Classification (1)

- Mechanics
  - Theoretical
  - Applied
  - Computational
  - Experimental

- Computational Mechanics $\xrightarrow{\text{physical scale}}$
  - Nanomechanics
  - Micromechanics
  - Continuum mechanics
    - Solids and Structures
    - Fluids
    - Multiphysics
  - Systems

- Computational Solid and Structural Mechanics $\xrightarrow{\text{inertia effects}}$
  - Statics
    - Time Invariant
    - Quasi-static
  - Dynamics

- Linear
- Nonlinear

Classification (2)

- Sptial Discretization
  - Finite Element Method (FEM)
  - Boundary Element Method (BEM)
  - Finite Difference Method (FDM)
  - Finite Volume Method (FVM)
  - Spectral Method
  - Mesh-Free Method
- FEM Formulation
  - Mechanics of Materials (MoM) Formulation
  - Convential Variational Formulation
  - Advanced Variational Formulation
  - Template Formulation
- Primary Unknown Variable Choice
  - Displacement
  - Force
  - Mixed
  - Hybrid

  $\leftrightarrow$ Solution Choice
  - Stiffness
  - Flexibility
  - Combined

What is a Finite Element?

- Archimedes' problem (circa 250 BC): rectification of the circle as limit of inscribed regular polygons

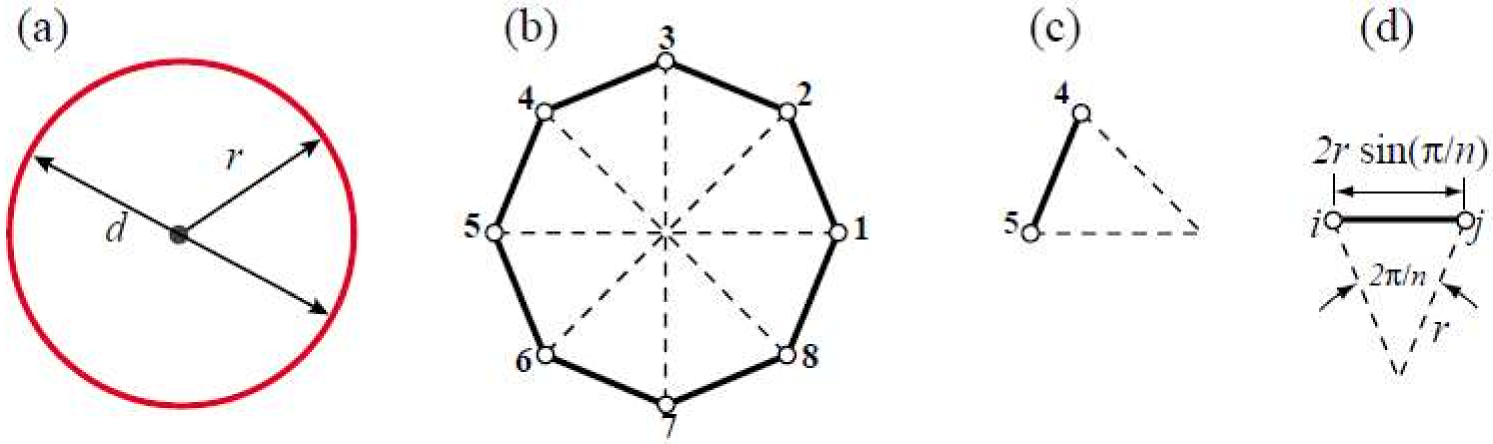
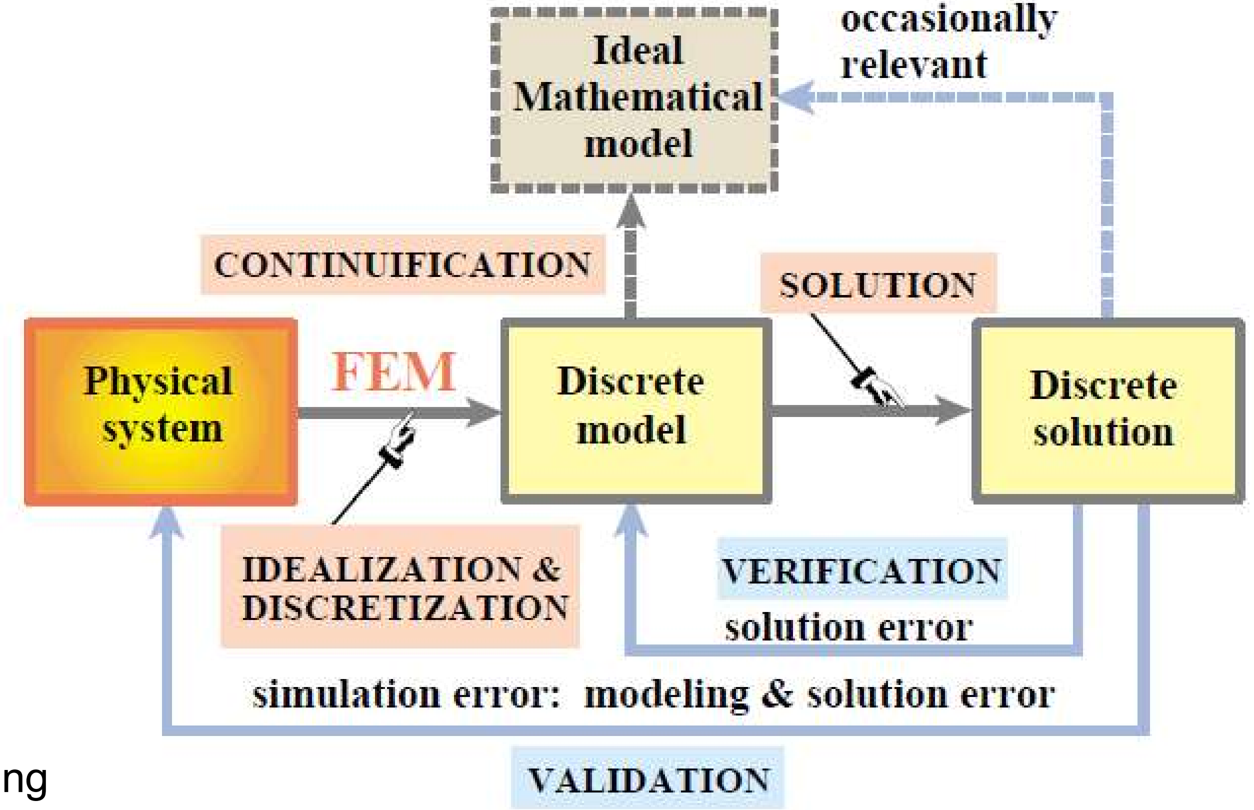
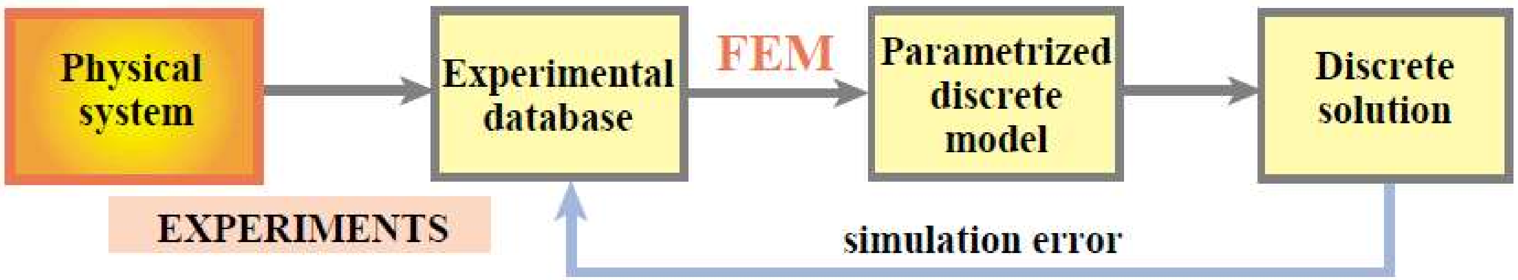


FIGURE 1.2. The "find π" problem treated with FEM concepts: (a) continuum object, (b) a discrete approximation by inscribed regular polygons, (c) disconnected element, (d) generic element.

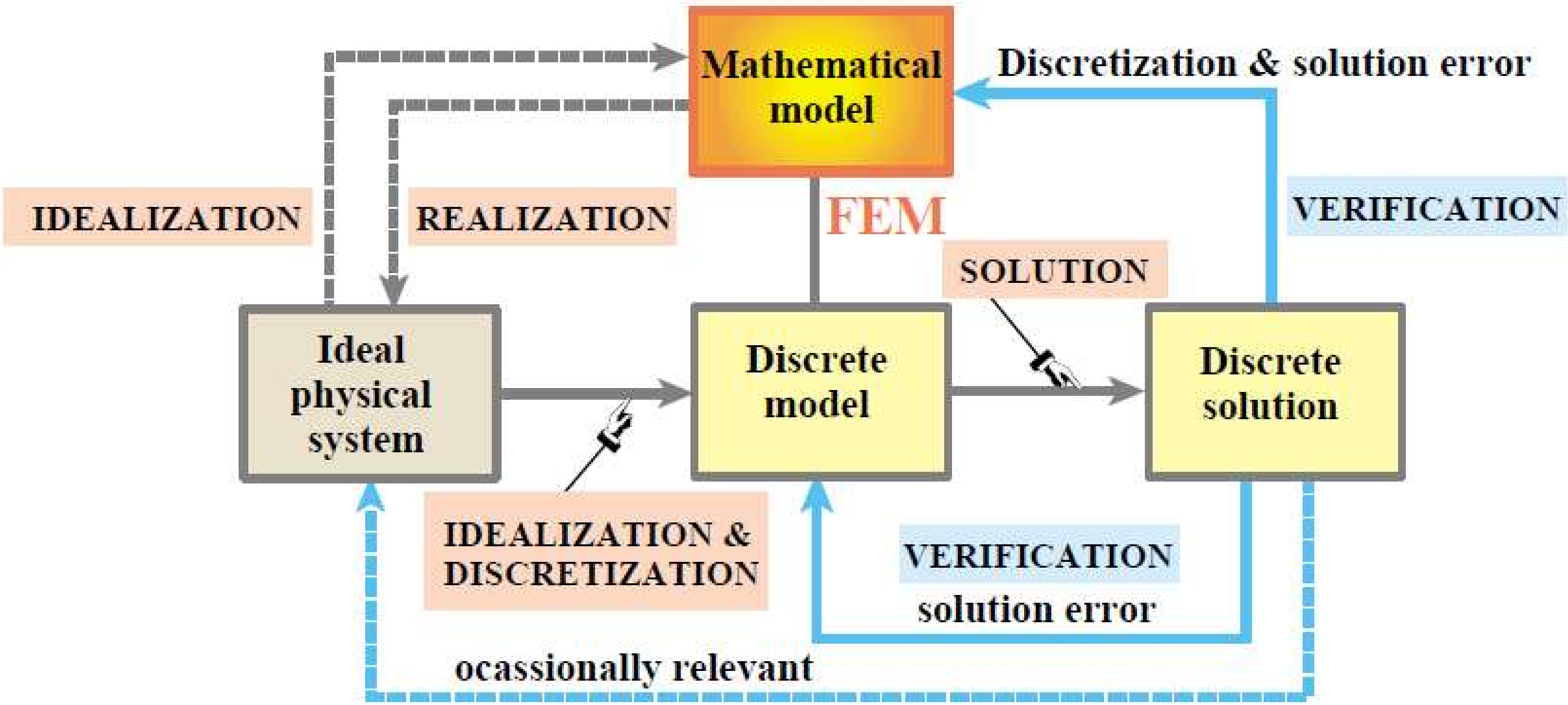
Physical FEM
model-based simulation of physical systems



Model updating

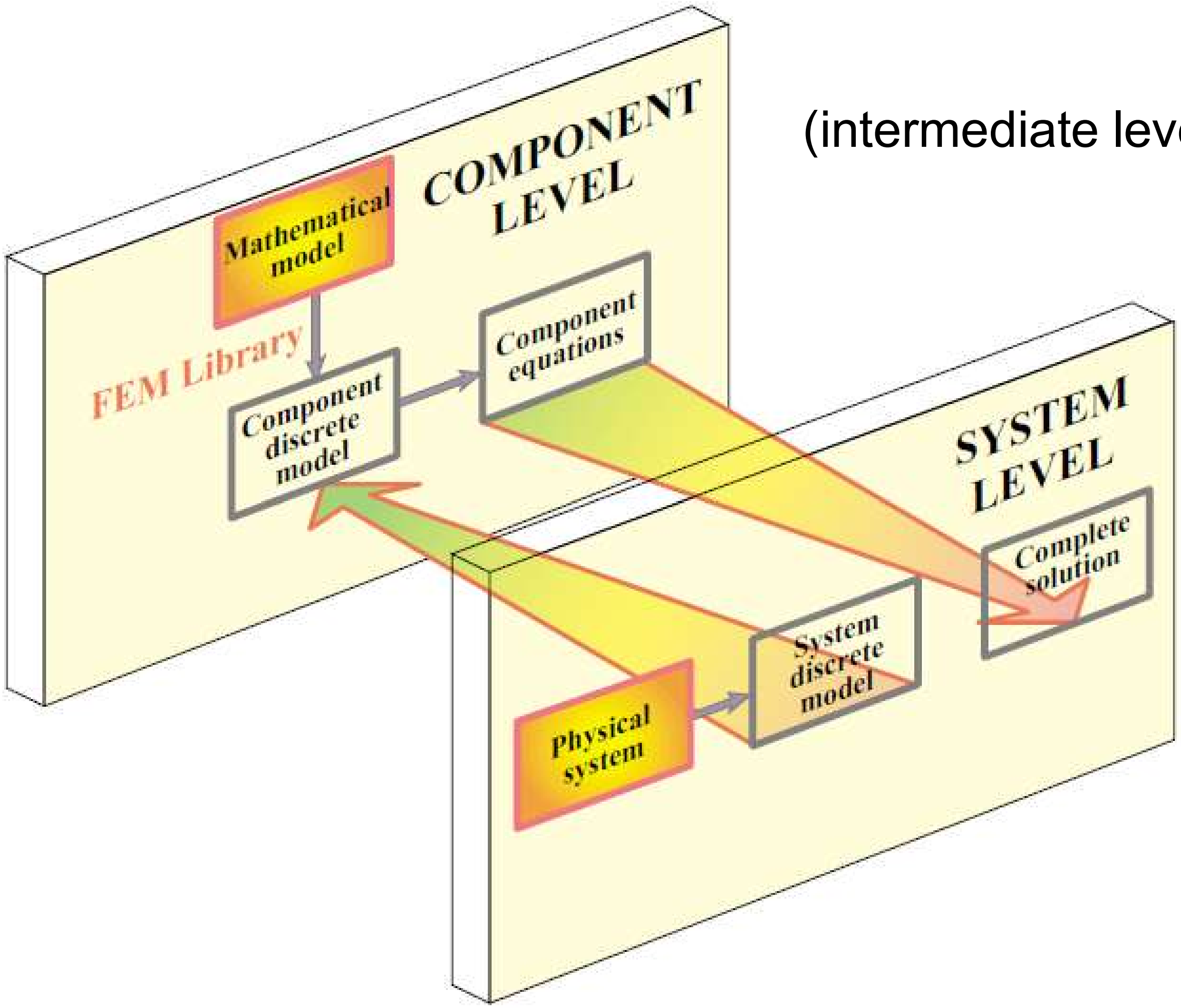


Mathematical FEM
numerical approximation to mathematical problems

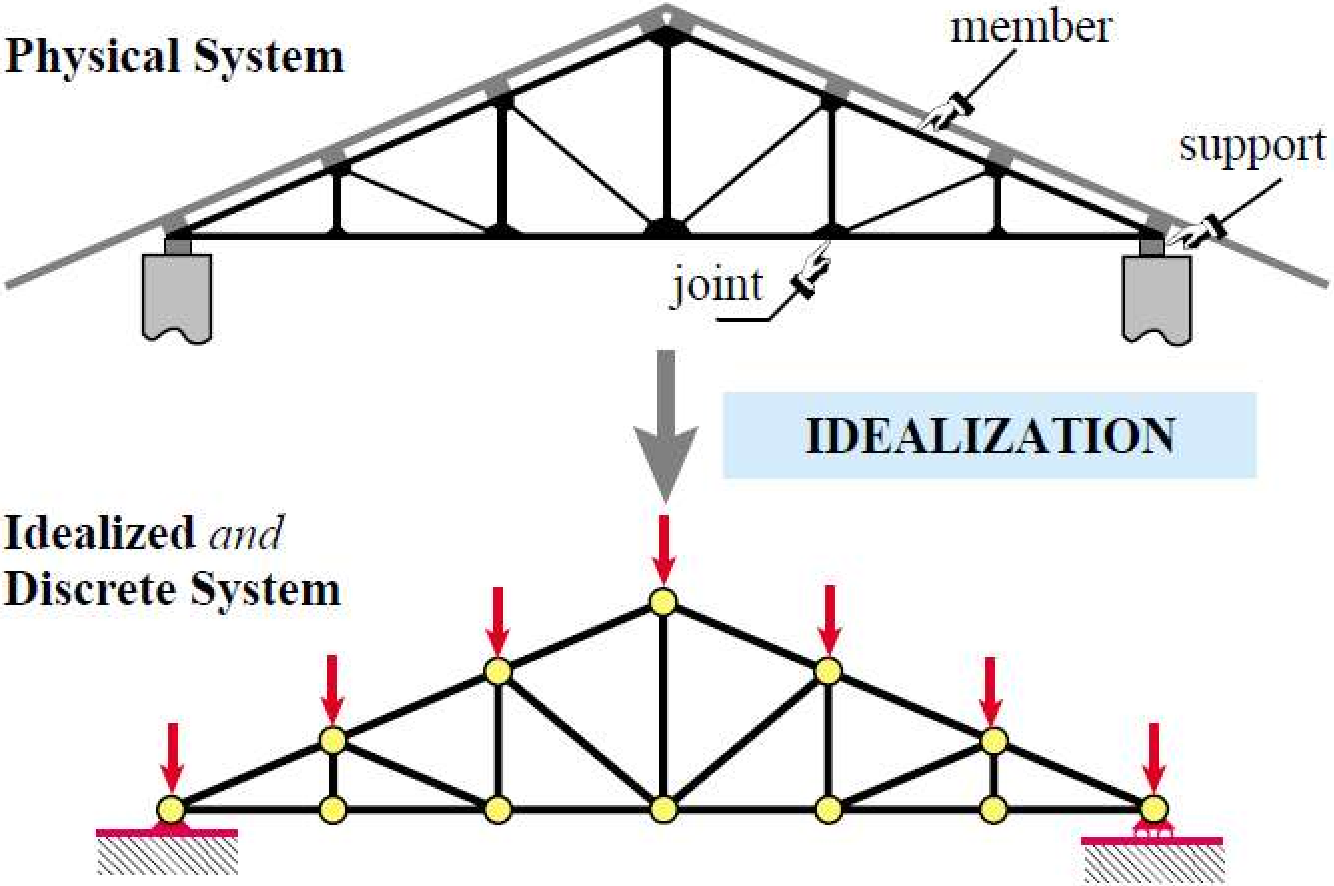


Synergy of Physical and Mathematical FEM

(intermediate levels omitted)



Idealization Process for a Simple Structure



Physical Interpretation

- Breakdown (disassembly, tearing, partition, separation, decomposition) of structural system into components (elements) and reconstruction by the assembly process
- Mechanical response of an element is characterized in terms of a finite number of degrees of freedom
  - DOF: values of the unknown functions as a set of node points

Mathematical Interpretation

- Numerical approximation of a boundary value problem by Ritz-Galerkin discretization with functions of local support
- geometry of Ω is only approximated by that of $\cup\Omega^{(e)}$
- unknown function (or functions) is locally approximated over each element by an interpolation formula
  - shape functions: states of the assumed unknown function(s) determined by unit node values

JOURNAL OF THE AERONAUTICAL SCIENCES

VOLUME 23 SEPTEMBER, 1956 NUMBER 9

Stiffness and Deflection Analysis of Complex Structures

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ABSTRACT

A method is developed for calculating stiffness influence coefficients of complex shell-type structures. The object is to provide a method that will yield structural data of sufficient accuracy to be adequate for subsequent dynamic and aeroelastic analyses.

Stiffness of the complete structure is obtained by summing stiffnesses of individual units. Stiffnesses of typical structural components are derived in the paper. Basic conditions of continuity and equilibrium are established at selected points (nodes) in the structure. Increasing the number of nodes increases the accuracy of results. Any physically possible support conditions can be taken into account. Details in setting up the analysis can be performed by nonengineering trained personnel; calculations are conveniently carried out on automatic digital computing equipment.

Method is illustrated by application to a simple truss, a flat plate, and a box beam. Due to shear lag and spar web deflection, the box beam has a 25 per cent greater deflection than predicted from beam theory. It is shown that the proposed method correctly accounts for these effects.

Considerable extension of the material presented in the paper is possible.

(I) INTRODUCTION

PRESENT CONFIGURATION TRENDS in the design of high-speed aircraft have created a number of difficult, fundamental structural problems for the worker in aeroelasticity and structural dynamics. The chief problem in this category is to predict, for a given elastic structure, a comprehensive set of load-deflection relations which can serve as structural basis for dynamic load calculations, theoretical vibration and flutter analyses, estimation of the effects of structural deflection on static air loads, and theoretical analysis of aeroelastic effects on stability and control. This is a problem of exceptional difficulty when thin wings and tail surfaces of low aspect ratio, either swept or unswept, are involved.

It is recognized that camber bending (or rib bending) is a significant feature of the vibration modes of the newer configurations, even of the low-order modes; in order to encompass these characteristics it seems likely that the load-deflection relations of a practical structure must be expressed in the form of either deflection or stiffness influence coefficients. One approach is to employ structural models and to determine the influence coefficients experimentally; it is anticipated that the experimental method will be employed extensively in the future, either in lieu of or as a final check on the result of analysis. However, elaborate models are expensive, they take a long time to build, and tend to become obsolete because of design changes; for these reasons it is considered essential that a continuing research effort should be applied to the development of analytical methods. It is to be expected that modern developments in high-speed digital computing machines will make possible a more fundamental approach to the problems of structural analysis; we shall expect to base our analysis on a more realistic and detailed conceptual model of the real structure than has been used in the past. As indicated by the title, the present paper is exclusively concerned with methods of theoretical analysis; also it is our object to outline the development of a method that is well adapted to the use of high-speed digital computing machinery.

(II) REVIEW OF EXISTING METHODS OF STRUCTURAL ANALYSIS

(1) *Elementary Theories of Flexure and Torsion*

The limitations of these venerable theories are too well known to justify extensive comment. They are

Received June 29, 1955. This paper is based on a paper presented at the Aeroelasticity Session, Twenty-Second Annual Meeting, IAS, New York, January 25 29, 1954.

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- Divide the solution domain into simply shaped regions or elements
- Develop an approximate solution for the PDE for each element
- Generate total solution by linking together, or “assembling,” the individual solutions taking care to ensure continuity at the interelement boundaries

Finite Element Method

- General PDE: $\mathrm{L}(u) - f = 0$
- Assume that $\mathrm{u} \approx \tilde{u} = \sum_i u_i \phi_i$ (1)

  Where ϕ_i is a set of basis functions.

(1) is a Fourier expansion

ϕ_i is a polynom in each mesh cell

ϕ_i is a constant value for each cell/node

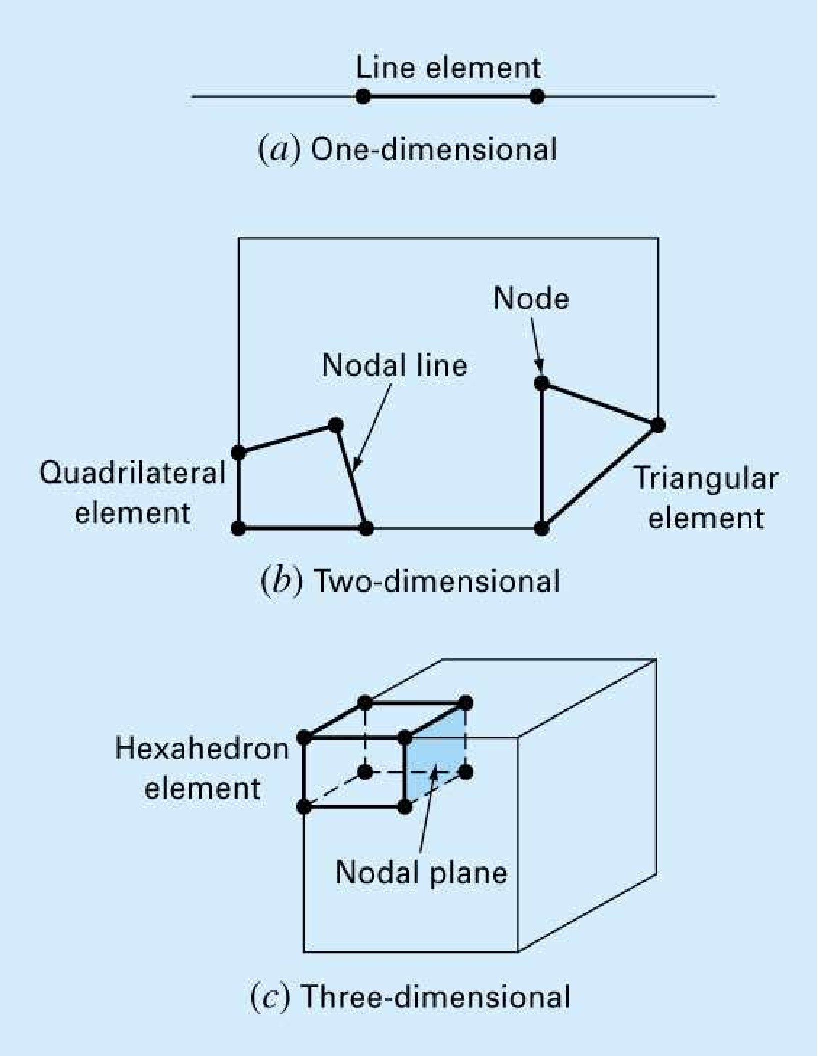
Spectral methods

Finite elements

ϕ_i is piecewise constant

Finite volume

Discretization



Element Equations

- Must choose an appropriate function with unknown coefficients that will be used to approximate the solution.
- Evaluation of the coefficients so that the function approximates the solution in an optimal fashion
- Choice of Approximation Functions:
  - For one dimensional case the simplest case is a first-order polynomial: $u(x) = a_0 + a_1 x$
- Obtaining an Optimal Fit of the Function to the Solution
  - Most common approaches are the direct approach, the method of weighted residuals, and the variational approach

$$\begin{cases} u_1 = a_0 + a_1 x_1 \\ u_2 = a_0 + a_1 x_2 \end{cases} \rightarrow \begin{cases} u_1 = u(x_1) \\ u_2 = u(x_2) \end{cases}$$

Using Cramer's rule

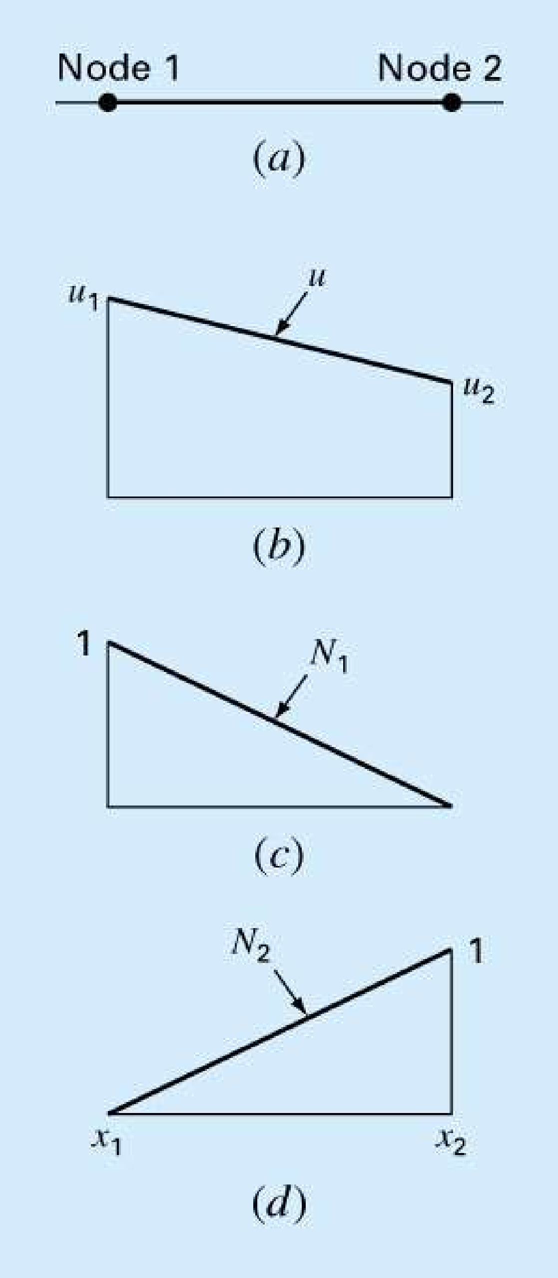
$$a_0 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \quad \text{and} \quad a_1 = \frac{u_2 - u_1}{x_2 - x_1}$$

$$u = a_0 + a_1 x = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} + \frac{u_2 - u_1}{x_2 - x_1} x = \frac{x_2 - x}{x_2 - x_1} u_1 + \frac{x - x_1}{x_2 - x_1} u_2$$

$$u = N_1 u_1 + N_2 u_2 \quad \text{where } N_1 = \frac{x_2 - x}{x_2 - x_1},\ N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2$$

$$\int_{x_1}^{x_2} u\, dx = \int_{x_1}^{x_2} \left(N_1 u_1 + N_2 u_2 \right) dx$$



- Mathematically, the resulting element equations will often consists of a set of linear algebraic equations that can be expressed in matrix form:

$$[k]\{u\} = \{F\}$$

$[k]$ = an element property or stiffness matrix

$\{u\}$ = a column vector of unknowns at the nodes

$\{F\}$ = a column vector reflecting the effect of any external influences applied at the nodes