1.

If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are corners of a parallelogram, then $\boldsymbol{z}$ = corner 4 can be $\boldsymbol{u}+\boldsymbol{v}-\boldsymbol{w}$ **or** $\boldsymbol{u}-\boldsymbol{v}+\boldsymbol{w}$ **or** $-\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$. Here those 4th corners are $\boldsymbol{z}=(4,0)$ **or** $\boldsymbol{z}=(-2,2)$ or $\boldsymbol{z}=(4,4)$.

Reasoning: The corners A, B, C, D around a parallelogram have $A+C=B+D$.

2.

If $\mathbf{C}(A)=\mathbf{R}^3$ then $m=3$ and $n \geq 3$ and $r=3$.

3.

An example is $A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Then C and R are 4 by 2 and 2 by 4.

4.

If $A=CR$ then $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}=\begin{bmatrix} C \\ C \end{bmatrix}\begin{bmatrix} 0 & R \end{bmatrix}$

5.

Yes, $\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$ is an m by n matrix. The number $a_i b_j$ is in row i, column j of $\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$. If $\boldsymbol{b}=\boldsymbol{a}$ then $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}$ is a *symmetric* matrix.

6.

$B=I$ has rows $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3=(1,0,0),(0,1,0),(0,0,1)$. The rank-1 matrices are

$\boldsymbol{a}_1\boldsymbol{b}_1=\begin{bmatrix} \boldsymbol{a}_1 & \mathbf{0} & \mathbf{0} \end{bmatrix}$ $\quad \boldsymbol{a}_2\boldsymbol{b}_2=\begin{bmatrix} \mathbf{0} & \boldsymbol{a}_2 & \mathbf{0} \end{bmatrix}$ $\quad \boldsymbol{a}_3\boldsymbol{b}_3=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{a}_3 \end{bmatrix}$.

The sum of those rank-1 matrices is $AI=A$.

7.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and the rank has dropped.}$$

$$\text{But } A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ has the same nullspace and rank as } A.$$

8.

$$PF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & i & & \\ & & i^2 & \\ & & & i^3 \end{bmatrix}$$

This says that P times the 4 columns of F gives those same 4 columns times $1, i, i^2, i^3 = \lambda_1, \lambda_2, \lambda_3, \lambda_4 =$ the 4 eigenvalues of P.

The columns of $F/2$ are orthonormal ! To check, remember that for the dot product of two *complex vectors*, we take complex conjugates of the first vector : *change* i *to* $-i$.

9.

$$W^{\mathrm{T}}W = \begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} \text{ so that the columns of } W \text{ are orthogonal but not orthonormal.}$$

$$\text{Then } W^{-1} = (W^{\mathrm{T}}W)^{-1}W^{\mathrm{T}} = \begin{bmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/2 & \\ & & & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$