II.4 2 *(for functions)* Given $a(x) > 0$ find $p(x) > 0$ by analogy with problem 1, so that

$$\int_0^1 p(x)dx = 1 \text{ and } \int_0^1 \frac{(a(x))^2}{p(x)}dx \text{ is a minimum}$$

$$L(p(x), \lambda) = \int_0^1 \frac{(a(x))^2}{p(x)}dx + \int_0^1 \lambda p(x)dx - \int_0^1 \lambda dx$$

$$L(p(x), \lambda) = \int_0^1 \Big[\frac{(a(x))^2}{p(x)}dx + \lambda p(x)dx - \lambda\Big]dx$$

$$\frac{\partial L}{\partial p(x)} = 0 = -\frac{(a(x))^2}{(p(x))^2} + \lambda$$

$$\frac{(a(x))^2}{(p(x))^2} = \lambda$$

$$p(x) = \frac{a(x)}{\sqrt{\lambda}} = \frac{a(x)}{C}$$

We take C such that the integral of $p(x)$ is 1:

$$C = \int_0^1 a(x)dx$$

II.4 4 If $M = 11^T$ is the $n \times n$ matrix of 1s, prove that $nI - M$ is positive semidefinite. Problem 3 was the energy test. For Problem 4, find the eigenvalues of $nI - M$.

The eigenvalues of $nI - M$ are the solutions to the equation $\det(nI - M - \lambda I) = 0$.

$$\det(nI - M - \lambda I) = \det((n - \lambda)I - M) = 0$$

We can see that $\lambda = n$ gives us just $\det(-M)$ and since this is the all ones matrix, this will be zero. The nullspace of M is going to be $n - 1$-dimensional, which gives us $n - 1$ eigenvectors with $\lambda = n$.

We can also see that every entry down the diagonal of $nI - M$ will be $n - 1$, and all the other $n - 1$ entries of each row will be -1, so the total sum of entries down each row will be zero. This gives us an eigenvector of the all ones vector, with an eigenvalue of 0.

In the end, all the eigenvalues are either n or 0, which makes $nI - M$ positive semidefinite.

problem VI.4 - 6 10/10

We derive the gradient descent equation $x_{k+1} = x_k - s_k \nabla f(x_k)$ for the least squares problem of minimizing $f(x) = \frac{1}{2} \|Ax-b\|^2$

In order to do so, we write $f(x)$ as :

$$f(x) = \frac{1}{2}(Ax-b)^T(Ax-b)$$

$$= \frac{1}{2} x^T A^T A x - \frac{1}{2} x^T A^T b - \frac{1}{2} b^T A x + \frac{1}{2} b^T b$$

since $x^T A^T b$ is a scalar quantity, then we know that the transpose of a scalar is the scalar itself, and we get:

$$x^T A^T b = (x^T A^T b)^T = b^T (x^T A^T)^T = b^T A x \quad \Rightarrow \quad x^T A^T b = b^T A x$$

Thus, we write $f(x)$ as :

$$f(x) = \frac{1}{2} x^T A^T A x - b^T A x + \frac{1}{2} b^T b$$

this is a quadratic form ($A^T A$ is a square, symmetric matrix) and therefore, we compute the gradient as follows :

$$\nabla f(x) = A^T A x - A^T b = A^T (Ax - b)$$

therefore, we have the following gradient descent equation :

$$x_{k+1} = x_k - s_k \nabla f(x_k) \quad \Rightarrow \quad \nabla f(x_k) = A^T (A x_k - b)$$

$$\Rightarrow \boxed{x_{k+1} = x_k - s_k A^T (A x_k - b)}$$