1.

$\det \begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = -1$.

The sum $2 - 1$ agrees with the trace $0 + 1$. A^{-1} has the same eigenvectors as A, with eigenvalues $\lambda_1^{-1} = \mathbf{\frac{1}{2}}$ and $\lambda_2^{-1} = \mathbf{-1}$.

2.

$(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. Pick $M = A - \lambda I$.

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have *different eigenvectors.*

3.

Eigenvectors in $\boldsymbol{X}$ and eigenvalues in Λ. Then $A = X\Lambda X^{-1}$ is given below.

The second matrix has $\lambda = 0$ (rank 1) and $\lambda = 4$ (trace $= 4$). A new $A = X\Lambda X^{-1}$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \quad \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

4.

Positive definite for $-3 < b < 3$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$$

Positive definite for $c > 8$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

Positive definite for $c^2 > b$

$$L = \begin{bmatrix} 1 & 0 \\ -b/c & 1 \end{bmatrix} \quad D = \begin{bmatrix} c & 0 \\ 0 & c - b/c \end{bmatrix} \quad S = LDL^{\mathrm{T}}.$$

5.

$$S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$$ has only one pivot $= 4$, rank $S = 1$, eigenvalues are $24, 0, 0$, $\det S = 0$.

6.

Corner determinants $|S_1| = 2$, $|S_2| = 6$, $|S_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

7.

$(c_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + c_n \boldsymbol{v}_n^{\mathrm{T}})(c_1 \boldsymbol{v}_1 + \cdots + c_n \boldsymbol{v}_n) = c_1^2 + \cdots + c_n^2$ because the $\boldsymbol{v}$'s are orthonormal.

$(c_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + c_n \boldsymbol{v}_n^{\mathrm{T}})\, S\, (c_1 \boldsymbol{v}_1 + \cdots + c_n \boldsymbol{v}_n) = (\qquad)(c_1 \lambda_1 \boldsymbol{v}_1 + \cdots + c_n \lambda_n \boldsymbol{v}_n)$
$= c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n$.

8.

Exchange $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (and keep $\sigma = \sqrt{45}$ and $\sigma = \sqrt{5}$) in equation (12) = the SVD of $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.