Contents

Introduction

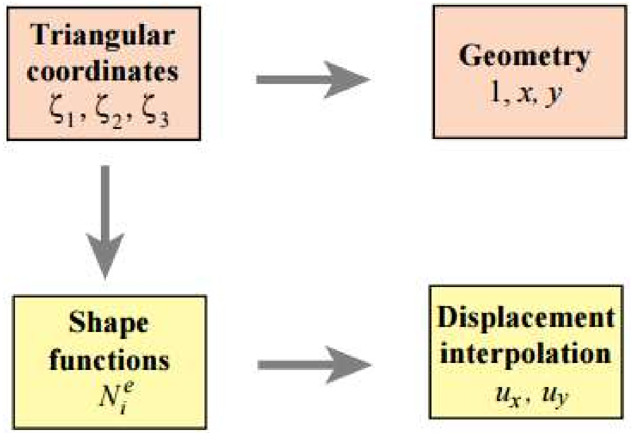
- Technical difficulties: linear triangle → quadrilateral as well as higher order triangles
  - The construction of shape functions that satisfy consistency requirements for higher order elements with curved boundaries becomes increasingly complicated → isoparametric elements
  - Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be evaluated in simple closed form → numerical quadrature
- Element geometry and displacements are represented by same set of shape functions (iso = equal)
- Before isoparametric concept was discovered, FEM developers did "SuperParametric" elements (cf. subparametric)
  - Element shape functions refined, more nodes and DOFs added
  - But element geometry was kept simple with straight sides

Isoparametric Formulations for Structural Mechanics

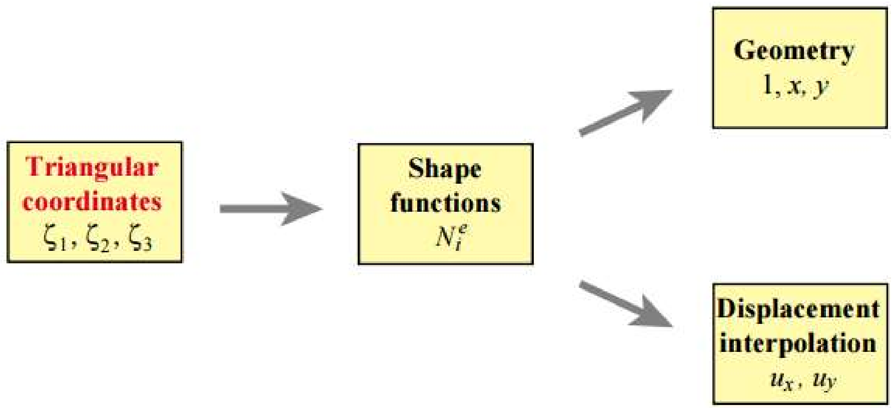
- Advantages
  - Unification: same steps for all iso-P elements
  - No need to distinguish straight vs. curved side elements
  - Quick construction of shape functions
- Disadvantages
  - Low-order iso-P elements may be poor performers (overstiff)
  - Method does not extend to problems with variational index higher than 1 (e.g., plate bending and shells)

Triangular Elements

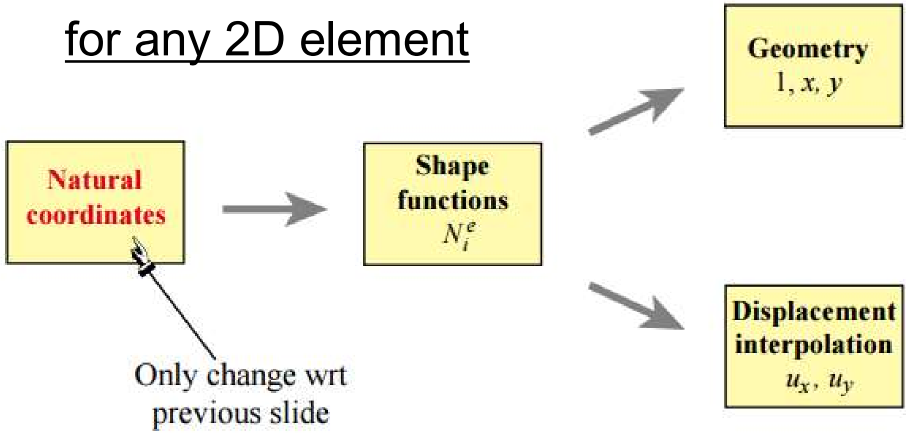
- Superparametric representation



- Isoparametric representation



for any 2D element



Three Node Linear Triangle

Geometry discription:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Displacement Interpolation:

$$u_x = u_{x1} N_1^e + u_{x2} N_2^e + u_{x3} N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$

$$u_y = u_{y1} N_1^e + u_{y2} N_2^e + u_{y3} N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

Equalizing Geometry and Displacements:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Iso-P Representation of 2D Plane Stress Elements with *n* Nodes

Element Geometry:

$$1 = \sum_{i=1}^{n} N_i^e, \quad x = \sum_{i=1}^{n} x_i N_i^e, \quad y = \sum_{i=1}^{n} y_i N_i^e$$

Displacement Interpolation:

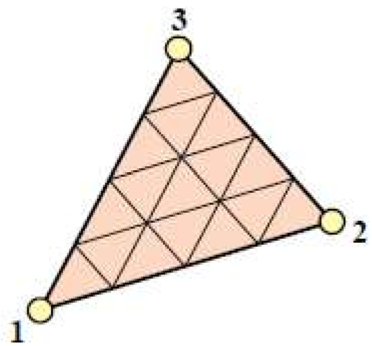
$$u_x = \sum_{i=1}^{n} u_{xi} N_i^e, \quad u_y = \sum_{i=1}^{n} u_{yi} N_i^e$$

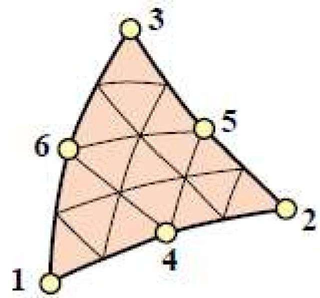
$$\xrightarrow{\text{m}} \begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

More rows may be added to interpolate other quantities from node values

$$\begin{matrix} \\ \\ \\ \\ \\ \text{thickness} \\ \text{temperature field} \end{matrix} \begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

Triangular Elements





Linear Triangle

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$N_1^e = \zeta_1,\ N_2^e = \zeta_2,\ N_3^e = \zeta_3$$

Quadratic Triangle

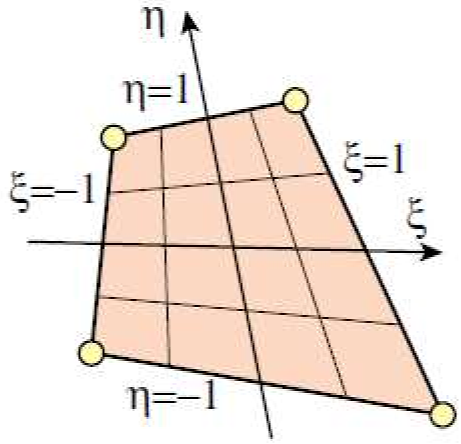
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix}$$

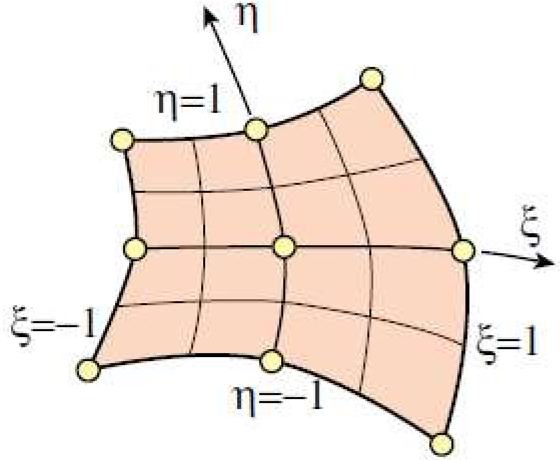
$$N_1^e = \zeta_1(2\zeta_1 - 1),\ N_2^e = \zeta_2(2\zeta_2 - 1),\ N_3^e = \zeta_3(2\zeta_3 - 1)$$

$$N_4^e = 4\zeta_1\zeta_2,\ N_5^e = 4\zeta_2\zeta_3,\ N_6^e = 4\zeta_3\zeta_1$$

Quadrilateral Coordinates (ξ, η)

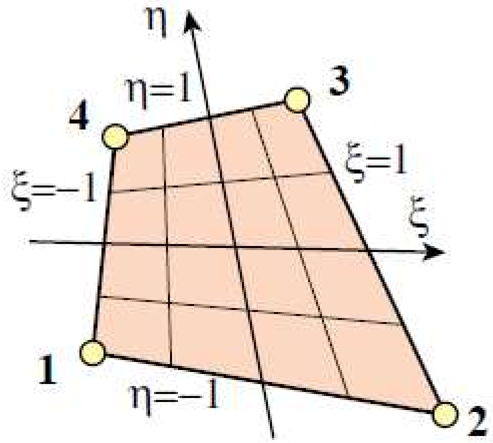
- Natural coordinate system (-1 ~ +1)





- Reference plane
  - quadrilateral coordinates plotted as Cartesian coordinates in the {ξ, η} plane
- Reference element: square of side 2
- Isoparametric mapping: {ξ, η} ↔ {x, y}

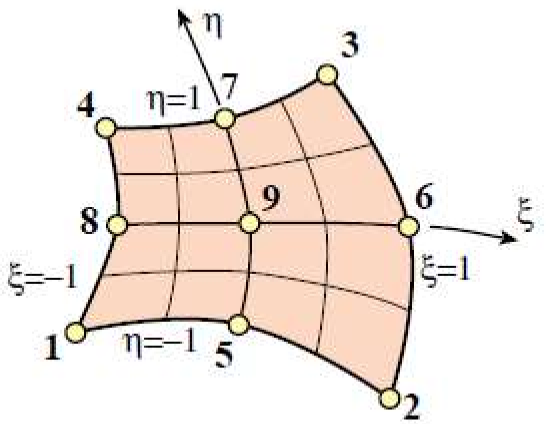
4-Node Bilinear Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix} \rightarrow \begin{cases} N_1^e = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2^e = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3^e = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4^e = \frac{1}{4}(1-\xi)(1+\eta) \end{cases}$$

These functions vary *linearly* on quadrilateral coordinate lines ξ = *const* and η = *const*, but are not linear polynomials as in the case of the three-node triangle.

9 Node Biquadratic Quadrilateral



Lagrangian quadrilateral

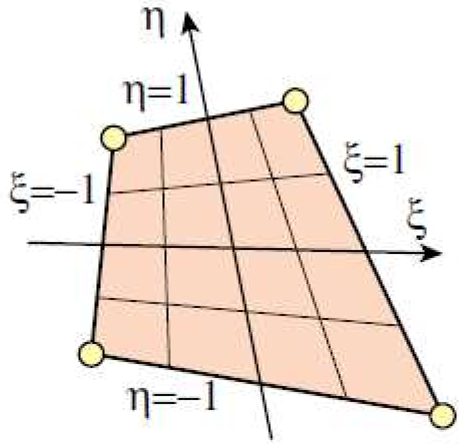
$$
\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_9 \\ y_1 & y_2 & \dots & y_9 \\ u_{x1} & u_{x2} & \dots & u_{x9} \\ u_{y1} & u_{y2} & \dots & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix} \rightarrow \begin{cases} N_1^e = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta, \; N_5^e = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \\ N_2^e = \frac{1}{4}(1+\xi)(1-\eta)\xi\eta, \; N_6^e = \frac{1}{2}(1+\xi)(1-\eta^2)\xi \\ N_3^e = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta, \; N_7^e = \frac{1}{2}(1-\xi^2)(1+\eta)\eta \\ N_4^e = \frac{1}{4}(1-\xi)(1+\eta)\xi\eta, \; N_8^e = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi \\ N_9^e = (1-\xi^2)(1-\eta^2) \end{cases}
$$

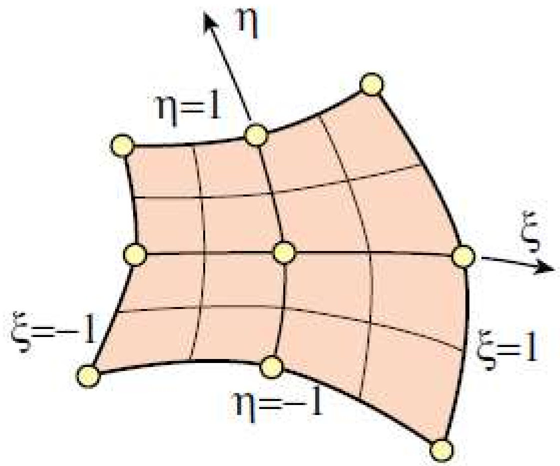
These functions vary *quadratically* along the coordinate lines ξ = *const* and η = *const*. The shape function associated with the internal node 9 is called a *bubble function* because of its geometric shape.

Isoparametric Quadrilaterals

- Implementation steps for element stiffness matrix
  - Construct shape functions in Quad coordinates
  - Compute x-y derivatives of shape functions and build strain-displacement matrix **B**
  - Integrate h $\mathbf{B}^T \mathbf{E} \mathbf{B}$ over element

- Quadrilateral Coordinates (ξ, η)





Partial Derivative Computation

- Shape functions are written in terms of ξ and η
- But Cartesian partials (with respect to x, y) are required to get strains & stresses
- Jacobian and Inverse Jacobian

$$\left(x(\xi,\eta),y(\xi,\eta)\right)\leftrightarrow\left(\xi(x,y),\eta(x,y)\right)$$

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta \\ dy &= \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta \end{aligned}\right\} \rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \mathbf{J}^T \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \text{ where } \underbrace{\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)}}_{\text{Jacobian}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\left.\begin{aligned} d\xi &= \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy \\ d\eta &= \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy \end{aligned}\right\} \rightarrow \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{J}^{-T} \begin{bmatrix} dx \\ dy \end{bmatrix} \text{ where } \underbrace{\mathbf{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,y)}}_{\text{inverse Jacobian}} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$J = |\mathbf{J}| = \det(\mathbf{J}) = J_{11}J_{22} - J_{12}J_{21}$$

$$\mathbf{J}^{-1}\mathbf{J} = \mathbf{I}$$

Shape Function Partial Derivatives

$$\left.\begin{aligned}\frac{\partial N_1^e}{\partial x} &= \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial x}\\ \frac{\partial N_1^e}{\partial y} &= \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial y}\end{aligned}\right\} \rightarrow \begin{bmatrix}\frac{\partial N_1^e}{\partial x}\\ \frac{\partial N_1^e}{\partial y}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x}\\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}\end{bmatrix}}_{\text{how to get?}}\begin{bmatrix}\frac{\partial N_1^e}{\partial \xi}\\ \frac{\partial N_1^e}{\partial \eta}\end{bmatrix} = \frac{\partial(\xi,\eta)}{\partial(x,y)}\begin{bmatrix}\frac{\partial N_1^e}{\partial \xi}\\ \frac{\partial N_1^e}{\partial \eta}\end{bmatrix} = \mathbf{J}^{-1}\begin{bmatrix}\frac{\partial N_1^e}{\partial \xi}\\ \frac{\partial N_1^e}{\partial \eta}\end{bmatrix}$$

$$\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} \rightarrow \mathbf{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{bmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x}\\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}\end{bmatrix}$$

Use the element geometry definition (n: number of nodes)

$$\left.\begin{aligned}x &= \sum_i^n x_i N_i^e\\ y &= \sum_i^n y_i N_i^e\end{aligned}\right\} \rightarrow \left\{\begin{aligned}\frac{\partial x}{\partial \xi} &= \sum_i^n x_i \frac{\partial N_i^e}{\partial \xi}, \quad \frac{\partial x}{\partial \eta} = \sum_i^n x_i \frac{\partial N_i^e}{\partial \eta}\\ \frac{\partial y}{\partial \xi} &= \sum_i^n y_i \frac{\partial N_i^e}{\partial \xi}, \quad \frac{\partial y}{\partial \eta} = \sum_i^n y_i \frac{\partial N_i^e}{\partial \eta}\end{aligned}\right\}$$

Store as entries of $\mathbf{J}$ and invert to get $\mathbf{J}^{-1}$

$$\mathbf{J} = \begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\sum_i^n x_i \frac{\partial N_i^e}{\partial \xi} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \xi}\\ \sum_i^n x_i \frac{\partial N_i^e}{\partial \eta} & \sum_i^n y_i \frac{\partial N_i^e}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_2^e}{\partial \xi} & \cdots & \frac{\partial N_n^e}{\partial \xi}\\ \frac{\partial N_1^e}{\partial \eta} & \frac{\partial N_2^e}{\partial \eta} & \cdots & \frac{\partial N_n^e}{\partial \eta}\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ \vdots & \vdots\\ x_n & y_n\end{bmatrix} = \mathbf{PX}$$

Partial Derivative Computation Sequence Summary

- At a specific point of quad coordinates ξ and η:
  - Compute from node coordinates and S.F.s
  - Form **J** and invert to get $\mathbf{J}^{-1}$ and J = det **J**
  - Apply the chain rule to get the x, y partials of the S.F.s

Strain Displacement Matrix B

- Use those S.F.s partials to build the strain-displacement matrix **B**

$$
e = \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{u}}{\partial x} \\ \dfrac{\partial \mathbf{v}}{\partial y} \\ \dfrac{\partial \mathbf{u}}{\partial y} + \dfrac{\partial \mathbf{v}}{\partial x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \dfrac{\partial N_i^e}{\partial x} u_{xi} \\ \sum_{i=1}^{n} \dfrac{\partial N_i^e}{\partial y} u_{yi} \\ \sum_{i=1}^{n} \dfrac{\partial N_i^e}{\partial y} u_{xi} + \sum_{i=1}^{n} \dfrac{\partial N_i^e}{\partial x} u_{yi} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & 0 & \dfrac{\partial N_2^e}{\partial x} & 0 & \cdots & \dfrac{\partial N_n^e}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1^e}{\partial y} & 0 & \dfrac{\partial N_2^e}{\partial y} & \cdots & 0 & \dfrac{\partial N_n^e}{\partial y} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_2^e}{\partial x} & \cdots & \dfrac{\partial N_n^e}{\partial y} & \dfrac{\partial N_n^e}{\partial x} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ \vdots \\ u_{xn} \\ u_{yn} \end{bmatrix} = \mathbf{B}\mathbf{u}^e
$$

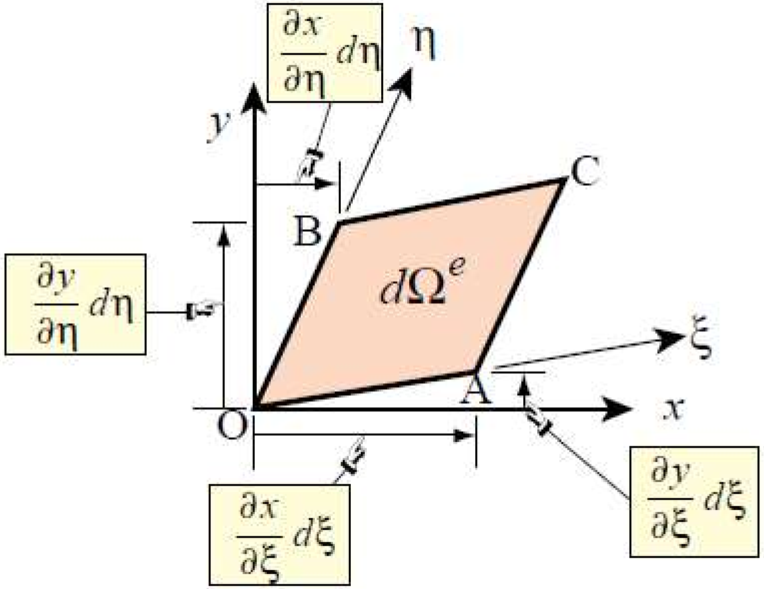
- Unlike the 3-node triangle, here **B** = **B**(ξ,η) varies over quad

Stiffness Matrix

$$\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\mathbf{E}\mathbf{B}d\Omega \xrightarrow{d\Omega = dxdy = \det\mathbf{J}d\xi d\eta} \mathbf{K}^e = \int_{-1}^{+1}\int_{-1}^{+1} h\mathbf{B}^T\mathbf{E}\mathbf{B}\det\mathbf{J}d\xi d\eta$$

$$\xrightarrow{h\mathbf{B}^T\mathbf{E}\mathbf{B}\det\mathbf{J} = \mathbf{F}(\xi,\eta)} \mathbf{K}^e = \int_{-1}^{+1}\int_{-1}^{+1} \mathbf{F}(\xi,\eta)d\xi d\eta$$

$$\int_{-1}^{+1}\int_{-1}^{+1} F(\xi,\eta)d\xi d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} \mathbf{F}(\xi,\eta)d\xi \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F(\xi_i,\eta_j)$$



$$d\Omega^e = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \frac{\partial x}{\partial \xi}d\xi & \frac{\partial y}{\partial \xi}d\xi \\ \frac{\partial x}{\partial \eta}d\eta & \frac{\partial y}{\partial \eta}d\eta \end{vmatrix}$$

$$= \frac{\partial x}{\partial \xi}d\xi\frac{\partial y}{\partial \eta}d\eta - \frac{\partial x}{\partial \eta}d\eta\frac{\partial y}{\partial \xi}d\xi$$

$$= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta = |\mathbf{J}|d\xi d\eta = \det\mathbf{J}d\xi d\eta$$

One Dimensional Gauss Integration Rules

$$\int_{-1}^{+1} F(\xi)d\xi \approx \sum_{i=1}^{p} w_i F(\xi_i)$$

$p(\geq 1)$: number of Gauss integration (sample) points

w_i: integration weights

ξ_i: sample-point abcissae in the interval $[-1,+1]$

integrate exactly polynomials of order up to $(2p-1)$

1 point: $\int_{-1}^{+1} F(\xi)d\xi \approx 2F(0)$

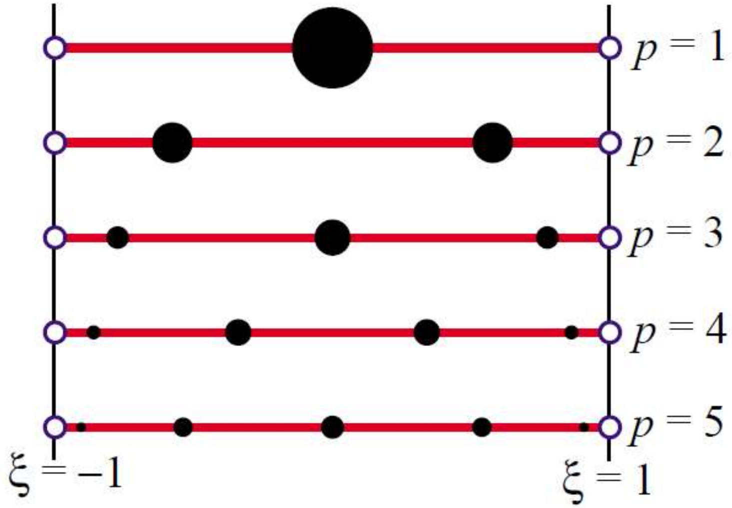
2 points: $\int_{-1}^{+1} F(\xi)d\xi \approx F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$

3 points: $\int_{-1}^{+1} F(\xi)d\xi \approx \frac{5}{9}F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}F(0) + \frac{5}{9}F\left(\sqrt{\frac{3}{5}}\right)$

4 points: $\int_{-1}^{+1} F(\xi)d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$

$$\left[\xi_{34} = -\xi_{24} = \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}},\ \xi_{44} = -\xi_{14} = \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}},\ w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{\frac{5}{6}},\ w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{\frac{5}{6}}\right]$$

Graphical Representation of One-Dimensional Gauss Integration Rules



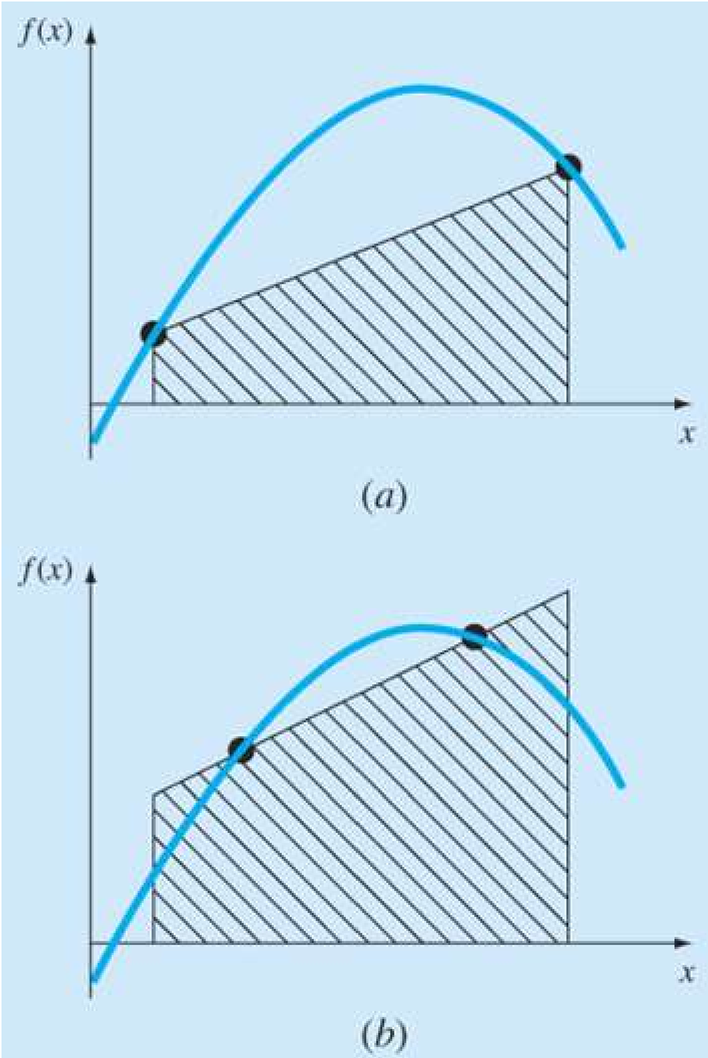
$$\int_a^b F(x)\,dx = \int_{-1}^{+1} F(\xi)\,J\,d\xi = \int_{-1}^{+1} F(\xi)\frac{1}{2}l\,d\xi$$

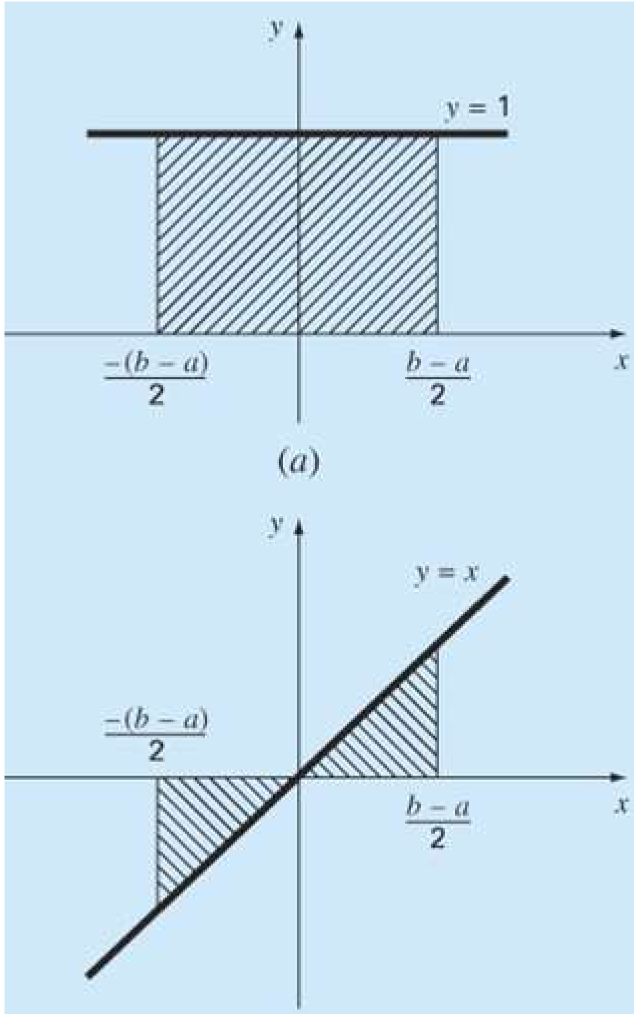
$$l = b - a > 0 \rightarrow [-1,+1]?$$

$$x = \frac{1}{2}a(1-\xi) + \frac{1}{2}b(1+\xi) = \frac{1}{2}(a+b) + \frac{1}{2}l\xi \quad \text{or} \quad \xi = \frac{2}{l}\left[x - \frac{1}{2}(a+b)\right]$$

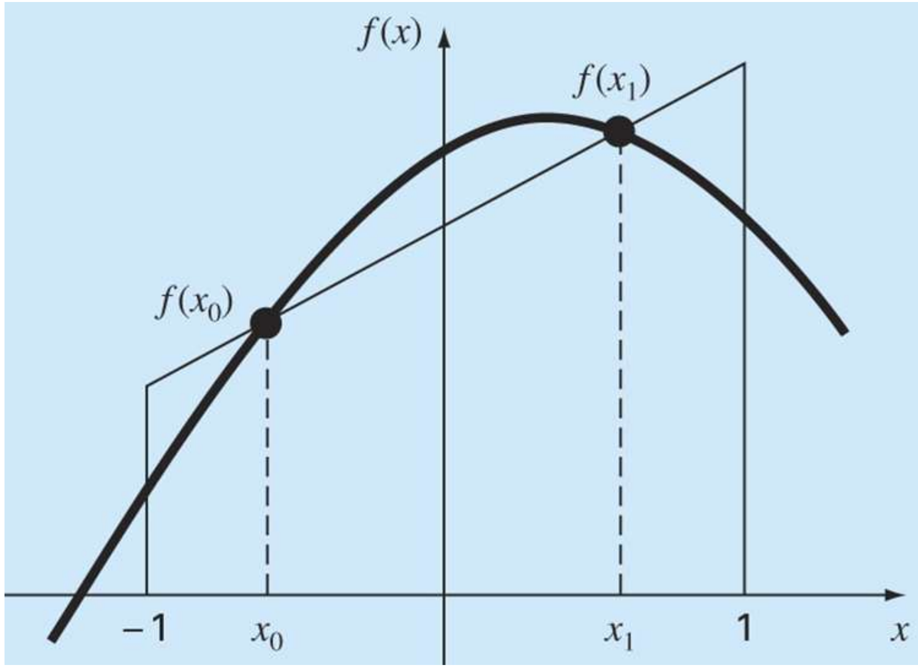
$$J = \frac{\partial x}{\partial \xi} = \frac{1}{2}l$$

Trapezoidal Rule vs. Improved Integral





Two-Point Gauss-Legendre Formula



$$\int_{-1}^{1} F(\xi)\, d\xi = w_1 F(\xi_1) + w_2 F(\xi_2)$$

unknown: w_1, w_2, ξ_1, ξ_2

$$\begin{cases} F(\xi) = 1 \Rightarrow w_1 + w_2 = 2 \\ F(\xi) = \xi \Rightarrow w_1\xi_1 + w_2\xi_2 = 0 \\ F(\xi) = \xi^2 \Rightarrow w_1\xi_1^{\,2} + w_2\xi_2^{\,2} = \dfrac{2}{3} \\ F(\xi) = \xi^3 \Rightarrow w_1\xi_1^{\,3} + w_2\xi_2^{\,3} = 0 \end{cases}$$

$$\rightarrow \begin{cases} \xi_1 = -\dfrac{1}{\sqrt{3}},\ \xi_2 = \dfrac{1}{\sqrt{3}} \\ w_1 = w_2 = 1 \end{cases}$$

$$\int_{-1}^{1} F(\xi)\, d\xi = 1 \times F\left(-\frac{1}{\sqrt{3}}\right) + 1 \times F\left(\frac{1}{\sqrt{3}}\right)$$

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$	$x_0 = -0.577350269$	$\cong f^{(4)}(\xi)$
	$c_1 = 1.0000000$	$x_1 = 0.577350269$	
3	$c_0 = 0.5555556$	$x_0 = -0.774596669$	$\cong f^{(6)}(\xi)$
	$c_1 = 0.8888889$	$x_1 = 0.0$	
	$c_2 = 0.5555556$	$x_2 = 0.774596669$	
4	$c_0 = 0.3478548$	$x_0 = -0.861136312$	$\cong f^{(8)}(\xi)$
	$c_1 = 0.6521452$	$x_1 = -0.339981044$	
	$c_2 = 0.6521452$	$x_2 = 0.339981044$	
	$c_3 = 0.3478548$	$x_3 = 0.861136312$	
5	$c_0 = 0.2369269$	$x_0 = -0.906179846$	$\cong f^{(10)}(\xi)$
	$c_1 = 0.4786287$	$x_1 = -0.538469310$	
	$c_2 = 0.5688889$	$x_2 = 0.0$	
	$c_3 = 0.4786287$	$x_3 = 0.538469310$	
	$c_4 = 0.2369269$	$x_4 = 0.906179846$	
6	$c_0 = 0.1713245$	$x_0 = -0.932469514$	$\cong f^{(12)}(\xi)$
	$c_1 = 0.3607616$	$x_1 = -0.661209386$	
	$c_2 = 0.4679139$	$x_2 = -0.238619186$	
	$c_3 = 0.4679139$	$x_3 = 0.238619186$	
	$c_4 = 0.3607616$	$x_4 = 0.661209386$	
	$c_5 = 0.1713245$	$x_5 = 0.932469514$	

Two Dimensional Product Gauss Rules

Canonical form of integral:

$$\int_{-1}^{+1}\int_{-1}^{+1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} F(\xi,\eta)\,d\xi$$

Gauss integration rules with p_1 points in the ξ direction and p_2 points in the η direction:

$$\int_{-1}^{+1}\int_{-1}^{+1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{+1} d\eta \int_{-1}^{+1} F(\xi,\eta)\,d\xi \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F\left(\xi_i,\eta_j\right)$$

usually $p_1 = p_2$