5. Orthogonal Matrices and Subspaces

- Orthogonal vectors x and y

$$\left.\begin{array}{l}\mathbf{x}^T\mathbf{y}=0\\ \overline{\mathbf{x}}^T\mathbf{y}=0\end{array}\right\}\rightarrow\left\{\begin{array}{l}\text{Pytagoras Law of right triangles}:\|\mathbf{x}-\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\\ \text{Law of cosines}:\|\mathbf{x}-\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta\end{array}\right.$$

- Orthogonal basis for a subspace
  – Standard basis is orthogonal (even orthonormal) in $\mathbf{R}^n$
  – Hadamard matrices H_n containing orthogonal bases of $\mathbf{R}^n$
    - Are those orthogonal matirces?
  – Every subspace of $\mathbf{R}^n$ has an orthogonal basis: Gram-Schmidt idea
    - Two independent vectors a and b in the plane: $a^Tc=0$

- Orthogonal subspace R (row space) and N (null space)
  - Ax=0: The row space of A is orthogonal to the nullspace of A
  - $A^T y=0$: The column space of A is orthogonal to the nullspace of A^T
- Tall thin matrices Q with orthonormal columns: $Q^T Q=I$

$$\mathbf{Q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{Q}_2 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{Q}_3 = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\mathbf{Q}_i \mathbf{Q}_i^T = \mathbf{I}\,?$$

$$\mathbf{P} = \mathbf{Q}\mathbf{Q}^T \rightarrow \text{projection matrix}: \mathbf{P}^2 = \mathbf{P} = \mathbf{P}^T \rightarrow \text{"least squares"}$$

$$\mathbf{Pb} \text{ is the orthogonal projection of } \mathbf{b} \text{ onto the column space of } \mathbf{P}$$

- Orthogonal matrices are square with orthonormal columns: $Q^T=Q^{-1}$

$$\mathbf{Q}\text{ is square}\rightarrow\begin{Bmatrix}\mathbf{Q}^T\mathbf{Q}=\mathbf{I}\\ \mathbf{Q}\mathbf{Q}^T=\mathbf{I}\end{Bmatrix}\rightarrow\mathbf{Q}^{-1}=\mathbf{Q}^T$$

$$\mathbf{Q}_{\text{rotate}}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},\ \mathbf{Q}_{\text{reflect}}=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

$$\text{Householer reflections}:\mathbf{Q}=\mathbf{H}_n=\mathbf{I}-2\mathbf{u}\mathbf{u}^T=\mathbf{I}-\frac{2}{n}\mathbf{ones}(n,n)\xrightarrow{\mathbf{u}\perp\mathbf{w}}\begin{cases}\mathbf{H}_n\mathbf{u}=-\mathbf{u}\\ \mathbf{H}_n\mathbf{w}=+\mathbf{w}\end{cases}$$

$$\text{eigenvalues of }\mathbf{H}_n\text{ are }-1\text{ (once) and }+1\ (n\text{-1 times})$$

$$\text{All reflection matrices have eigenvalues }-1\text{ and }1$$

- Orthogonal basis = orthogonal axes in $\mathbf{R}^n$

$$\mathbf{v}=c_1\mathbf{q}_1+\cdots+c_n\mathbf{q}_n\rightarrow c_i=\mathbf{q}_i^{\ T}\mathbf{v}$$

$$\mathbf{v}=\mathbf{Q}\mathbf{c}\rightarrow\mathbf{Q}^T\mathbf{v}=\mathbf{Q}^T\mathbf{Q}\mathbf{c}=\mathbf{c}$$

- Examples
  - Rotations
  - Reflections
  - Hadamard matrices
  - Haar wavelets
  - Discrete Fourier Transform (DFT)
  - Complex inner product

6. Eigenvalues and Eigenvectors

- Ax = (eigenvalue) times x
- A^2x = (eigenvalue)^2 times x

$$\text{matrix } \mathbf{A}(\text{real}), \mathbf{S}(\text{symmetric}), \mathbf{Q}(\text{orthogonal})$$

$$\sum_{i=1}^{n} \lambda_i = \text{trace of matrix}$$

$$\prod \lambda_i = \text{determinant of matrix}$$

$$\text{eigenvectors of } \mathbf{A} \text{ are orthogonal iff } \mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$$

$$\mathbf{S} : \text{real eigenvalues, orthogonal eigenvectors}$$

- Write over vectors as combination of eigenvectors
- Similar matrix $B=M^{-1}AM$ has the same eigenvalues of A
  - Compute eigenvalues of large matrices
  - Make B gradually into a triangular matrix
  - Gradually show up on the main diagonal

- Diagonalizing a Matrix

$$\mathbf{AX} = \mathbf{X\Lambda} \rightarrow \mathbf{A} = \mathbf{X\Lambda X}^{-1}$$

$$\mathbf{A}^k\mathbf{v} = ?$$

$$\mathbf{v} = \mathbf{Xc} \rightarrow \underbrace{\mathbf{c} = \mathbf{X}^{-1}\mathbf{v}}_{c_i} \rightarrow \underbrace{\Lambda^k\mathbf{X}^{-1}\mathbf{v}}_{c_i\lambda_i^k} \rightarrow \underbrace{\mathbf{X\Lambda X}^{-1}\mathbf{v}}_{\sum c_i\lambda_i^k x_i}$$

$$\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 10 & \\ & 5 \end{bmatrix} \frac{1}{5} \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$\underbrace{\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}}_{\text{Markov matrix}} \xrightarrow{\lambda=1,\frac{1}{2}} \mathbf{A}^k\mathbf{v} = c_1(1)^k x_1 + c_2\left(\frac{1}{2}\right)^k x_2$$

  - Nondiagonalizable matrices: GM < AM
    - GM(Gometric Multiplicity): independent eigenvectors
    - AM(Algebraic Multiplicity): repetition of eigenvalues

7. Symmetric Positive Definite Matrices

- Test
  - All eigenvalues of S are positive
  - Energy $x^T S x$ is positive for $x \neq 0$
  - All pivots are positive
  - $S = A^T A$ with independent columns of A
  - All leading determinants are positive
- Second derivative matrix is positive definite @ a minimum point
- Semidefinite allows zero eigenvalues/energy/pivots/determinants

- The graph of $x^T S x=1$ is an ellipse, with its axes pointing along the eigenvectors of S

$$\mathbf{S} = \mathbf{Q}\Lambda\mathbf{Q}^T \ : \text{principal axis theorem}$$

$$\mathbf{x}^T\mathbf{S}\mathbf{x} = \mathbf{x}^T\left(\mathbf{Q}\Lambda\mathbf{Q}^T\right)\mathbf{x} = \left(\mathbf{x}^T\mathbf{Q}\right)\Lambda\left(\mathbf{Q}^T\mathbf{x}\right)$$

$$\mathbf{x}^T\mathbf{S}\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\mathbf{Q}\Lambda\mathbf{Q}^T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix}\Lambda\begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 9 & \\ & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$5x^2 + 8xy + 5y^2 = 1 \rightarrow 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1 \rightarrow 9X^2 + Y^2 = 1$$