)' + q(x) = 0 \xrightarrow{EA \text{ is constant}} EAu''(x) + q(x) = 0 \\ r(x) = \left(EAu'(x)\right)' + q(x) \xrightarrow{EA \text{ is constant}} r(x) = EAu''(x) + q(x) \\ r(x) = 0: \text{ at each point over the member span, } x \in [0, L]\end{cases}$$

$$[\text{Weak Form}]\begin{cases}\text{relax the condition } \left(r(x) = 0 \text{ everywhere}\right) \rightarrow \text{satisfy in an average sense} \\ J = \int_0^L r(x)v(x)\,dx = 0 \\ v(x) = \begin{cases}\text{test function in a genenral mathematical context} \\ \text{weight(ing) function in the approximation method}\end{cases}\end{cases}$$

Example (1)

$$J=\int_0^L\left[EAu''(x)+q_0\right]v(x)\,dx=0\ \text{ with } u(0)=0,\ F(L)=EAu'(L)=0$$

[method 1]

$$\left.\begin{aligned}u(x)&=a_0+a_1x+a_2x^2\rightarrow\text{trial function}\\ v(x)&=b_0+b_1x+b_2x^2\rightarrow\text{weight function}\end{aligned}\right\}\xrightarrow[\text{bases}]{\text{same}}\text{Galerkin method}$$

(apply BCs *a posteri*)

$$J=\int_0^L\left[EA(2a_2)+q_0\right]\left(b_0+b_1x+b_2x^2\right)dx=\frac{L}{6}\left(6b_0+3b_1L+2b_2L^2\right)\left(2EAa_2+q_0\right)=0$$

$$\rightarrow u(x)=a_0+a_1x-\frac{q_0}{2EA}x^2\xrightarrow[F(L)=EAu'(L)=0]{u(0)=0}u(x)=\frac{q_0}{2EA}x(2L-x)$$

(apply BCs *a priori*)

$$u(x)=a_0+a_1x+a_2x^2\xrightarrow[F(L)=EAu'(L)=0]{u(0)=0}u(x)=a_2x(x-2L)$$

$$J=\int_0^L\left[EA(2a_2)+q_0\right]\left(b_0+b_1x+b_2x^2\right)dx=0\rightarrow a_2=-\frac{q_0}{2EA}$$

Example (2)

[method 2] balanced-derivative

$$J=\int_0^L\left[EAu''(x)v(x)+q_0v(x)\right]dx=\left[EAu'(x)v(x)\right]_0^L-\int_0^L EAu'(x)v'(x)\,dx+\int_0^L q_0v(x)\,dx=0$$

(i) same smoothness requirements for assumed u and v

(ii) BC appear explicitly in the non-integral term

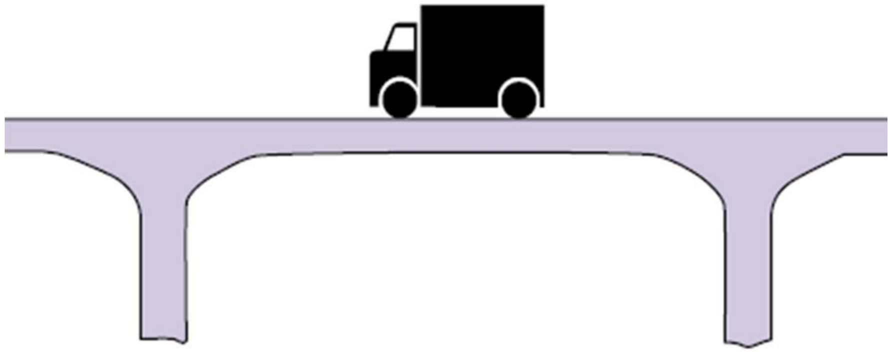
$$\xrightarrow{v(x)=\delta u(x)} J=\int_0^L EAu'(x)\delta u'(x)\,dx-\int_0^L q_0\delta u(x)\,dx-\left[EAu'(x)\delta u(x)\right]_0^L\equiv\delta\Pi$$

$$\Pi=U-W=\frac{1}{2}\int_0^L u'(x)EAu'(x)\,dx-\int_0^L q_0u(x)\,dx$$

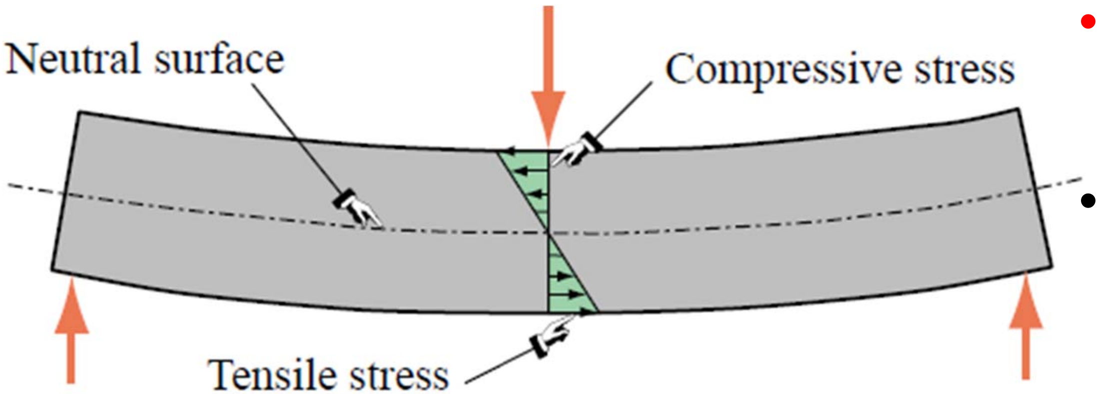
$$J=0\leftrightarrow\delta\Pi=0\leftrightarrow\delta U=\delta W$$

Galerkin method $\xleftrightarrow[\text{the Euler-Lagrange equation of a functional}]{\text{if the residual is}}$ variational formulation

What is a Beam?



Resist primarily transverse loads
General beam > beam-column > beam

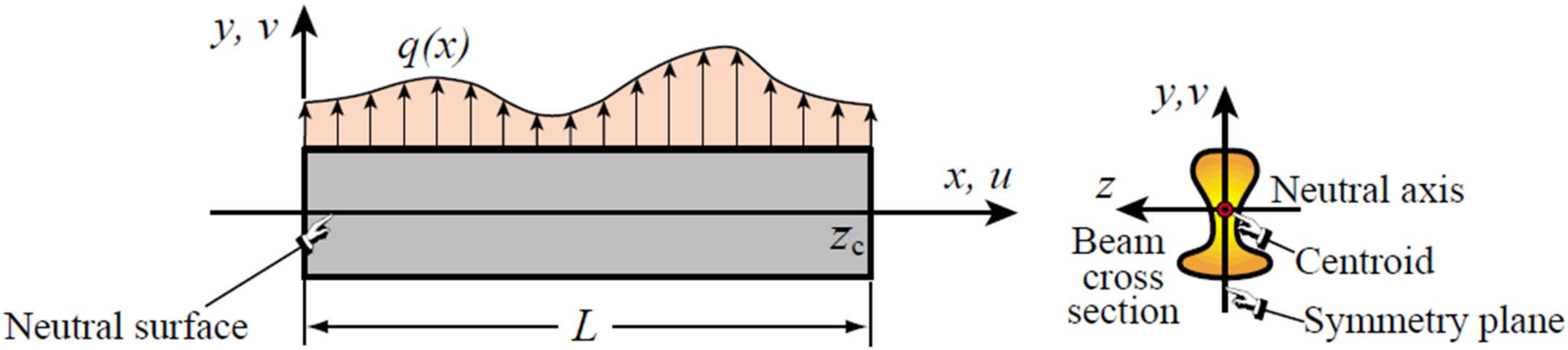


transverse loads → (flexural action) → supports

- Terminology
  - Straight (longitudinal axis)
  - Prismatic (const cross-sec)
- Configuration
  - Spatial
  - Plane
- Model (beam theory)
  - Bernoulli-Euler
    - Hermitian beam element
    - C^1 element
  - Timoshenko
    - C^0 element

Assumptions of Classical Beam Theory

- Planar symmetry
- Cross section variation
- Normality
- Strain energy: only for bending moment deformations
- Linearization
  - So small transverse deflections, rotations and deformations
- Material model: elastic and isotropic

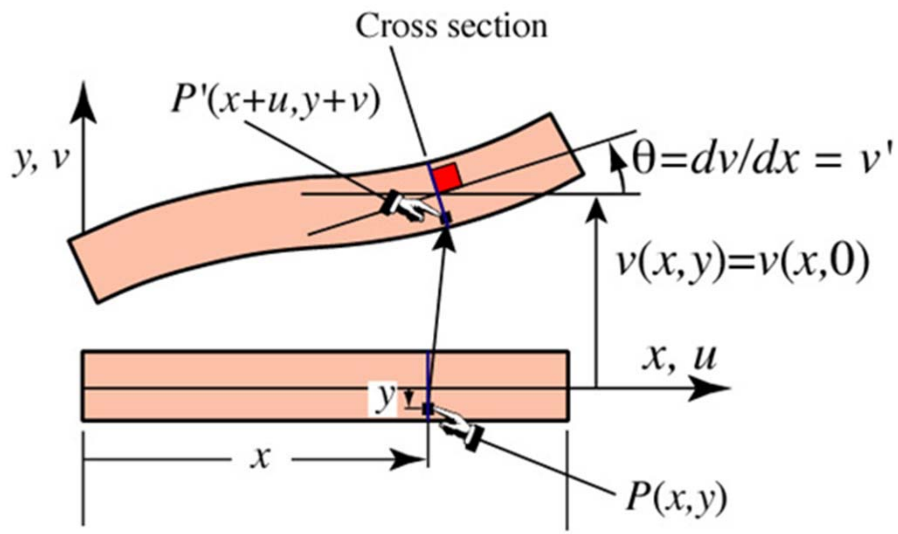


Bernoulli-Euler Beam Theory

[Kinematics]

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y\dfrac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

$$\kappa = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}} \approx \frac{\partial^2 v}{\partial x^2}$$

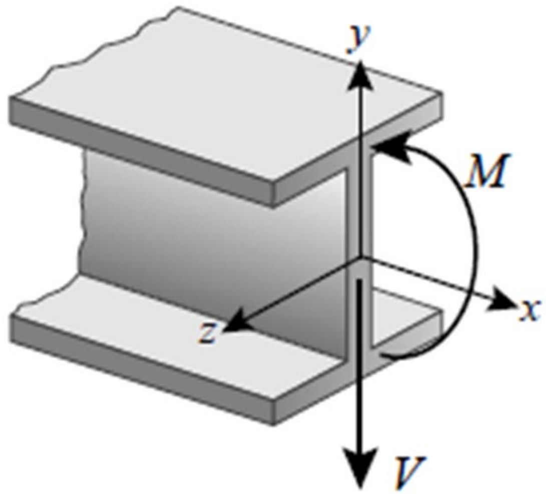


[Strains, Stresses, Bending Moments]

$$e = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = -yv'' = -y\kappa$$

$$\sigma = Ee = -Ey\frac{\partial^2 v}{\partial x^2} = -Ey\kappa$$

$$M = \int_A -y\sigma dA = E\frac{\partial^2 v}{\partial x^2}\int_A y^2 dA = EI\kappa$$

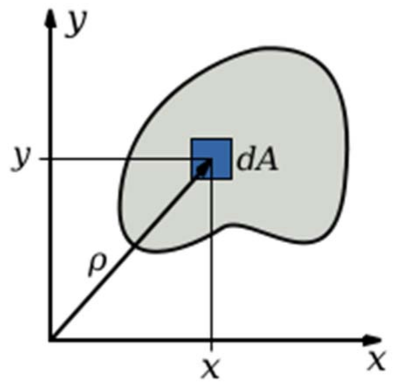


Moment of Inertia

- Mass moment of inertia (관성모멘트)

$$I = kmr^2 = \sum_{i=1}^{n} m_i r_i^2 = \int r^2 dm = \iiint_V r^2 \rho(r) dV \to I = I_{cm} + md^2$$

- Area moment of inertia
  - Second moment of area (단면이차모멘트): bending
  - Polar moment of inertia (극관성모멘트): torsion
  - Product of inertia: unsymmetric geometry



$$I_{xx} = \int_A y^2 dA \to I_{xx} = I_{xx\_c} + \bar{x}^2 A \text{ where } \bar{x}A = \int_A x dA$$

$$I_{yy} = \int_A x^2 dA$$

$$J(= I_z) = \int_A \rho^2 dA = \int_A \left(x^2 + y^2\right) dA = \int_A x^2 dA + \int_A y^2 dA = I_{xx} + I_{yy}$$

$$I_{xy} = \int_A xy dA$$

Curvature

– Rate of change of the slope angle of the curve w.r.t. distance along the curve

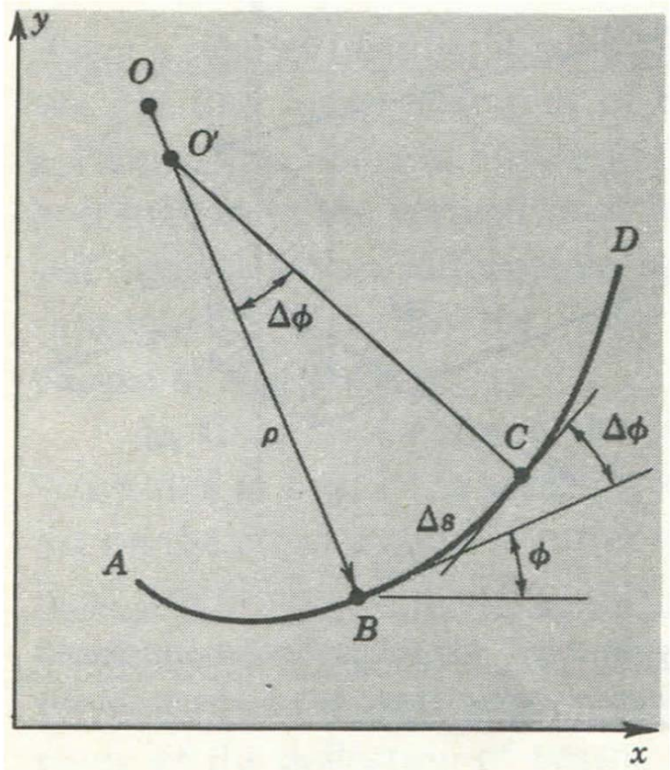
$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} == \lim_{\Delta s \to 0} \frac{1}{O'B} = \frac{1}{\rho}$$

ρ : radius of curvature @B

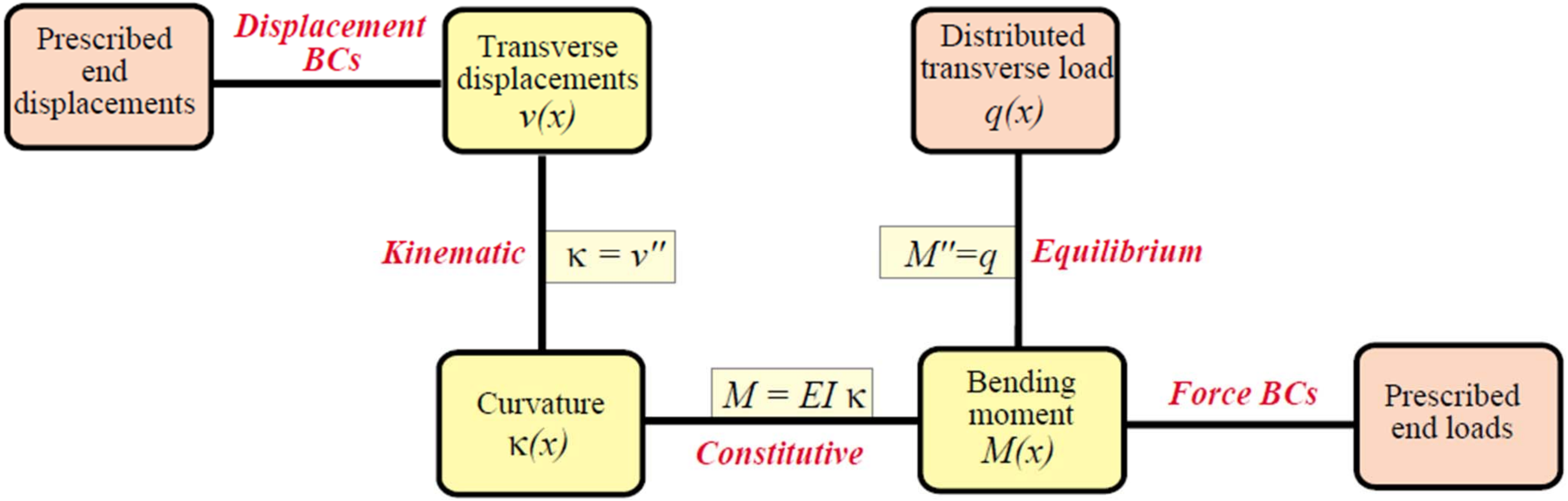
$$\frac{dy}{dx} = \tan\phi \rightarrow \frac{d^2 y}{dx^2} \underbrace{\frac{dx}{ds}}_{\cos\phi} = \sec^2\phi \frac{d\phi}{ds}$$

$$\cos\phi = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$\frac{d\phi}{ds} = \frac{d^2 y}{dx^2} \cos^3\phi = \frac{d^2 y}{dx^2} \Bigg/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$



Tonti Diagram of the Bernoulli-Euler beam model



[Internal energy due to bending]

$$U = \frac{1}{2}\int_V \sigma e dV = \frac{1}{2}\int_V (-Ey\kappa)(-y\kappa)dV = \frac{1}{2}\int_0^L E\kappa^2 dx \int_A y^2 dA = \frac{1}{2}\int_0^L EI\kappa^2 dx$$

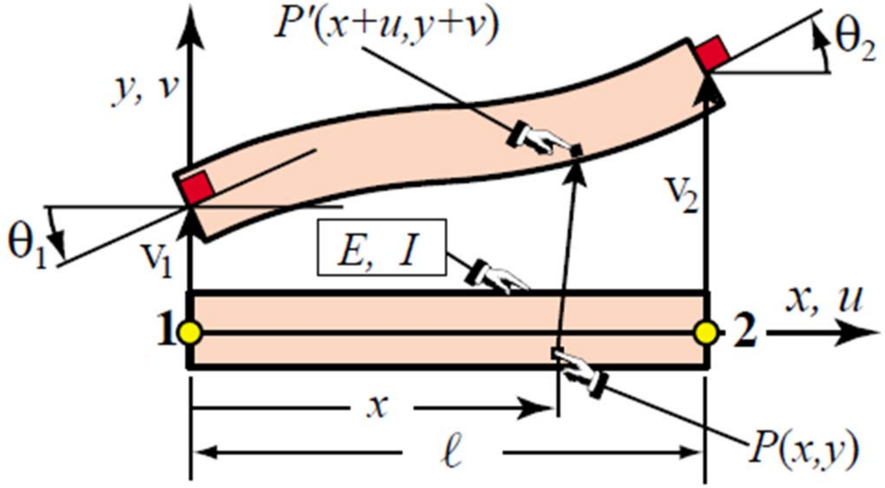
$$= \frac{1}{2}\int_0^L M\kappa dx = \frac{1}{2}\int_0^L EI(v'')^2 dx = \frac{1}{2}\int_0^L v'' EI v'' dx$$

[External energy due to transverse load q] $W = \int_0^L qv dx$

$$\Pi = U - W$$

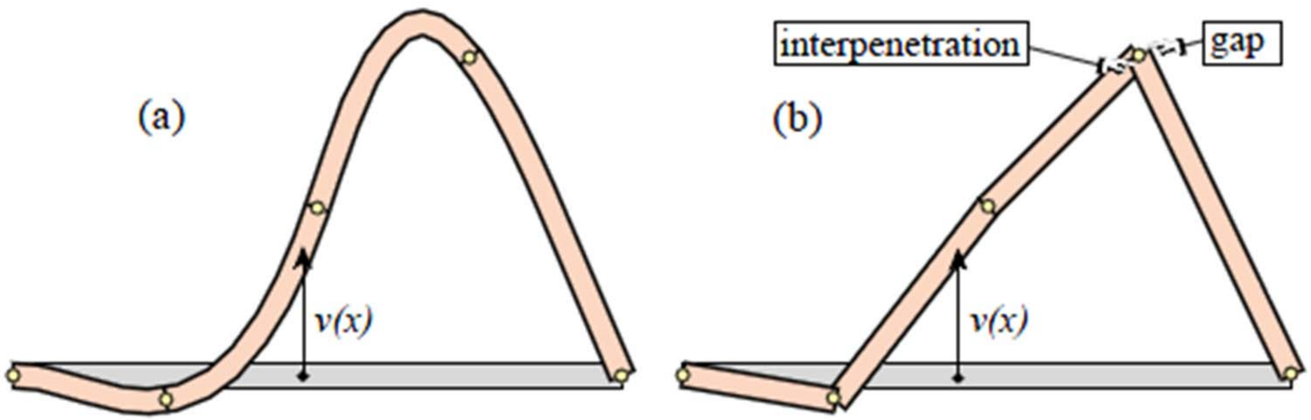
Beam Finite Elements

$$\mathbf{u}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$



C^1 continuity requirement:

$v(x)$ and $\theta = v'(x) = \dfrac{dv(x)}{dx}$ must be continuous over the entire member and between elements



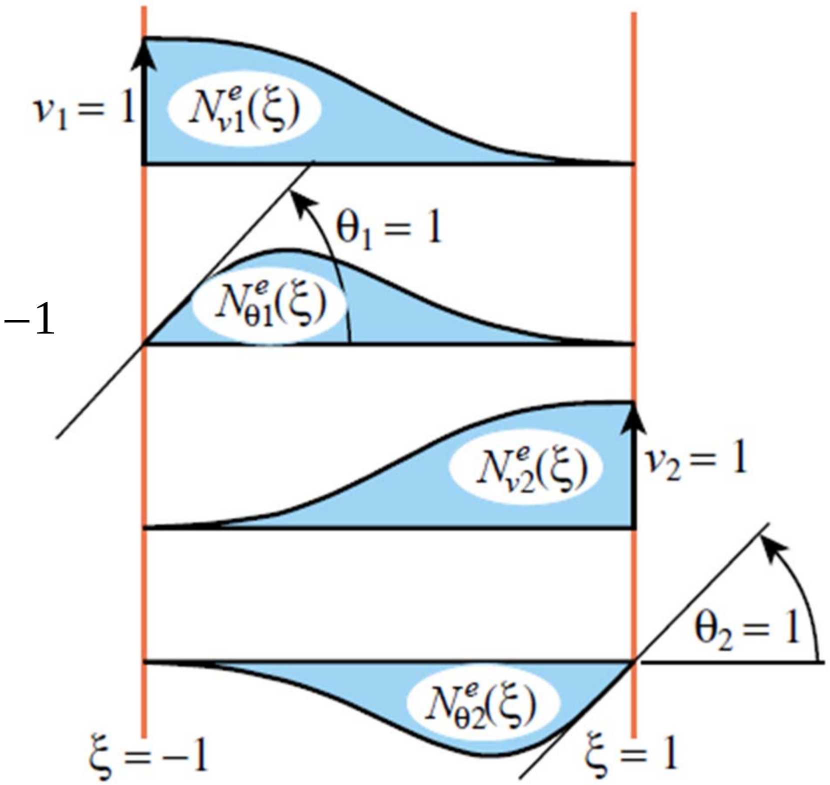
Hermitian Cubic Shape Functions (1)

$$v^e = \begin{bmatrix} N_{v_1}^e & N_{\theta_1}^e & N_{v_2}^e & N_{\theta_2}^e \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{N}^e \mathbf{u}^e$$

introduce the natural (isoparametric) coordinate

$$\left.\begin{matrix} x : 0 \sim l \\ \xi : -1 \sim +1 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} \xi(x) = ax + b \\ \xi(0) = -1 \text{ and } \xi(l) = +1 \end{matrix}\right\} \rightarrow \xi = \frac{2x}{l} - 1$$

	$N^e(-1)$	$\frac{dN^e}{dx}(-1)$	$N^e(+1)$	$\frac{dN^e}{dx}(+1)$
$N_{v_1}^e$	1	0	0	0
$N_{\theta_1}^e$	0	1	0	0
$N_{v_2}^e$	0	0	1	0
$N_{\theta_2}^e$	0	0	0	1



Hermitian Cubic Shape Functions (2)

$$N_{v_1}^e = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \rightarrow \left(N_{v_1}^e\right)' = a_1 + 2a_2\xi + 3a_3\xi^2$$

$$\left.\begin{array}{l} \left.\begin{array}{l} N_{v_1}^e(-1) = a_0 - a_1 + a_2 - a_3 = 1 \\ N_{v_1}^e(+1) = a_0 + a_1 + a_2 + a_3 = 0 \end{array}\right\} 2a_1 + 2a_3 = -1 \\ \left.\begin{array}{l} \left(N_{v_1}^e\right)'(-1) = a_1 - 2a_2 + 3a_3 = 0 \\ \left(N_{v_1}^e\right)'(+1) = a_1 + 2a_2 + 3a_3 = 0 \end{array}\right\} 2a_1 + 6a_3 = 0 \end{array}\right\} a_3 = \frac{1}{4}, a_1 = -\frac{3}{4} \rightarrow \begin{cases} a_0 + a_2 = \frac{1}{2} \\ a_2 = 0 \end{cases}$$

$$N_{v_1}^e = \frac{1}{4}\left(2 - 3\xi + \xi^3\right) = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta_1}^e = \frac{1}{8}l(1-\xi)^2(1+\xi)$$

$$N_{v_2}^e = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta_2}^e = -\frac{1}{8}l(1+\xi)^2(1-\xi)$$

Curvatures from Displacements

$$\kappa = \frac{d^2 v(x)}{dx^2} = \frac{d^2 \mathbf{N}^e}{dx^2}\mathbf{u}^e + \mathbf{N}^e \frac{d^2 \mathbf{u}^e}{dx^2} = \frac{d^2 \mathbf{N}^e}{dx^2}\mathbf{u}^e = \begin{bmatrix} \frac{d^2 N^e_{v_1}}{dx^2} & \frac{d^2 N^e_{\theta_1}}{dx^2} & \frac{d^2 N^e_{v_2}}{dx^2} & \frac{d^2 N^e_{\theta_2}}{dx^2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{B}\mathbf{u}^e$$

$$\mathbf{B} = \frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} & 3\xi - 1 & -6\frac{\xi}{l} & 3\xi + 1 \end{bmatrix}$$

$$\frac{df(x)}{dx} = \frac{df(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{df(\xi)}{d\xi}\frac{2}{l} \rightarrow \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df(\xi)}{d\xi}\right)\frac{2}{l} + \frac{df(\xi)}{d\xi}\frac{d}{dx}\left(\frac{2}{l}\right) = \frac{4}{l^2}\frac{d^2 f(\xi)}{d\xi^2}$$

$$N^e_{v_1} = \frac{1}{4}(1-\xi)^2(2+\xi) \rightarrow \frac{d^2 N^e_{v_1}}{dx^2} = \frac{4}{l^2}\left[\frac{1}{4}\frac{d}{d\xi^2}\left(-2(1-\xi)(2+\xi) + (1-\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{1}{4}6\xi\right)$$

$$N^e_{\theta_1} = \frac{1}{8}l(1-\xi)^2(1+\xi) \rightarrow \frac{d^2 N^e_{\theta_1}}{dx^2} = \frac{4}{l^2}\left[\frac{l}{8}\frac{d}{d\xi^2}\left(-2(1-\xi)(1+\xi) + (1-\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{l}{8}(6\xi - 2)\right)$$

$$N^e_{v_2} = \frac{1}{4}(1+\xi)^2(2-\xi) \rightarrow \frac{d^2 N^e_{v_2}}{dx^2} = \frac{4}{l^2}\left[\frac{1}{4}\frac{d}{d\xi^2}\left(2(1+\xi)(2-\xi) - (1+\xi)^2\right)\right] = \frac{4}{l^2}\left(\frac{1}{4}(-6\xi)\right)$$

$$N^e_{\theta_2} = -\frac{1}{8}l(1+\xi)^2(1-\xi) \rightarrow \frac{d^2 N^e_{\theta_2}}{dx^2} = \frac{4}{l^2}\left[-\frac{l}{8}\frac{d}{d\xi^2}\left(2(1+\xi)(1-\xi) - (1+\xi)^2\right)\right] = \frac{4}{l^2}\left(-\frac{l}{8}(-6\xi - 2)\right)$$

Element Stiffness and Consistent Nodal Forces

$$\mathbf{N} = \left[\frac{1}{4}(1-\xi)^2(2+\xi) \quad \frac{1}{8}l(1-\xi)^2(1+\xi) \quad \frac{1}{4}(1+\xi)^2(2-\xi) \quad -\frac{1}{8}l(1+\xi)^2(1-\xi) \right]$$

$$\mathbf{B} = \frac{1}{l}\left[6\frac{\xi}{l} \quad 3\xi - 1 \quad -6\frac{\xi}{l} \quad 3\xi + 1 \right]$$

$$\mathbf{v}^e = \mathbf{N}\mathbf{u}^e \rightarrow \mathbf{v}'' = \mathbf{N}''\mathbf{u}^e = \mathbf{B}\mathbf{u}^e$$

[Internal energy due to bending]

$$U^e = \frac{1}{2}\int_0^L v'' EI v'' dx = \frac{1}{2}\int_0^L (\mathbf{u}^e)^T \mathbf{B}^T EI \mathbf{B}\mathbf{u}^e dx = \frac{1}{2}(\mathbf{u}^e)^T \left[\int_0^L \mathbf{B}^T EI \mathbf{B} dx \right] \mathbf{u}^e$$

[External energy due to transverse load q] $W^e = \int_0^L qv dx = \int_0^L q\mathbf{N}^T \mathbf{u}^e dx = (\mathbf{u}^e)^T \int_0^L q\mathbf{N}^T dx$

$$\Pi^e = U^e - W^e = \frac{1}{2}(\mathbf{u}^e)^T \underbrace{\left[\int_0^L \mathbf{B}^T EI \mathbf{B} dx \right]}_{\mathbf{K}^e} \mathbf{u}^e - (\mathbf{u}^e)^T \underbrace{\int_0^L q\mathbf{N}^T dx}_{\mathbf{f}^e}$$

$$\mathbf{K}^e = \int_0^L EI\mathbf{B}^T\mathbf{B} dx = \int_{-1}^{+1} EI\mathbf{B}^T\mathbf{B}\frac{1}{2}l d\xi$$

$$\mathbf{f}^e = \int_0^L q\mathbf{N}^T dx = \int_{-1}^{+1} q\mathbf{N}^T \frac{1}{2}l d\xi$$

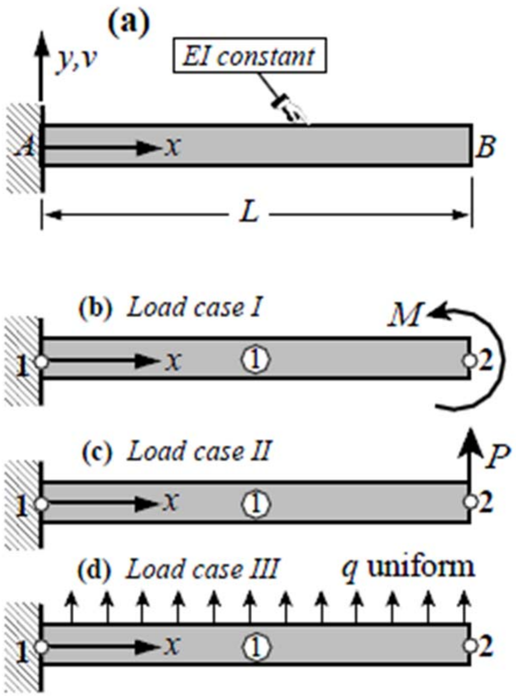
Prismatic Beam and Uniform Load

$$\mathbf{K}^e = \int_{-1}^{+1} EI\mathbf{B}^T\mathbf{B}\frac{1}{2}l\,d\xi = \frac{1}{2}EIl\int_{-1}^{+1}\mathbf{B}^T\mathbf{B}\,d\xi = \frac{1}{2}EIl\int_{-1}^{+1}\frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} \\ 3\xi-1 \\ -6\frac{\xi}{l} \\ 3\xi+1 \end{bmatrix}\frac{1}{l}\begin{bmatrix} 6\frac{\xi}{l} & 3\xi-1 & -6\frac{\xi}{l} & 3\xi+1 \end{bmatrix}d\xi$$

$$= \frac{1}{2l}EI\int_{-1}^{+1}\begin{bmatrix} 36\xi^2 & 6\xi(3\xi-1)l & -36\xi^2 & 6\xi(3\xi+1)l \\ & (3\xi-1)^2 l^2 & -6\xi(3\xi-1)l & (9\xi^2-1)l^2 \\ & & 36\xi^2 & -6\xi(3\xi+1)l \\ sym & & & (3\xi+1)^2 l^2 \end{bmatrix}d\xi = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ sym & & & 4l^2 \end{bmatrix}$$

$$\mathbf{f}^e = \int_{-1}^{+1} q\mathbf{N}^T\frac{1}{2}l\,d\xi = \frac{1}{2}ql\int_{-1}^{+1}\mathbf{N}^T d\xi = \frac{1}{2}ql\int_{-1}^{+1}\begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{1}{8}l(1-\xi)^2(1+\xi) \\ \frac{1}{4}(1+\xi)^2(2-\xi) \\ -\frac{1}{8}l(1+\xi)^2(1-\xi) \end{bmatrix}d\xi = \frac{1}{2}ql\begin{bmatrix} 1 \\ \frac{1}{6}l \\ 1 \\ -\frac{1}{6}l \end{bmatrix} \rightarrow \begin{cases} \text{two transverse nodal loads: } \frac{1}{2}ql \\ \text{two nodal moments: } \pm\frac{1}{12}ql^2 \end{cases}$$

Example 1: Cantilever Beam



$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\M\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\dfrac{ML^2}{2EI}\\ \dfrac{ML}{EI}\end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\\P\\0\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\dfrac{PL^3}{3EI}\\ \dfrac{PL^2}{2EI}\end{bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{bmatrix}=\frac{1}{2}qL\begin{bmatrix}1\\ \frac{1}{6}\beta L\\ 1\\ -\frac{1}{6}\beta L\end{bmatrix}\rightarrow\begin{bmatrix}v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}\dfrac{qL^4(4-\beta)}{24EI}\\ \dfrac{qL^3(3-\beta)}{12EI}\end{bmatrix}$$

Load case I: $v=\dfrac{Mx^2}{2EI},\ \theta=\dfrac{Mx}{EI}$

Load case II: $\begin{cases} v=\dfrac{Px^2}{6EI}(3L-x)\\ \theta=\dfrac{Px}{2EI}(2L-x)\end{cases}$

$\begin{cases}\beta=1: \text{ energy consistent load lumping}\\ \beta=0: \text{ EbE (here same as NbN) load lumping}\end{cases}$

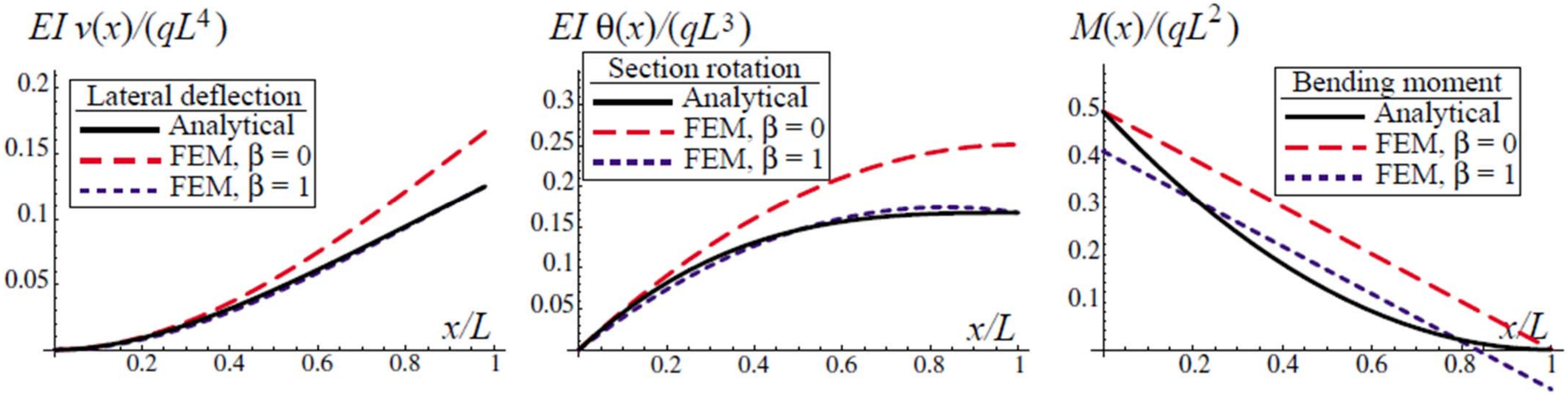
Example 1: Lode case III

$\left[\text{Analytic Solution}\right]\ v=\frac{qx^2}{24EI}\left(x^2+6l^2-4lx\right),\ \theta=\frac{qx}{6EI}\left(x^2+3l^2-3lx\right),\ M=\frac{q}{2}\left(x^2-l^2-2lx\right)$

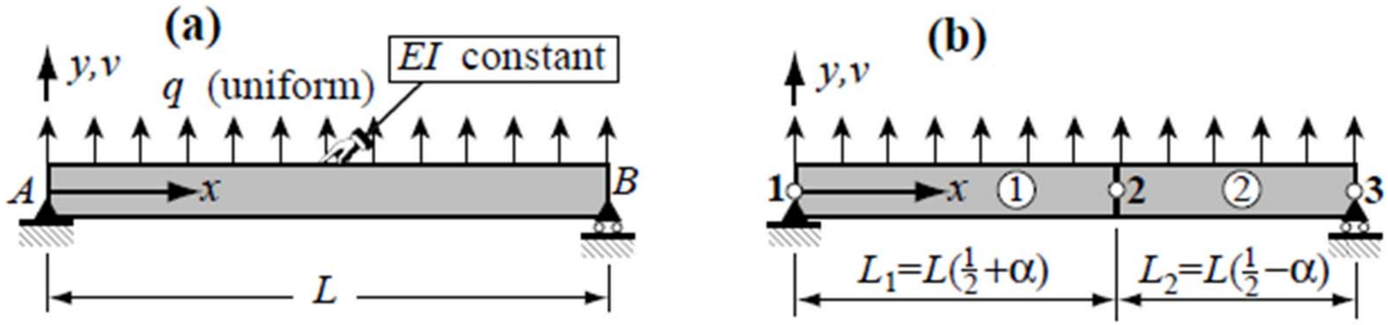
$\left[\text{FEM}\right]$

$$\mathbf{v}^e=\mathbf{N}\mathbf{u}^e\rightarrow v=\begin{bmatrix}(1-\xi)^2(2+\xi)/4\\ L(1-\xi)^2(1+\xi)/8\\ (1+\xi)^2(2-\xi)/4\\ -L(1+\xi)^2(1-\xi)/8\end{bmatrix}^T\begin{Bmatrix}0\\0\\v_2\\\theta_2\end{Bmatrix}=\frac{1}{4}\left(\frac{2x}{L}\right)^2\left(3-\frac{2x}{L}\right)\frac{qL^4(4-\beta)}{24EI}-\frac{1}{8}\left(\frac{2x}{L}\right)^2\left(2-\frac{2x}{L}\right)\frac{qL^3(3-\beta)}{12EI}=qL^2x^2\frac{L(6-\beta)-2x}{24EI}$$

$$\theta=\frac{dv}{dx}=qL^2x\frac{L(6-\beta)-3x}{12EI},\ M=EI\frac{d^2v}{dx^2}=\frac{qL}{12}\left[L(6-\beta)-6x\right]$$



Example 2: SS Beam



$$\frac{EI}{L^3}\begin{bmatrix} \frac{12}{(1/2+\alpha)^3} & \frac{6L}{(1/2+\alpha)^2} & \frac{-12}{(1/2+\alpha)^3} & \frac{6L}{(1/2+\alpha)^2} & 0 & 0 \\ \frac{6L}{(1/2+\alpha)^2} & \frac{4L^2}{1/2+\alpha} & -\frac{6L}{(1/2+\alpha)^2} & \frac{2L^2}{1/2+\alpha} & 0 & 0 \\ \frac{-12}{(1/2+\alpha)^3} & -\frac{6L}{(1/2+\alpha)^2} & \frac{12}{(1/2+\alpha)^3}+\frac{12}{(1/2-\alpha)^3} & -\frac{6L}{(1/2+\alpha)^2}+\frac{6L}{(1/2-\alpha)^2} & \frac{-12}{(1/2-\alpha)^3} & \frac{6L}{(1/2-\alpha)^2} \\ \frac{6L}{(1/2+\alpha)^2} & \frac{2L^2}{1/2+\alpha} & -\frac{6L}{(1/2+\alpha)^2}+\frac{6L}{(1/2-\alpha)^2} & \frac{4L^2}{1/2+\alpha}+\frac{4L^2}{1/2-\alpha} & \frac{-6L}{(1/2-\alpha)^2} & \frac{2L^2}{1/2-\alpha} \\ 0 & 0 & \frac{-12}{(1/2-\alpha)^3} & \frac{-6L}{(1/2-\alpha)^2} & \frac{12}{(1/2-\alpha)^3} & \frac{-6L}{(1/2-\alpha)^2} \\ 0 & 0 & \frac{6L}{(1/2-\alpha)^2} & \frac{2L^2}{1/2-\alpha} & \frac{-6L}{(1/2-\alpha)^2} & \frac{4L^2}{1/2-\alpha} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \frac{1}{2}qL\begin{bmatrix} 1/2+\alpha \\ \frac{1}{6}L(1/2+\alpha)^2 \\ 1/2+\alpha+1/2-\alpha \\ -\frac{1}{6}L(1/2+\alpha)^2+\frac{1}{6}L(1/2-\alpha)^2 \\ 1/2-\alpha \\ -\frac{1}{6}L(1/2-\alpha)^2 \end{bmatrix}$$

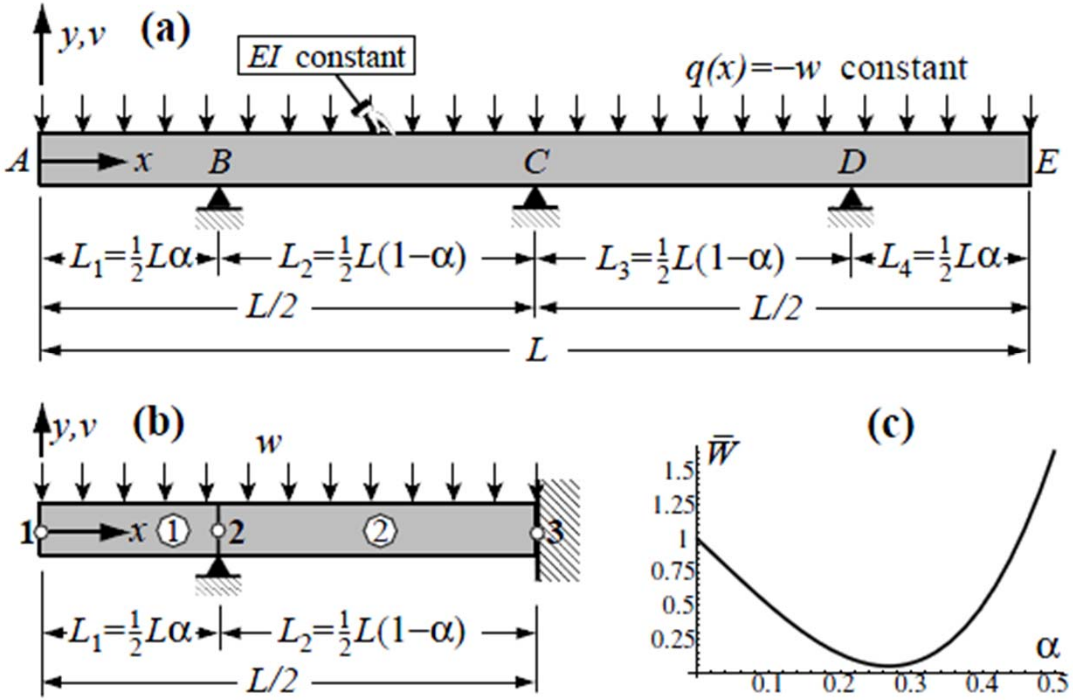
$$\rightarrow v_2 = \frac{qL^4\left(5-24\alpha^2+16\alpha^4\right)}{384EI}$$

$$v(x) = \frac{qL^4\left(\zeta-2\zeta^3+\zeta^4\right)}{24EI} \text{ where } \zeta = \frac{x}{L} \xrightarrow{x=L_1=L\left(\frac{1}{2}+\alpha\right)} v_2^{\text{exact}} = \frac{qL^4\left(5-24\alpha^2+16\alpha^4\right)}{384EI}$$

(v and θ inside elements will NOT agree with the exact one.)

Example 3: Continuum Beam

Optimal location of supports ?



$$
\text{best } \alpha\,? \rightarrow \begin{cases} \text{Minimum external energy: } W(\alpha) = \mathbf{f}^T\mathbf{u} \rightarrow dW/d\alpha = 0 \rightarrow \alpha \approx 0.27 \\ \text{Equal reactions: } R_B = R_C \rightarrow \alpha = 0.30546 \\ \text{Minimum relative deflection: } v_{ji}^{\max}(\alpha) = \max\left|v_j - v_i\right| \rightarrow \alpha = 0.26681 \\ \text{Minimum absolute moment: } M^{\max}(\alpha) = \max\left|M(x,\alpha)\right| \rightarrow \alpha = 0.25540 \end{cases}
$$