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- Multifreedom constraints
- Master-slave method
- Penalty method
- Lagrange multiplier method

Multifreedom Constraints (1)

- Single freedom constraint examples

$$u_{x4} = 0 \quad \text{(linear, homogeneous)}$$

$$u_{y9} = 0.6 \quad \text{(linear, non-homogeneous)}$$

- Multifreedom constraint examples

$$u_{x2} = \frac{1}{2} u_{y2} \quad \text{(linear, homogeneous)}$$

$$u_{x2} - 2u_{x4} + u_{x6} = 0.25 \quad \text{(linear, non-homogeneous)}$$

$$\left(x_5 + u_{x5} - x_3 + u_{x3}\right)^2 + \left(y_5 + u_{y5} - y_3 + u_{y3}\right)^2 = 0 \quad \text{(nonlinear, homogeneous)}$$

Multifreedom Constraints (2)

- Sources
  - "skew" displacement BCs
  - Coupling nonmatched FEM meshes
  - Global-local and multiscale analysis
  - Incompressibility
  - Model reduction

- MFC application methods
  - Master-slave elimination
  - Penalty function augmentation
  - Lagrange multiplier adjunction

Procedure Summary in Static Analysis

Unmodified master stiffness equations $\mathbf{Ku} = \mathbf{f}$ before applying MFCs

↓

Apply MFCs
- master-slave
- penalty function
- Lagrange multiplier

↓

Modified stiffness equations $\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$

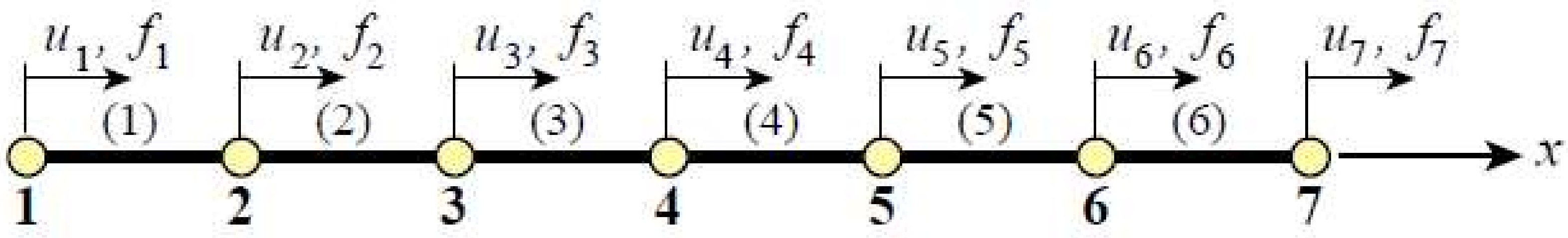
↓

Equation solver gives $\hat{\mathbf{u}}$

↓

Recover $\mathbf{u}$ if necessary

Example: 1D structure



multifreedom constraint: $u_2 = u_6$ or $u_2 - u_6 = 0$ (rigid link)

unconstrained master stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} \Leftrightarrow \mathbf{Ku} = \mathbf{f}$$

Example: Master-Slave Method

taking u_2 as master and u_6 as slave

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} \Leftrightarrow \mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

unconstrained master stiffness equations: $\mathbf{K}\mathbf{u} = \mathbf{f}$

master-slave transformation: $\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$

replace $\mathbf{u}$ and premultiply both sides by $\mathbf{T}^T$: $\mathbf{T}^T\mathbf{K}\mathbf{T}\hat{\mathbf{u}} = \mathbf{T}^T\mathbf{f}$

modified stiffness equations: $\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$

Example: Master-Slave Method

modified stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2+f_6 \\ f_3 \\ f_4 \\ f_5 \\ f_7 \end{bmatrix}$$

$$\Leftrightarrow \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}} \xrightarrow{\text{solve for } \hat{\mathbf{u}}} \mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

taking u_6 as master and u_2 as slave $(u_2 = u_6)$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} \xrightarrow[\mathbf{T}^T\mathbf{K}\mathbf{T}\hat{\mathbf{u}}=\mathbf{T}^T\mathbf{f}]{\mathbf{u}=\mathbf{T}\hat{\mathbf{u}}} \begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{12} & 0 \\ 0 & K_{33} & K_{34} & 0 & K_{23} & 0 \\ 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ K_{12} & K_{23} & 0 & K_{56} & K_{22}+K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_2+f_6 \\ f_7 \end{bmatrix}$$

Example: Multiple MFCs (1)

Suppose
$$\left.\begin{aligned} u_2 - u_6 &= 0 \\ u_1 + 4u_4 &= 0 \\ 2u_3 + u_4 + u_5 &= 0 \end{aligned}\right\} \xrightarrow[\text{slave: 3,4,6}]{\text{master: 1,2,5,7}} \begin{cases} u_6 = u_2 \\ u_4 = -\dfrac{1}{4} u_1 \\ u_3 = -\dfrac{1}{2}\left(u_4 + u_5\right) = \dfrac{1}{8} u_1 - \dfrac{1}{2} u_5 \end{cases}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/8 & 0 & -1/2 & 0 \\ -1/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \end{bmatrix} \Leftrightarrow \mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

Example: Multiple MFCs (2)

$$\left.\begin{aligned} u_2 - u_6 &= 0 \\ u_1 + 4u_4 &= 0 \\ 2u_3 + u_4 + u_5 &= 0 \end{aligned}\right\} \xrightarrow[\text{slave: 3,4,6}]{\text{master: 1,2,5}} \left\{\begin{aligned} u_6 &= u_2 \\ 4u_4 &= -u_1 \\ 2u_3 + u_4 &= -u_5 \end{aligned}\right\}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \end{bmatrix} \rightarrow \mathbf{A}_s \mathbf{u}_s + \mathbf{A}_m \mathbf{u}_m = 0 \rightarrow \mathbf{u}_s = -\mathbf{A}_s^{-1} \mathbf{A}_m \mathbf{u}_m = \mathbf{T} \mathbf{u}_m$$

$$u = \begin{bmatrix} u_7 \\ u_1 \\ u_2 \\ u_5 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ 0 & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \end{bmatrix}$$

Example: Non-homogeneous MFCs

$$u_2 - u_6 = 0.2$$

Pick again u_6 as slave, put into matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.2 \\ 0 \end{bmatrix} \Leftrightarrow \mathbf{u} = \mathbf{T}\hat{\mathbf{u}} + \underbrace{\mathbf{g}}_{\text{gap vector}}$$

premultiply both sides by $\mathbf{T}^T\mathbf{K}$, replace $\mathbf{Ku} = \mathbf{f}$ and pass data to RHS

$$\mathbf{T}^T\mathbf{Ku} = \mathbf{T}^T\mathbf{K}\left(\mathbf{T}\hat{\mathbf{u}} + \mathbf{g}\right) \rightarrow \mathbf{T}^T\mathbf{KT}\hat{\mathbf{u}} = \mathbf{T}^T\left(\mathbf{Ku} - \mathbf{Kg}\right)$$

$$\rightarrow \mathbf{T}^T\mathbf{KT}\hat{\mathbf{u}} = \underbrace{\mathbf{T}^T\left(\mathbf{f} - \mathbf{Kg}\right)}_{\text{modified force vector}} \xrightarrow[\hat{\mathbf{f}}=\mathbf{T}^T\left(\mathbf{Ku}-\mathbf{Kg}\right)]{\hat{\mathbf{K}}=\mathbf{T}^T\mathbf{KT}} \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

Example: Non-homogeneous MFCs

modified stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 - 0.2K_{66} \\ f_3 \\ f_4 \\ f_5 - 0.2K_{56} \\ f_7 - 0.2K_{67} \end{bmatrix}$$

$$\Leftrightarrow \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}} \xrightarrow{\text{solve for } \hat{\mathbf{u}}} \mathbf{u} = \mathbf{T}\hat{\mathbf{u}} + \mathbf{g}$$

General Case

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_u & \mathbf{u}_m & \mathbf{u}_s \end{bmatrix}^T$$

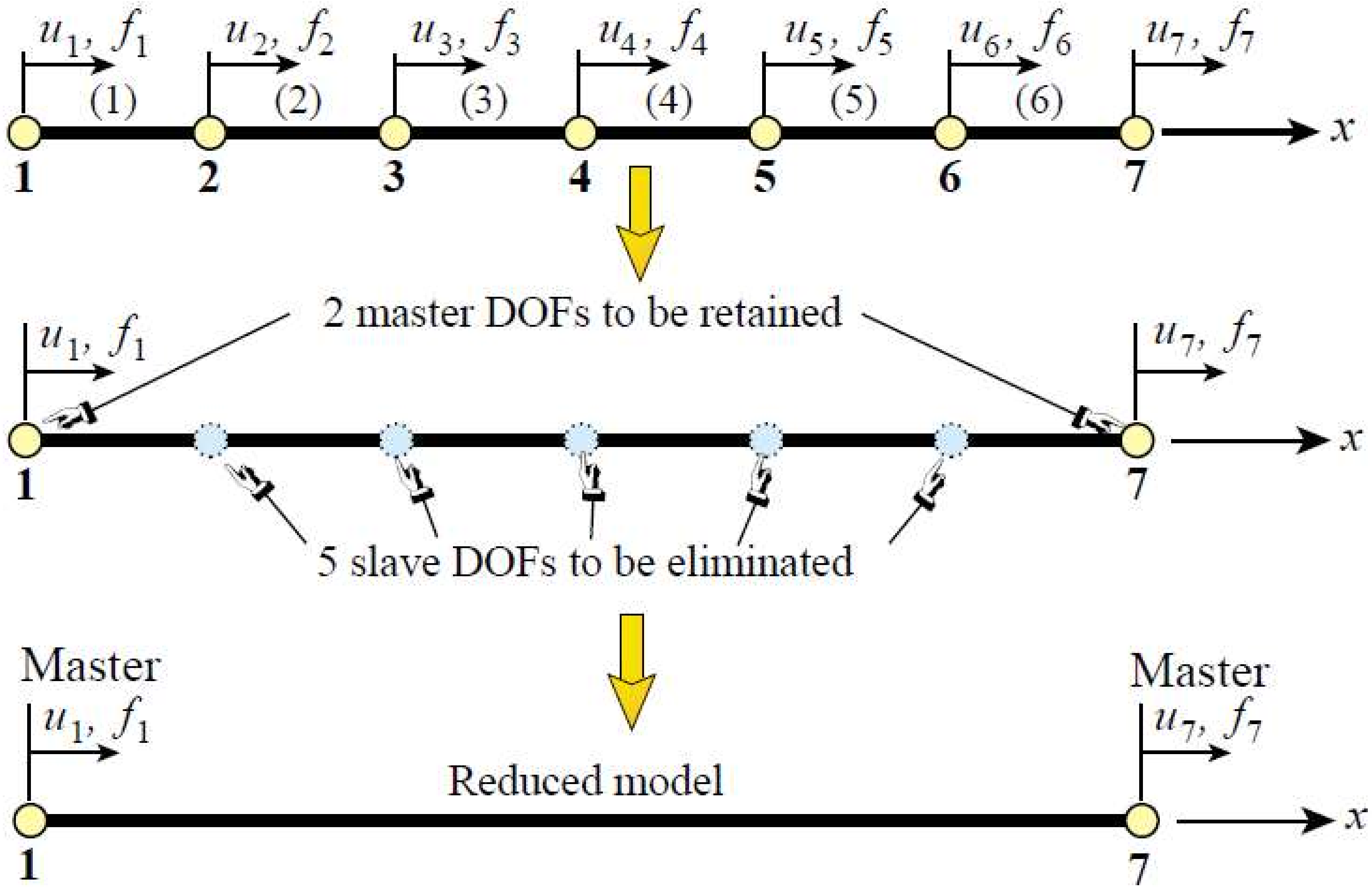
$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{um} & \mathbf{K}_{us} \\ \mathbf{K}_{um}^T & \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{us}^T & \mathbf{K}_{ms}^T & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_m \\ \mathbf{f}_s \end{bmatrix}$$

$$\mathbf{A}_m \mathbf{u}_m + \mathbf{A}_s \mathbf{u}_s = \mathbf{g}_A \rightarrow \mathbf{u}_s = -\mathbf{A}_s^{-1} \mathbf{A}_m \mathbf{u}_m + \mathbf{A}_s^{-1} \mathbf{g}_A = \mathbf{T}\mathbf{u}_m + \mathbf{g}$$

$$\begin{cases} \mathbf{K}_{uu}\mathbf{u}_u + \mathbf{K}_{um}\mathbf{u}_m + \mathbf{K}_{us}\left(\mathbf{T}\mathbf{u}_m + \mathbf{g}\right) = \mathbf{f}_u \rightarrow \mathbf{K}_{uu}\mathbf{u}_u + \left(\mathbf{K}_{um} + \mathbf{K}_{us}\mathbf{T}\right)\mathbf{u}_m = \mathbf{f}_u - \mathbf{K}_{us}\mathbf{g} \\ \left.\begin{array}{l} \mathbf{K}_{um}^T\mathbf{u}_u + \mathbf{K}_{mm}\mathbf{u}_m + \mathbf{K}_{ms}\left(\mathbf{T}\mathbf{u}_m + \mathbf{g}\right) = \mathbf{f}_m \\ \mathbf{T}^T\left[\mathbf{K}_{us}^T\mathbf{u}_u + \mathbf{K}_{ms}^T\mathbf{u}_m + \mathbf{K}_{ss}\left(\mathbf{T}\mathbf{u}_m + \mathbf{g}\right) = \mathbf{f}_s\right] \end{array}\right\} \\ \rightarrow \left(\mathbf{K}_{um}^T + \mathbf{T}^T\mathbf{K}_{us}^T\right)\mathbf{u}_u + \left(\mathbf{K}_{mm} + \mathbf{T}^T\mathbf{K}_{ms}^T + \mathbf{K}_{ms}\mathbf{T} + \mathbf{T}^T\mathbf{K}_{ss}\mathbf{T}\right)\mathbf{u}_m = \mathbf{f}_m - \mathbf{K}_{ms}\mathbf{g} + \mathbf{f}_s - \mathbf{T}^T\mathbf{K}_{ss}\mathbf{g} \end{cases}$$

$$\rightarrow \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{um} + \mathbf{K}_{us}\mathbf{T} \\ \mathbf{K}_{um}^T + \mathbf{T}^T\mathbf{K}_{us}^T & \mathbf{K}_{mm} + \mathbf{T}^T\mathbf{K}_{ms}^T + \mathbf{K}_{ms}\mathbf{T} + \mathbf{T}^T\mathbf{K}_{ss}\mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u - \mathbf{K}_{us}\mathbf{g} \\ \mathbf{f}_m - \mathbf{K}_{ms}\mathbf{g} + \mathbf{f}_s - \mathbf{T}^T\mathbf{K}_{ss}\mathbf{g} \end{bmatrix}$$

Example: Model Reduction



Example: Model Reduction

Lots of slaves, few masters. Only masters are left.

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}}_{5\text{ slaves}} = \begin{bmatrix} 1 & 0 \\ 5/6 & 1/6 \\ 4/6 & 2/6 \\ 3/6 & 3/6 \\ 2/6 & 4/6 \\ 1/6 & 5/6 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} u_1 \\ u_7 \end{bmatrix}}_{2\text{ masters}} \Leftrightarrow \mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

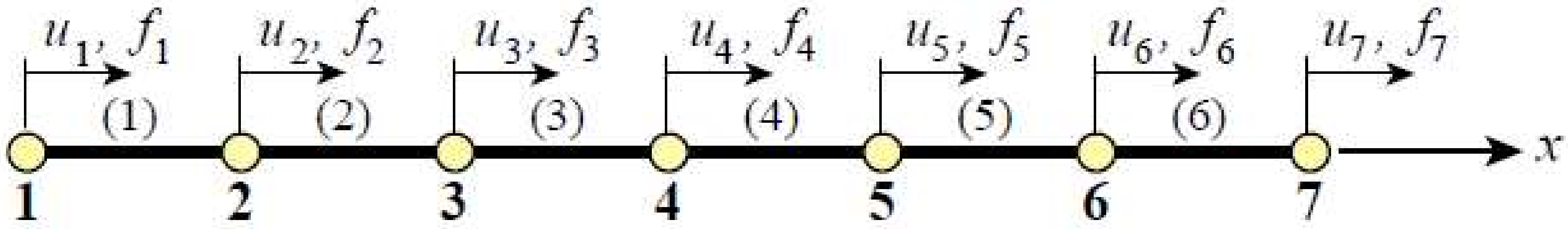
apply the congruential transformation we get the reduced stiffness equations

$$\mathbf{Ku} = \mathbf{f} \rightarrow \mathbf{K}\left(\mathbf{T}\hat{\mathbf{u}}\right) = \mathbf{f} \rightarrow \mathbf{T}^T\mathbf{KT}\hat{\mathbf{u}} = \mathbf{T}^T\mathbf{f} \rightarrow \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}} \rightarrow \begin{bmatrix} \hat{K}_{11} & \hat{K}_{17} \\ \hat{K}_{17} & \hat{K}_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_7 \end{bmatrix}$$

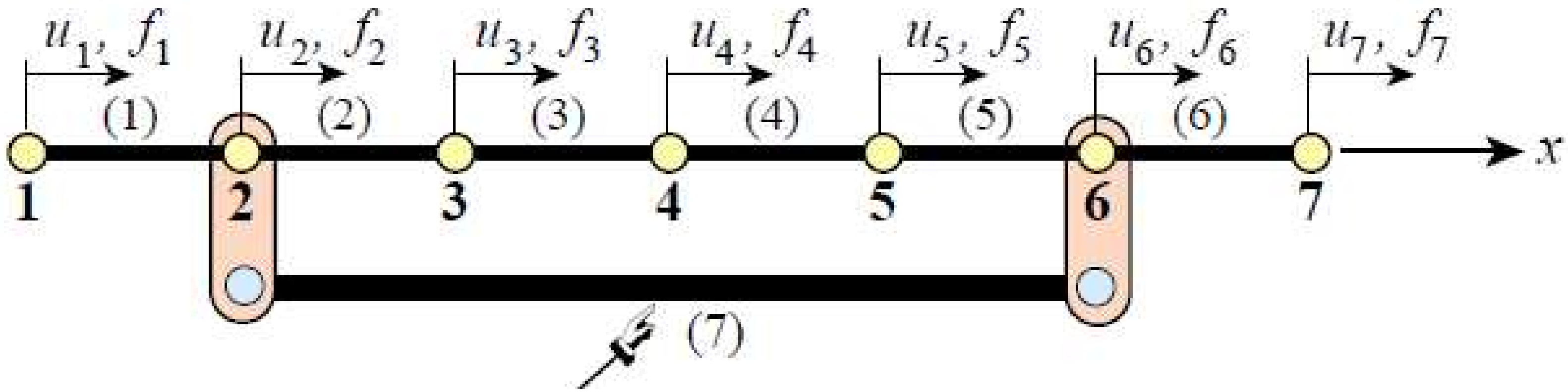
Master-Slave Method

- Advantages
  - Exact of precaution taken
  - Easy to understand
  - Retains positive definiteness
  - Important applications to model reduction
- Disadvantages
  - Requires user decisions
    - Larger coefficients → slaves
  - Messy implementation for general MFCs
  - Hinders sparsity of master stiffness equations
  - Sensitive to constraint dependence
  - Restricted to linear constraints

Penalty Function Method: Physical Interpretation



$u_2 = u_6$



add “penalty element” of axial rigidity *w*

$$w\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}u_2\\ u_6\end{bmatrix} = 0 \xrightarrow[\begin{bmatrix}1 & -1\end{bmatrix}^T]{\text{premultiply}} w\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_2\\ u_6\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \rightarrow \mathbf{K}^{(7)}\mathbf{u}^{(7)} = \mathbf{f}^{(7)}$$

w: "penalty weight" assigned to the constraint

Penalty Function Method

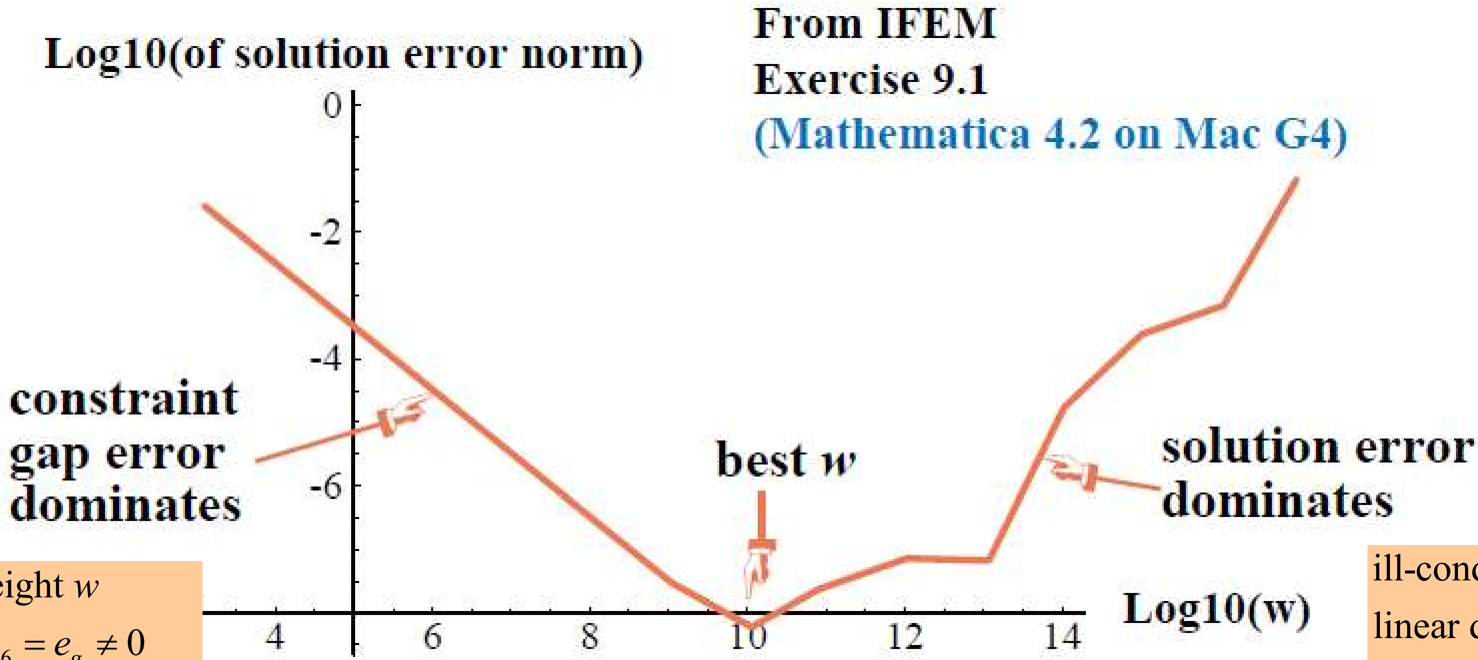
upon merging the penalty element, the modified stiffness equations are

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+w & K_{23} & 0 & 0 & -w & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & -w & 0 & 0 & K_{56} & K_{66}+w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This modified system is submitted to the equation solver.

Note that **u** remains the same arrangement of DOFs

But which penalty weight to use?



finite weight w
$\rightarrow u_2 - u_6 = e_g \neq 0$
(constraint violation)
$|e_g| \propto 1/w$

ill-conditioned
linear dependency
$\rightarrow$ singular

$$\left[\text{Square Root Rule: } w = 10^k \sqrt{10^p} = 10^{k+p/2}\right]$$

k : order of the largest stiffness coefficient before adding penalty elements

p : digits of the working machine precison

Penalty Function Method: General MFCs

$$3u_3 + u_5 - 4u_6 = 1 \rightarrow \begin{bmatrix} 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = 1$$

premultiply both sides by $\begin{bmatrix} 3 & 1 & -4 \end{bmatrix}^T$:

$$\underbrace{\begin{bmatrix} 9 & 3 & -12 \\ 3 & 1 & -4 \\ -12 & -4 & 16 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}}_{\text{"penalty element" stiffness equations}}$$

scale by w and merge:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33}+9w & K_{34} & 3w & -12w & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 3w & K_{45} & K_{55}+w & K_{56}-4w & 0 \\ 0 & 0 & -12w & 0 & K_{56}-4w & K_{66}+16w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3+3w \\ f_4 \\ f_5+w \\ f_6-4w \\ f_7 \end{bmatrix}$$

Theory behind Penalty Method

set of m linear MFCs $\rightarrow \mathbf{a}_p \mathbf{u} = \mathbf{b}_p,\ p = 1,\ldots,m \rightarrow w_p \mathbf{a}_p^{\ T} \left(\mathbf{a}_p \mathbf{u} - \mathbf{b}_p \right) = 0$

Courant quadratic penalty function or penalty energy $\rightarrow P = \sum_{p=1}^{m} \mathbf{u}^T \left(\frac{1}{2} \underbrace{w_p \mathbf{a}_p^{\ T} \mathbf{a}_p}_{\mathbf{K}^{(p)}} \mathbf{u} - \underbrace{w_p \mathbf{a}_p^{\ T} \mathbf{b}_p}_{\mathbf{f}^{(p)}} \right)$

augmented potential energy $\rightarrow \Pi_a = \Pi + P = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{f} + \sum_{p=1}^{m} \left(\frac{1}{2} \mathbf{u}^T \mathbf{K}^{(p)} \mathbf{u} - \mathbf{u}^T \mathbf{f}^{(p)} \right)$

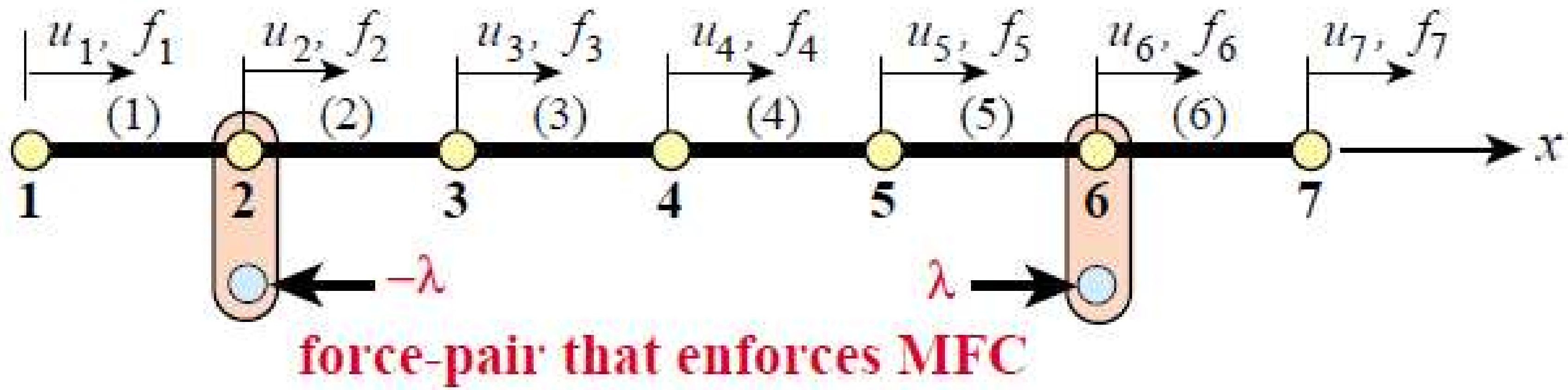
$$\frac{\partial \Pi_a}{\partial \mathbf{u}} = \mathbf{K}\mathbf{u} - \mathbf{f} + \sum_{p=1}^{m} \left(\mathbf{K}^{(p)} \mathbf{u} - \mathbf{f}^{(p)} \right) = 0 \rightarrow \left[\mathbf{K} + \sum_{p=1}^{m} \mathbf{K}^{(p)} \right] \mathbf{u} = \mathbf{f} + \sum_{p=1}^{m} \mathbf{f}^{(p)}$$

$$\left. \begin{aligned} &\mathbf{A}\mathbf{u} = \mathbf{b} \rightarrow \mathbf{W}\mathbf{A}^T \left(\mathbf{A}\mathbf{u} - \mathbf{b} \right) = 0 \\ &\mathbf{K}\mathbf{u} = \mathbf{f} \end{aligned} \right\} \rightarrow \left(\mathbf{K} + \mathbf{A}^T \mathbf{W} \mathbf{A} \right) \mathbf{u} = \mathbf{f} + \mathbf{W}\mathbf{A}^T \mathbf{b}$$

Penalty Function Method

- Advantages
  - General application including nonlinear MFCs
  - Easy to implement using FE library and standard assembler
  - No change in vector of unknowns
  - Retains positive definiteness
  - Insensitive to constraint dependence
- Disadvantages
  - Selection of weights left to users: big burden
  - Accuracy limited by ill-conditioning

Lagrange Multiplier Method: Physical Interpretation



$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \lambda \\ f_3 \\ f_4 \\ f_5 \\ f_6 + \lambda \\ f_7 \end{bmatrix}$$

Lagrange Multiplier Method

Because λ is unknown, it is passed to the LHS and appended to the node-displacement vector:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This is now a system of 7 equations and 8 unknowns.

Need an extra equation: MFC

Lagrange Multiplier Method

Appended MFC as an additional equation (adjunction):

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \end{bmatrix}$$

This is the *multiplier-augmented system*.

The new coefficient matrix is called the *bordered stiffness*.

Lagrange Multiplier Method: Multiple MFCs

Three MFCs: $u_2 - u_6 = 0, \quad 5u_2 - 8u_7 = 3, \quad 3u_3 + u_5 - 4u_6 = 1$

Step#1: append the 3 constraints $\left(\text{adjoin}\right)$

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Lagrange Multiplier Method: Multiple MFCs

Three MFCs: $\underbrace{u_2 - u_6}_{\lambda_1} = 0, \quad \underbrace{5u_2 - 8u_7}_{\lambda_2} = 3, \quad \underbrace{3u_3 + u_5 - 4u_6}_{\lambda_3} = 0$

Step#2: append multipliers, symmetrize and fill

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Theory behind Lagrange Multiplier

set of m MFCs $\rightarrow \mathbf{Au} = \mathbf{b}$

potential energy of the unconstrained FE model $\rightarrow \Pi = \frac{1}{2}\mathbf{u}^T\mathbf{Ku} - \mathbf{u}^T\mathbf{f}$

$$L(\mathbf{u}, \boldsymbol{\lambda}) = \Pi + \boldsymbol{\lambda}^T(\mathbf{Au} - \mathbf{b}) = \frac{1}{2}\mathbf{u}^T\mathbf{Ku} - \mathbf{u}^T\mathbf{f} + \boldsymbol{\lambda}^T(\mathbf{Au} - \mathbf{b})$$

$$\left.\begin{aligned} \frac{\partial L}{\partial \mathbf{u}} &= \mathbf{Ku} - \mathbf{f} + \mathbf{A}^T\boldsymbol{\lambda} = 0 \\ \frac{\partial L}{\partial \boldsymbol{\lambda}} &= \mathbf{Au} - \mathbf{b} = 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{b} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{A}^T \\ \mathbf{A} & \varepsilon\mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{b} + \varepsilon\mathbf{S}^{-1}\boldsymbol{\lambda}^P \end{bmatrix}$$

$$\rightarrow \begin{cases} \mathbf{Au} + \varepsilon\mathbf{S}^{-1}\boldsymbol{\lambda} = \mathbf{b} + \varepsilon\mathbf{S}^{-1}\boldsymbol{\lambda}^P \xrightarrow{w=1/\varepsilon} \boldsymbol{\lambda} = \boldsymbol{\lambda}^P + w\mathbf{S}(\mathbf{b} - \mathbf{Au}) \\ \mathbf{Ku} + \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{f} \rightarrow \mathbf{Ku} + \mathbf{A}^T\left[\boldsymbol{\lambda}^P + w\mathbf{S}(\mathbf{b} - \mathbf{Au})\right] = \mathbf{f} \rightarrow (\mathbf{K} - w\mathbf{A}^T\mathbf{SA})\mathbf{u} = \mathbf{f} - w\mathbf{A}^T\mathbf{Sb} - \mathbf{A}^T\boldsymbol{\lambda}^P \end{cases}$$

$$\xrightarrow{\mathbf{W}=-w\mathbf{S}} \begin{cases} (\mathbf{K} + \mathbf{A}^T\mathbf{WA})\mathbf{u}^k = \mathbf{f} + \mathbf{A}^T\mathbf{Wb} - \mathbf{A}^T\boldsymbol{\lambda}^k \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k - \mathbf{W}(\mathbf{b} - \mathbf{Au}^k) \end{cases}$$

Lagrange Multiplier Method

- Advantages
  - General application
  - Exact
  - No user decisions: black-box
- Disadvantages
  - Difficult implementation
  - Additional unknowns
  - Loses positive definiteness
  - Sensitive to constraint dependence

MFC Application Methods: Summary

	Master-slave Elimination	Penalty function	Lagrange multiplier
Physical interpretation	Model reduction	Penalty element (flexible link)	Rigid link (reaction force)
Generality	fair	Excellent	Excellent
Ease of implementation	Poor to fair	Good	Fair
Sensitivity to user decisions	High	High	Small to none
Accuracy	Variable	Mediocre	Excellent
Sensitivity as regards constraint dependence	High	None	High
Retains positive definiteness	Yes	Yes	No
Modifies unknown vector	Yes	No	Yes