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Introduction

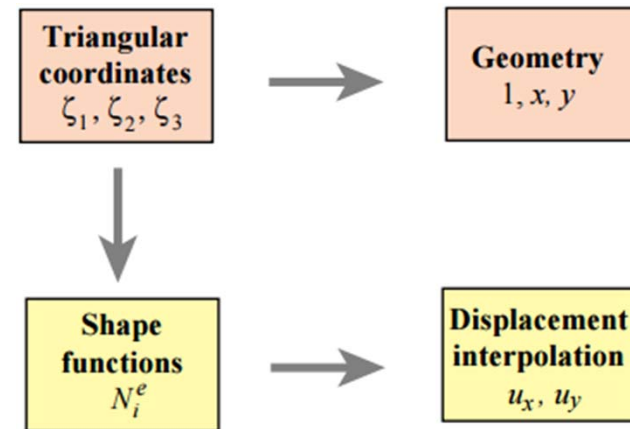
- Technical difficulties: linear triangle → quadrilateral as well as higher order triangles
 - The construction of shape functions that satisfy consistency requirements for higher order elements with curved boundaries becomes increasingly complicated: *isoparametric elements*
 - Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be evaluated in simple closed form: *numerical quadrature*
- Element **geometry** and **displacements** are represented by **same** set of shape functions (iso = equal)
- Before isoparametric concept was discovered, FEM developers did "SuperParametric" Elems
 - Element shape functions refined, more nodes and DOFs added
 - But element geometry was kept simple with straight sides

Isoparametric Formulations for Structural Mechanics

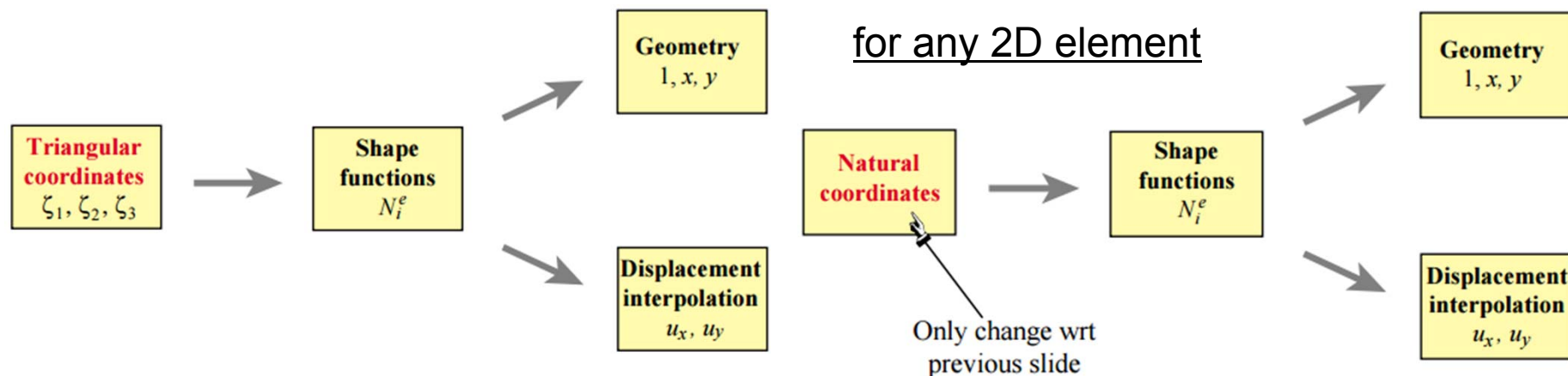
- Advantages
 - Unification: same steps for all iso-P elements
 - No need to distinguish straight vs. curved side elements
 - Quick construction of shape functions
- Disadvantages
 - Low-order iso-P elements may be poor performers (overstiff)
 - Method does not extend to problems with variational index higher than 1 (e.g., plate bending and shells)

Triangular Elements

- Superparametric representation



- Isoparametric representation



Three Node Linear Triangle

Geometry discription:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Displacement Interpolation:

$$u_x = u_{x1}N_1^e + u_{x2}N_2^e + u_{x3}N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$

$$u_y = u_{y1}N_1^e + u_{y2}N_2^e + u_{y3}N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

Equalizing Geometry and Displacements:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

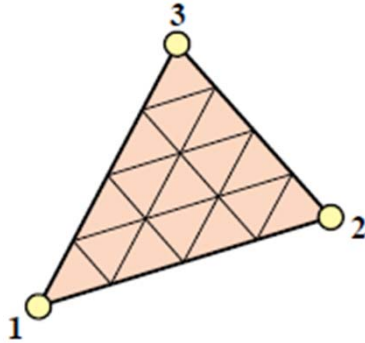
Iso-P Representation of 2D Plane Stress Elements with n Nodes

$$\left. \begin{array}{l} \text{Element Geometry:} \\ 1 = \sum_{i=1}^n N_i^e, \quad x = \sum_{i=1}^n x_i N_i^e, \quad y = \sum_{i=1}^n y_i N_i^e \\ \text{Displacement Interpolation:} \\ u_x = \sum_{i=1}^n u_{xi} N_i^e, \quad u_y = \sum_{i=1}^n u_{yi} N_i^e \end{array} \right\} \xrightarrow{m} \begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

More Rows May be Added to Interpolate other Quantities from Node Values

$$\begin{array}{l} \text{thickness} \\ \text{temperature field} \end{array} \begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ \textcolor{red}{h} \\ \textcolor{red}{T} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ \textcolor{red}{h_1} & \textcolor{red}{h_2} & \dots & \textcolor{red}{h_n} \\ \textcolor{red}{T_1} & \textcolor{red}{T_2} & \dots & \textcolor{red}{T_n} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

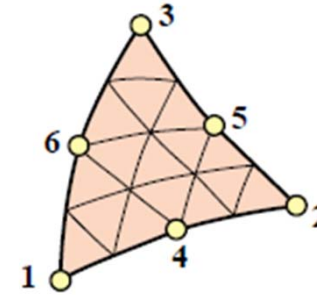
Triangular Elements



Linear Triangle

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$N_1^e = \zeta_1, N_2^e = \zeta_2, N_3^e = \zeta_3$$



Quadratic Triangle

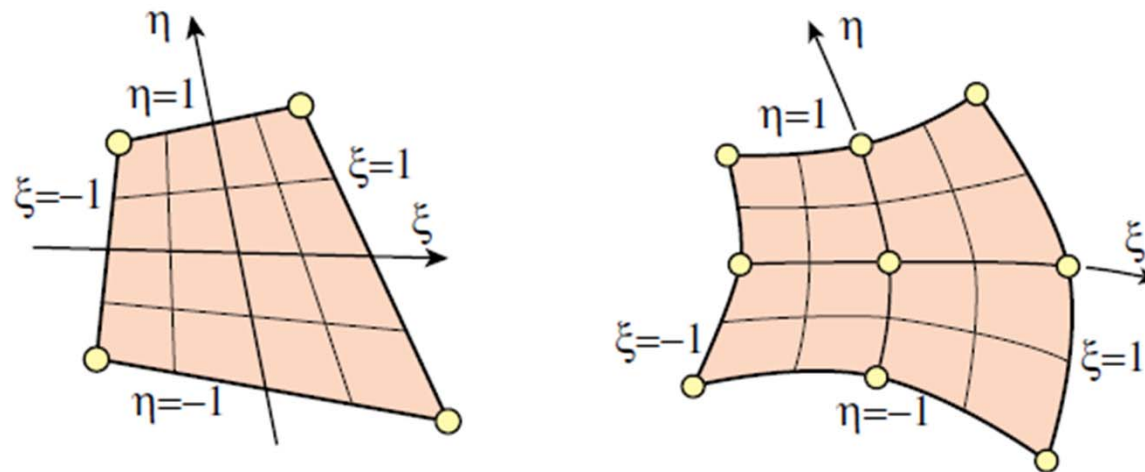
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix}$$

$$N_1^e = \zeta_1(2\zeta_1 - 1), N_2^e = \zeta_2(2\zeta_2 - 1), N_3^e = \zeta_3(2\zeta_3 - 1)$$

$$N_4^e = 4\zeta_1\zeta_2, N_5^e = 4\zeta_2\zeta_3, N_6^e = 4\zeta_3\zeta_1$$

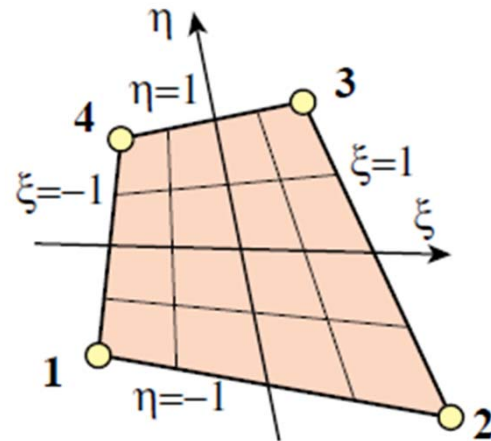
Quadrilateral Coordinates (ξ, η)

- Natural coordinate system



- Reference plane
 - quadrilateral coordinates plotted as Cartesian coordinates in the $\{\xi, \eta\}$ plane
- Reference element: square of side 2
- Isoparametric mapping: $\{\xi, \eta\} \leftrightarrow \{x, y\}$

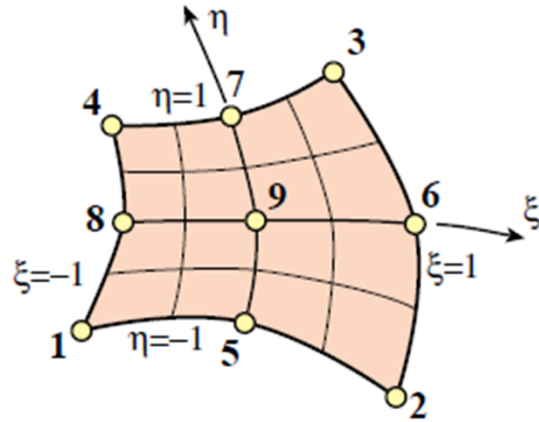
4-Node Bilinear Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix} \rightarrow \begin{cases} N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2^e = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4^e = \frac{1}{4}(1-\xi)(1+\eta) \end{cases}$$

These functions vary *linearly* on quadrilateral coordinate lines $\xi = \text{const}$ and $\eta = \text{const}$, but are not linear polynomials as in the case of the three-node triangle.

9 Node Biquadratic Quadrilateral



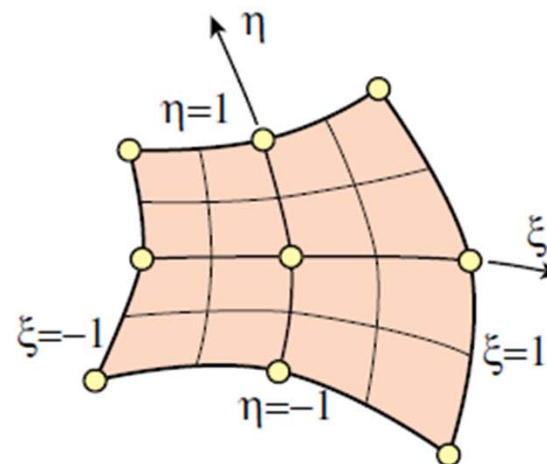
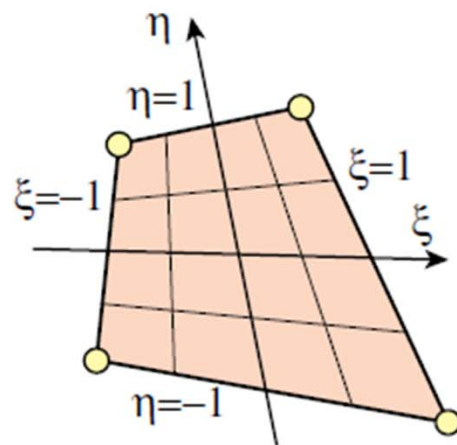
Lagrangian quadrilateral

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_9 \\ y_1 & y_2 & \dots & y_9 \\ u_{x1} & u_{x2} & \dots & u_{x9} \\ u_{y1} & u_{y2} & \dots & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix} \rightarrow \begin{cases} N_1^e = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta, & N_5^e = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \\ N_2^e = \frac{1}{4}(1+\xi)(1-\eta)\xi\eta, & N_6^e = \frac{1}{2}(1+\xi)(1-\eta^2)\xi \\ N_3^e = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta, & N_7^e = \frac{1}{2}(1-\xi^2)(1+\eta)\eta \\ N_4^e = \frac{1}{4}(1-\xi)(1+\eta)\xi\eta, & N_8^e = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi \\ N_9^e = (1-\xi^2)(1-\eta^2) \end{cases}$$

These functions vary *quadratically* along the coordinate lines $\xi = \text{const}$ and $\eta = \text{const}$. The shape function associated with the internal node 9 is called a *bubble function* because of its geometric shape.

Isoparametric Quadrilaterals

- Implementation steps for element stiffness matrix
 - Construct shape functions in Quad coordinates
 - Compute x-y derivatives of shape functions and build strain-displacement matrix $\mathbf{B}$
 - Integrate $h \mathbf{B}^T \mathbf{E} \mathbf{B}$ over element
- Quadrilateral Coordinates (ξ, η)



Partial Derivative Computation

- Shape functions are written in terms of ξ and η
- But Cartesian partials (with respect to x, y) are required to get strains & stresses
- Jacobian and Inverse Jacobian

$$(x(\xi, \eta), y(\xi, \eta)) \leftrightarrow (\xi(x, y), \eta(x, y))$$

$$\left. \begin{aligned} dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \\ dy &= \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \end{aligned} \right\} \rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \mathbf{J}^T \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \quad \text{where } \underbrace{\mathbf{J} = \frac{\partial(x, y)}{\partial(\xi, \eta)}}_{\text{Jacobian}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\left. \begin{aligned} d\xi &= \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \\ d\eta &= \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy \end{aligned} \right\} \rightarrow \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{J}^{-T} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{where } \underbrace{\mathbf{J}^{-1} = \frac{\partial(\xi, \eta)}{\partial(x, y)}}_{\text{inverse Jacobian}} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$J = |\mathbf{J}| = \det(\mathbf{J}) = J_{11}J_{22} - J_{12}J_{21}$$

$$\mathbf{J}^{-1}\mathbf{J} = \mathbf{I}$$

Shape Function Partial Derivatives

$$\left. \begin{aligned} \frac{\partial N_1^e}{\partial x} &= \frac{\partial N_1^e}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_1^e}{\partial \eta} \frac{\partial \eta}{\partial x} \\ \frac{\partial N_1^e}{\partial y} &= \frac{\partial N_1^e}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_1^e}{\partial \eta} \frac{\partial \eta}{\partial y} \end{aligned} \right\} \rightarrow \begin{bmatrix} \frac{\partial N_1^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{\text{how to get?}} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} \end{bmatrix} = \frac{\partial(\xi, \eta)}{\partial(x, y)} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} \end{bmatrix}$$

$$\mathbf{J} = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \rightarrow \mathbf{J}^{-1} = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

Use the element geometry definition (n : number of nodes)

$$\left. \begin{aligned} x &= \sum_i^n x_i N_i^e \\ y &= \sum_i^n y_i N_i^e \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \frac{\partial x}{\partial \xi} &= \sum_i^n x_i \frac{\partial N_i^e}{\partial \xi}, & \frac{\partial x}{\partial \eta} &= \sum_i^n x_i \frac{\partial N_i^e}{\partial \eta} \\ \frac{\partial y}{\partial \xi} &= \sum_i^n y_i \frac{\partial N_i^e}{\partial \xi}, & \frac{\partial y}{\partial \eta} &= \sum_i^n y_i \frac{\partial N_i^e}{\partial \eta} \end{aligned} \right\}$$

Store as entries of $\mathbf{J}$ and invert to get $\mathbf{J}^{-1}$

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_i^n x_i \frac{\partial N_i^e}{\partial \xi} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \xi} \\ \sum_i^n x_i \frac{\partial N_i^e}{\partial \eta} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_2^e}{\partial \xi} & \dots & \frac{\partial N_n^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} & \frac{\partial N_2^e}{\partial \eta} & \dots & \frac{\partial N_n^e}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = \mathbf{P}\mathbf{X}$$

Partial Derivative Computation Sequence

Summary

- At a specific point of quad coordinates ξ and η :
 - Compute from node coordinates and S.F.s
 - Form $\mathbf{J}$ and invert to get $\mathbf{J}^{-1}$ and $J = \det \mathbf{J}$
 - Apply the chain rule to get the x, y partials of the S.F.s

Strain Displacement Matrix B

- Use those S.F.s partials to build the strain-displacement matrix **B**

$$e = \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial x} \\ \frac{\partial \mathbf{v}}{\partial y} \\ \frac{\partial \mathbf{u}}{\partial y} + \frac{\partial \mathbf{v}}{\partial x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n \frac{\partial N_i^e}{\partial x} u_{xi} \\ \sum_{i=1}^n \frac{\partial N_i^e}{\partial y} u_{yi} \\ \sum_{i=1}^n \frac{\partial N_i^e}{\partial y} u_{xi} + \sum_{i=1}^n \frac{\partial N_i^e}{\partial x} u_{yi} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ \vdots \\ u_{xn} \\ u_{yn} \end{bmatrix} = \mathbf{B} \mathbf{u}^e$$

- Unlike the 3-node triangle, here **B** = **B**(ξ,η) varies over quad

One Dimensional Gauss Integration Rules

$$\int_{-1}^{+1} F(\xi) d\xi \approx \sum_{i=1}^p w_i F(\xi_i)$$

$p(\geq 1)$: number of Gauss integration (sample) points

w_i : integration weights

ξ_i : sample-point abscissae in the interval $[-1, +1]$

integrate exactly polynomials of order up to $(2p-1)$

1 point: $\int_{-1}^{+1} F(\xi) d\xi \approx 2F(0)$

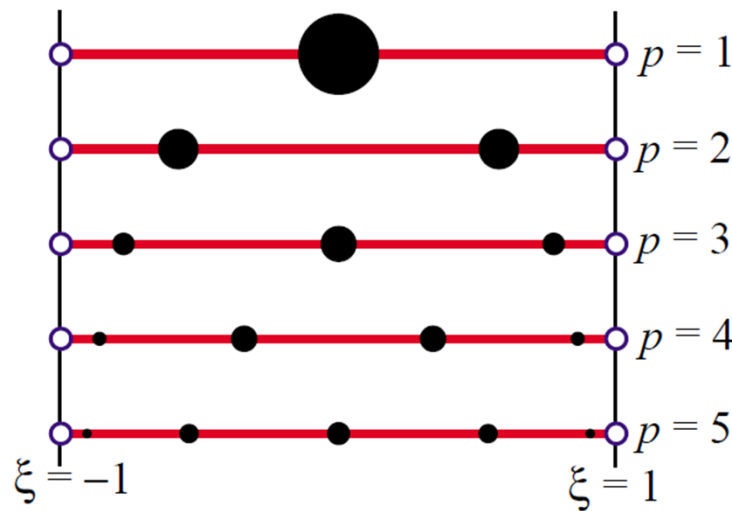
2 points: $\int_{-1}^{+1} F(\xi) d\xi \approx F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$

3 points: $\int_{-1}^{+1} F(\xi) d\xi \approx \frac{5}{9}F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}F(0) + \frac{5}{9}F\left(\sqrt{\frac{3}{5}}\right)$

4 points: $\int_{-1}^{+1} F(\xi) d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$

$$\left[\xi_{34} = -\xi_{24} = \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \xi_{44} = -\xi_{14} = \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}, w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{\frac{5}{6}}, w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{\frac{5}{6}} \right]$$

Graphical Representation of One-Dimensional Gauss Integration Rules

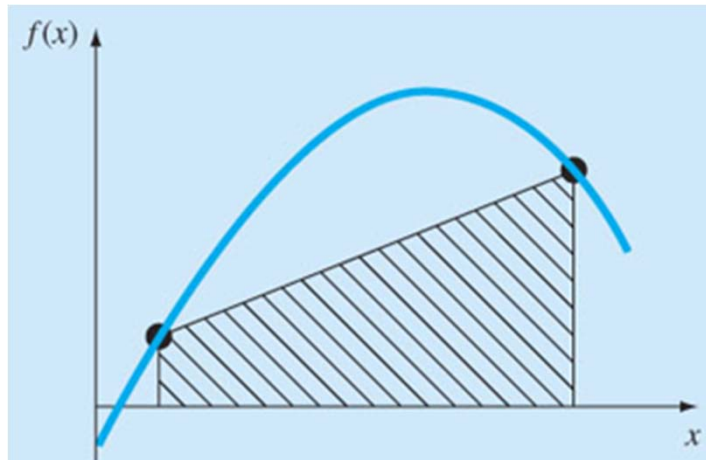


$$\int_a^b F(x) dx = \int_{-1}^{+1} F(\xi) J d\xi = \int_{-1}^{+1} F(\xi) \frac{1}{2} l d\xi$$

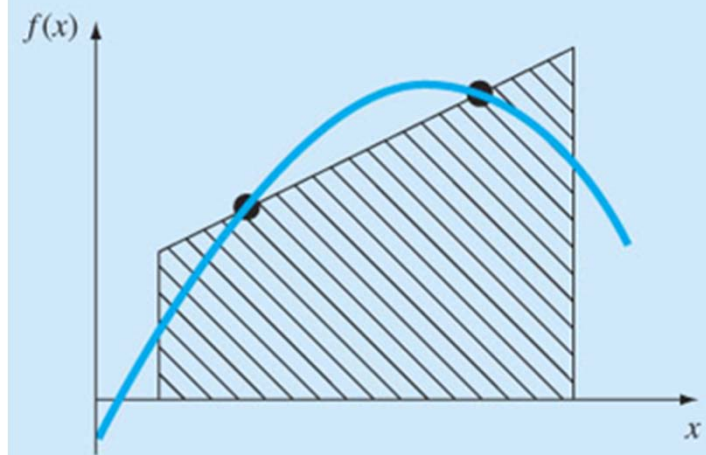
$$l = b - a > 0 \rightarrow [-1, +1]?$$

$$x = \frac{1}{2}a(1-\xi) + \frac{1}{2}b(1+\xi) = \frac{1}{2}(a+b) + \frac{1}{2}l\xi \quad \text{or} \quad \xi = \frac{2}{l} \left[x - \frac{1}{2}(a+b) \right]$$

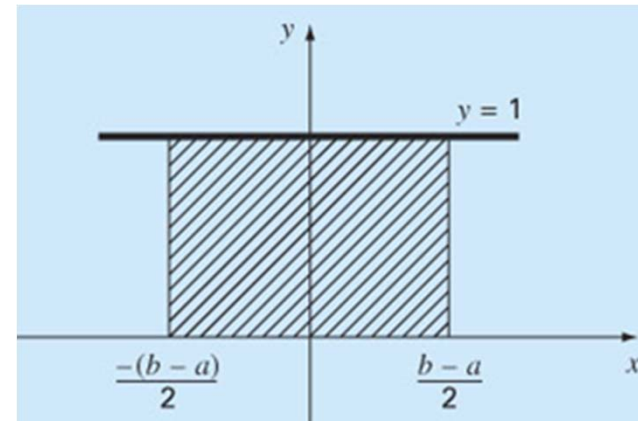
$$J = \frac{\partial x}{\partial \xi} = \frac{1}{2}l$$



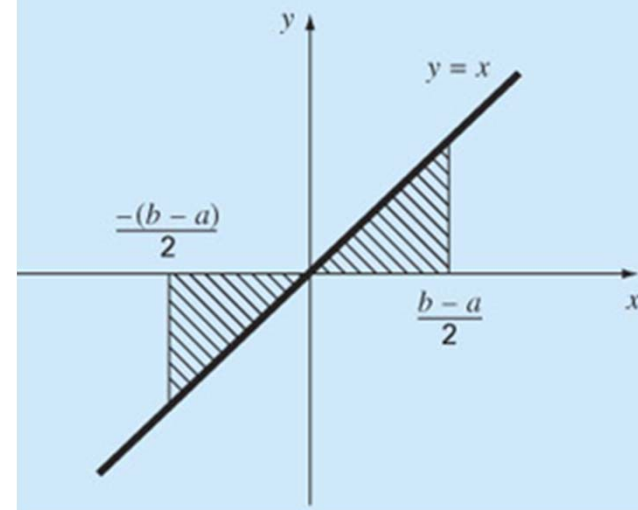
(a)

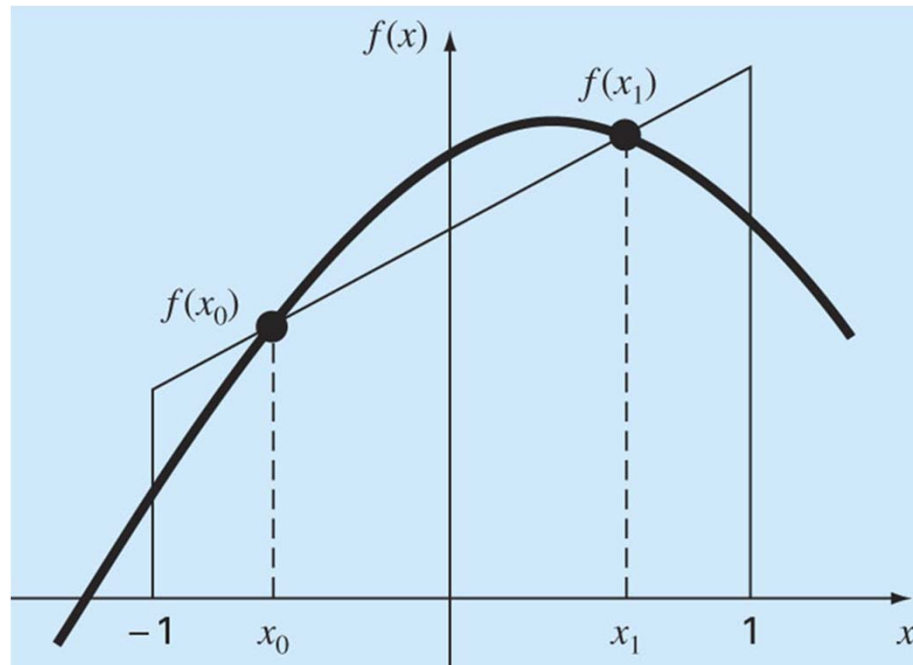


(b)



(a)





$$\int_{-1}^1 F(\xi) d\xi = w_1 F(\xi_1) + w_2 F(\xi_2)$$

unknown: w_1, w_2, ξ_1, ξ_2

$$\begin{cases} F(\xi) = 1 \Rightarrow w_1 + w_2 = 2 \\ F(\xi) = \xi \Rightarrow w_1 \xi_1 + w_2 \xi_2 = 0 \\ F(\xi) = \xi^2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = \frac{2}{3} \\ F(\xi) = \xi^3 \Rightarrow w_1 \xi_1^3 + w_2 \xi_2^3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}} \\ w_1 = w_2 = 1 \end{cases}$$

$$\int_{-1}^1 F(\xi) d\xi = 1 \times F\left(-\frac{1}{\sqrt{3}}\right) + 1 \times F\left(\frac{1}{\sqrt{3}}\right)$$

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$ $c_1 = 1.0000000$	$x_0 = -0.577350269$ $x_1 = 0.577350269$	$\cong f^{(4)}(\xi)$
3	$c_0 = 0.5555556$ $c_1 = 0.8888889$ $c_2 = 0.5555556$	$x_0 = -0.774596669$ $x_1 = 0.0$ $x_2 = 0.774596669$	$\cong f^{(6)}(\xi)$
4	$c_0 = 0.3478548$ $c_1 = 0.6521452$ $c_2 = 0.6521452$ $c_3 = 0.3478548$	$x_0 = -0.861136312$ $x_1 = -0.339981044$ $x_2 = 0.339981044$ $x_3 = 0.861136312$	$\cong f^{(8)}(\xi)$
5	$c_0 = 0.2369269$ $c_1 = 0.4786287$ $c_2 = 0.5688889$ $c_3 = 0.4786287$ $c_4 = 0.2369269$	$x_0 = -0.906179846$ $x_1 = -0.538469310$ $x_2 = 0.0$ $x_3 = 0.538469310$ $x_4 = 0.906179846$	$\cong f^{(10)}(\xi)$
6	$c_0 = 0.1713245$ $c_1 = 0.3607616$ $c_2 = 0.4679139$ $c_3 = 0.4679139$ $c_4 = 0.3607616$ $c_5 = 0.1713245$	$x_0 = -0.932469514$ $x_1 = -0.661209386$ $x_2 = -0.238619186$ $x_3 = 0.238619186$ $x_4 = 0.661209386$ $x_5 = 0.932469514$	$\cong f^{(12)}(\xi)$

Two Dimensional Product Gauss Rules

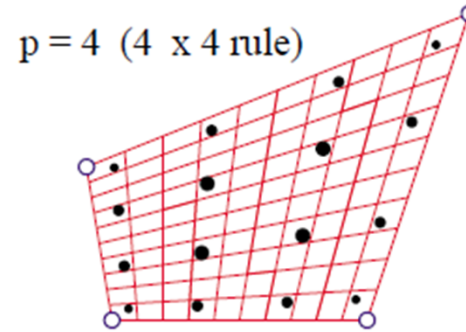
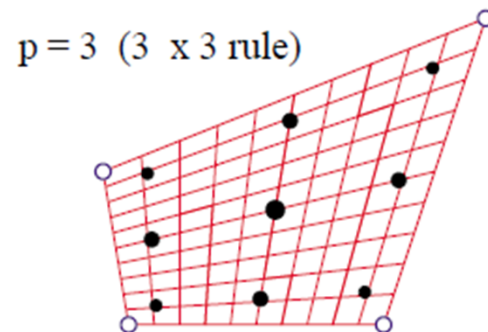
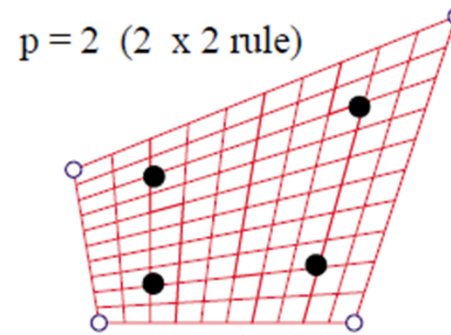
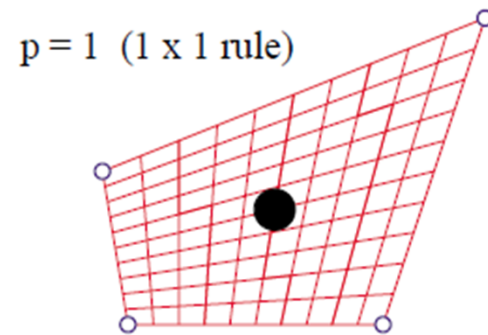
Canonical form of integral:

$$\int_{-1}^{+1} \int_{-1}^{+1} F(\xi, \eta) d\xi d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} F(\xi, \eta) d\xi$$

Gauss integration rules with p_1 points in the ξ direction and p_2 points in the η direction:

$$\int_{-1}^{+1} \int_{-1}^{+1} F(\xi, \eta) d\xi d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} F(\xi, \eta) d\xi \approx \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} w_i w_j F(\xi_i, \eta_j)$$

usually $p_1 = p_2$

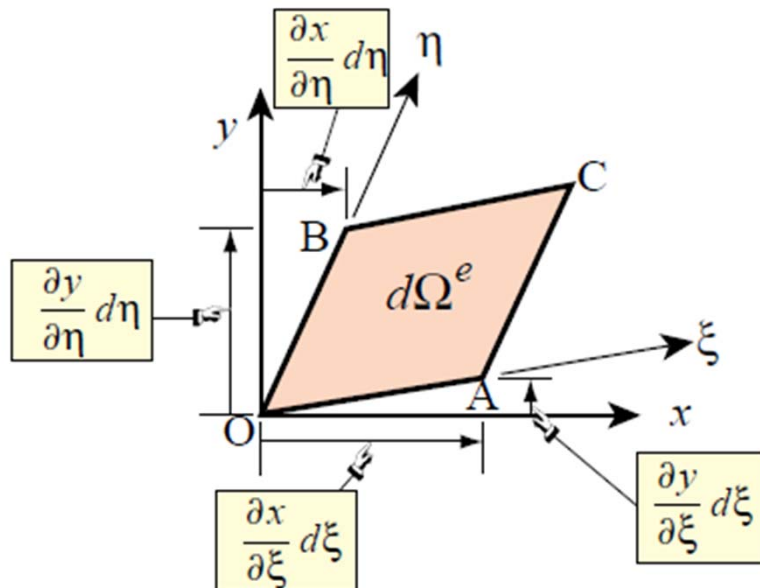


Stiffness Matrix

$$\mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega \xrightarrow{d\Omega = dx dy = \det \mathbf{J} d\xi d\eta} \mathbf{K}^e = \int_{-1}^{+1} \int_{-1}^{+1} h \mathbf{B}^T \mathbf{E} \mathbf{B} \det \mathbf{J} d\xi d\eta$$

$$\xrightarrow{h \mathbf{B}^T \mathbf{E} \mathbf{B} \det \mathbf{J} = \mathbf{F}(\xi, \eta)} \mathbf{K}^e = \int_{-1}^{+1} \int_{-1}^{+1} \mathbf{F}(\xi, \eta) d\xi d\eta$$

$$\int_{-1}^{+1} \int_{-1}^{+1} F(\xi, \eta) d\xi d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} F(\xi, \eta) d\xi \approx \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} w_i w_j F(\xi_i, \eta_j)$$



$$\begin{aligned} d\Omega^e &= |\overline{OA} \times \overline{OB}| = \begin{vmatrix} \frac{\partial x}{\partial \xi} d\xi & \frac{\partial y}{\partial \xi} d\xi \\ \frac{\partial x}{\partial \eta} d\eta & \frac{\partial y}{\partial \eta} d\eta \end{vmatrix} \\ &= \frac{\partial x}{\partial \xi} d\xi \frac{\partial y}{\partial \eta} d\eta - \frac{\partial x}{\partial \eta} d\eta \frac{\partial y}{\partial \xi} d\xi \\ &= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta = |\mathbf{J}| d\xi d\eta = \det \mathbf{J} d\xi d\eta \end{aligned}$$