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Shape Function Requirements

- (A) Interpolation
  - Takes a unit value at node i , and is zero at all other nodes
- (B) Local Support
  - Vanishes over any element boundary (a side in 2D, a face in 3D) that does not include node *i*
- (C) Continuity (Intra- & Inter-Element)
  - Satisfies C^0 continuity between adjacent elements over any element boundary that includes node *i*
- (D) Completeness
  - The interpolation is able to represent exactly any displacement field which is a linear polynomial in x and y; in particular, a constant value

Direct Construction of Shape Functions: Are Conditions Automatically Satisfied?

- (A) Interpolation
  - Yes: by construction except scale factor
- (B) Local Support
  - Yes: by construction except scale factor
- (C) Continuity (Intra- & Inter-Element)
  - No: a posteriori check necessary
- (D) Completeness
  - Satisfied if (B,C) are met and the sum of shape functions is identically one

products of fairly simple polynomial expressions in the natural coordinates:

$$N_i^e = c_i L_1 L_2 \cdots L_m$$

$L_k = 0$: equations of "lines" expressed in natural coordinates that cross all nodes except i

Rules

– R1: Select the L_j as the minimal number of lines or curves linear in the natural coordinates that cross all nodes except the *i* th node. Primary choices in 2D are the element sides and medians.

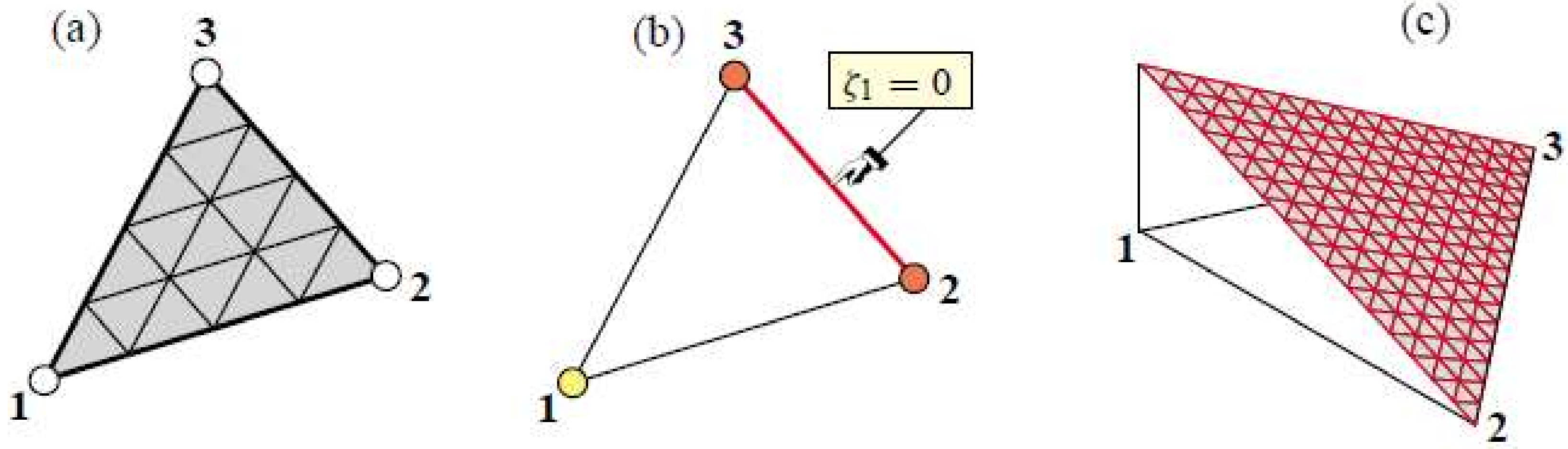
– R2: Set coefficient c_i so that N_i has the value 1 at the *i* th node.

– R3: Check that N_i vanishes over all element sides that do not contain node *i* .

– R4: Check the polynomial order over each side that contains node *i* . If the order is *n*, there must be exactly *n* + 1 nodes on the side for compatibility to hold.

– R5: If local support (R3) and interelement compatibility (R4) are satisfied, check that the sum of shape functions is identically one.

Three Node Linear Triangle



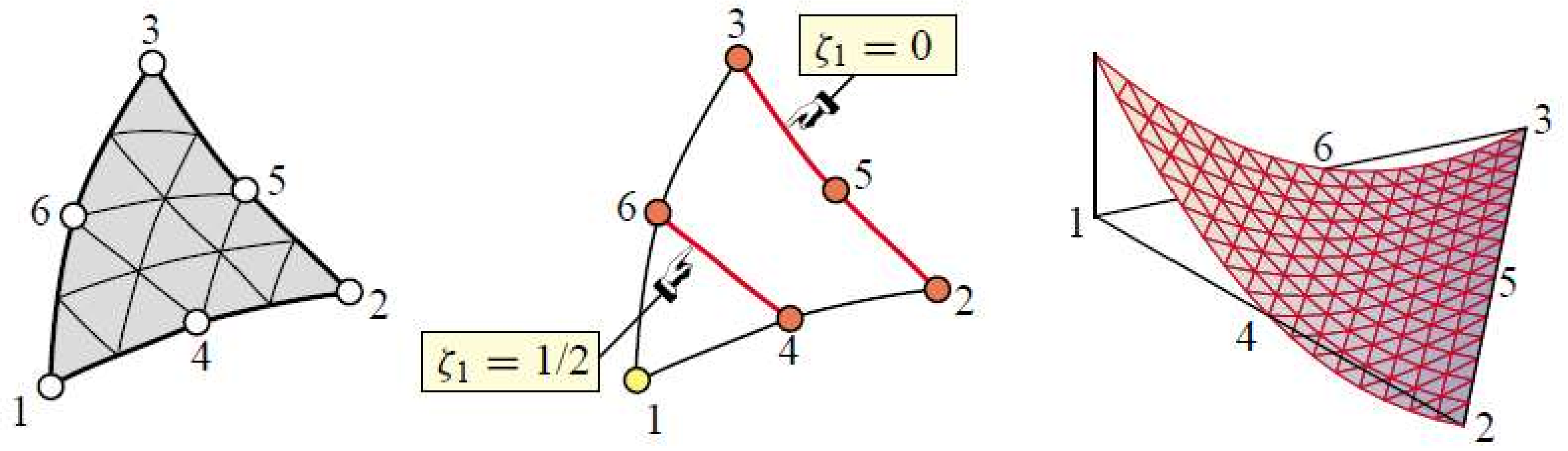
(A) $N_i^e = c_i L_i = c_i L_{j-k} = c_i \zeta_i \ \left(\zeta_i = 0 \text{ over side } j-k \right)$

(B) $c_i = 1 \ \left(N_i^e = c_i \zeta_i = 1 \text{ at node } i \right)$

(C) variation of ζ_i along the 2 sides meeting at node i is linear and that there are two nodes on each side

(D) $N_1^e + N_2^e + N_3^e = \zeta_1 + \zeta_2 + \zeta_3 = 1$

Six Node Quadratic Triangle: Corner Node



(A) $N_1^e = c_1 L_{2-3} L_{4-6} = c_1 \zeta_1 \left(\zeta_1 - \frac{1}{2} \right)$

(B) $N_1^e \left(\zeta_1, \zeta_2, \zeta_3 \right) = N_1^e \left(1, 0, 0 \right) = c_1 \left(1 \right) \left(1 - \frac{1}{2} \right) = 1 \rightarrow c_1 = 2 \rightarrow N_1^e = 2\zeta_1 \left(\zeta_1 - \frac{1}{2} \right) = \zeta_1 \left(2\zeta_1 - 1 \right)$

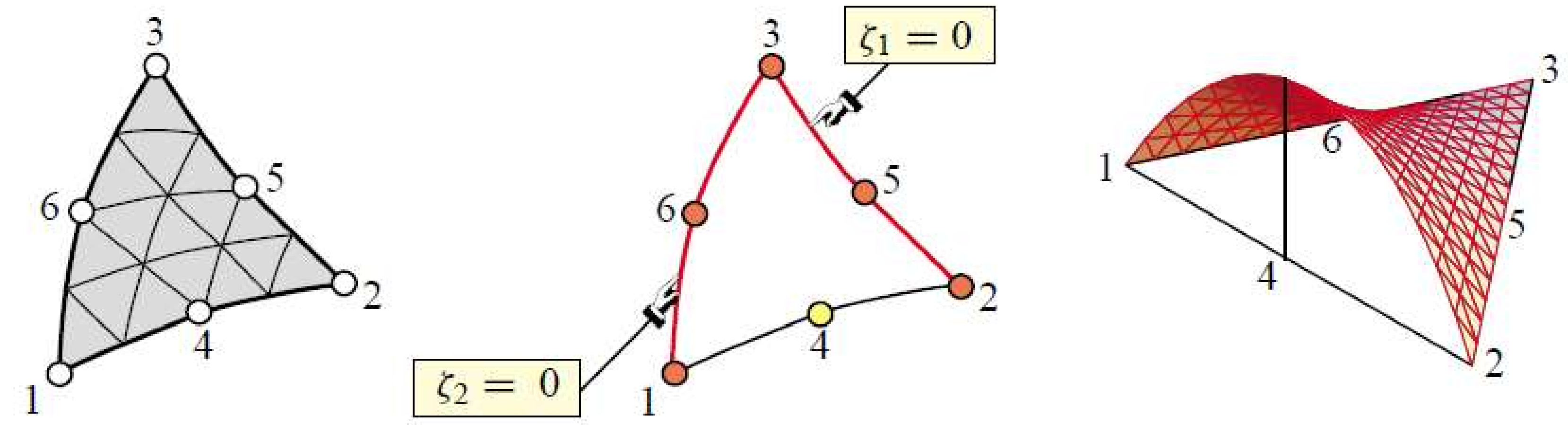
(C) Over each one the variation of N_1^e is quadratic in ζ_1.

Therefore the polynomial order over each side is 2.

Because there are three nodes on each boundary, the compatibility condition is verified.

(D) $\sum_{i=1}^{6} N_i^e = 1$

Six Node Quadratic Triangle: Midside Node



(A) $N_4^e = c_4 L_{2-3} L_{1-3} = c_4 \zeta_1 \zeta_2$

(B) $N_1^e\left(\zeta_1, \zeta_2, \zeta_3\right) = N_1^e\left(\frac{1}{2}, \frac{1}{2}, 0\right) = c_4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1 \rightarrow c_4 = 4 \rightarrow N_1^e = 4\zeta_1\zeta_2$

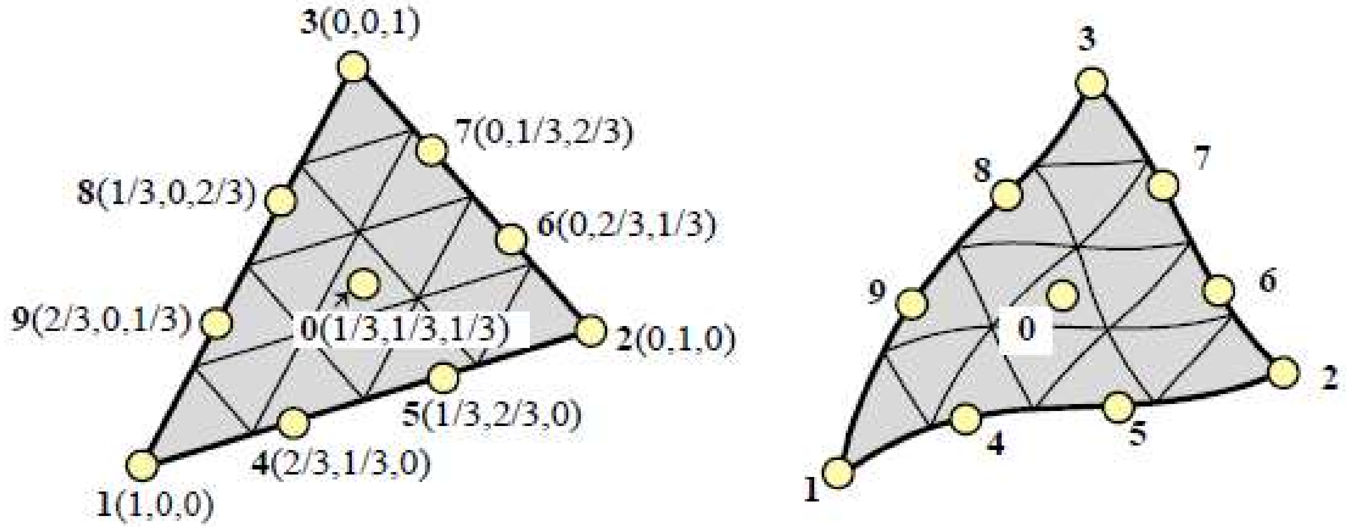
(C) Over each one the variation of N_4^e is quadratic in ζ_1 and ζ_2.

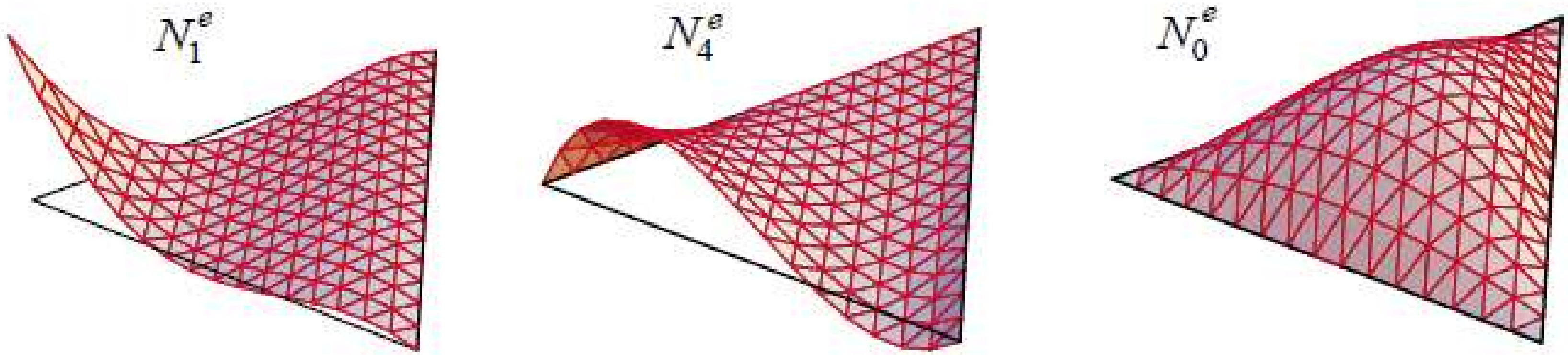
Therefore the polynomial order over each side is 2.

Because there are three nodes on each boundary, the compatibility condition is verified.

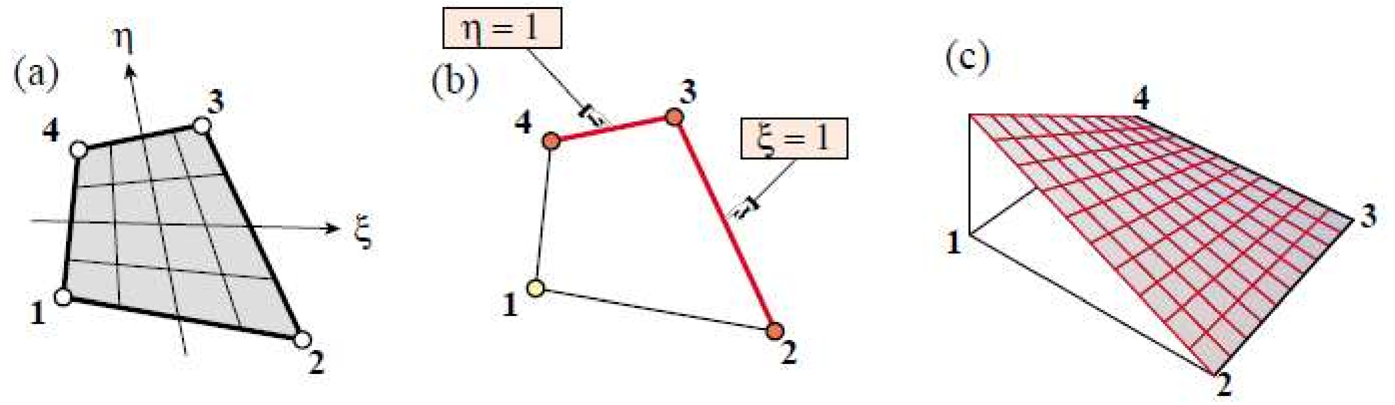
(D) $\sum_{i=1}^{6} N_i^e = 1$

Cubic Triangle (Exercise 18.1)





Four-Node Bilinear Quadrilateral



(A) $N_1^e(\xi,\eta) = c_1 L_{2-3} L_{3-4} = c_1(\xi-1)(\eta-1) = c_1(1-\xi)(1-\eta)$

(B) $N_1^e(-1,-1) = c_1(2)(2) = 1 \rightarrow c_1 = \dfrac{1}{4} \rightarrow N_1^e = \dfrac{1}{4}(1-\xi)(1-\eta)$

(C) Over side 1-2$(\eta=-1)$ N_1^e is a linear function of ξ.

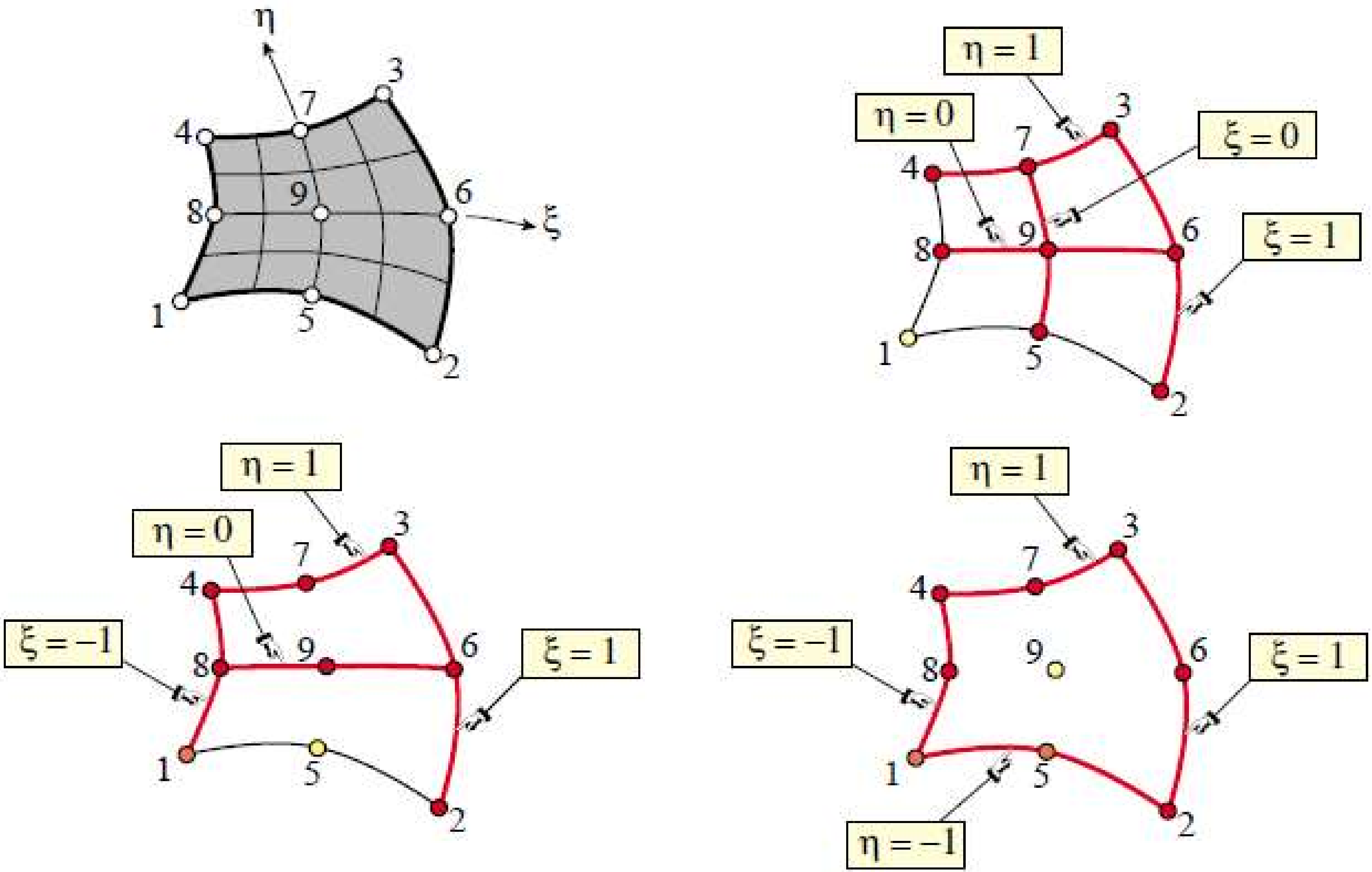
Over side 1-4$(\xi=-1)$ N_1^e is a linear function of η.

Therefore the polynomial order over both sides is 1.

Because there are two nodes on each boundary, the compatibility condition is verified.

(D) $N_i^e = \dfrac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \rightarrow \sum_{i=1}^{4} N_i^e = 1$

Nine-Node Biquadratic Quadrilateral (1)



Nine-Node Biquadratic Quadrilateral (2)

$$
\text{(A)}\ \begin{cases}
\text{corner: } N_1^e(\xi,\eta) = c_1 L_{2-3} L_{3-4} L_{5-7} L_{6-8} = c_1(\xi-1)(\eta-1)\xi\eta \\
\text{midside: } N_5^e(\xi,\eta) = c_5 L_{2-3} L_{1-4} L_{6-8} L_{3-4} = c_5(\xi-1)(\xi+1)\eta(\eta-1) = c_5\left(1-\xi^2\right)\eta(1-\eta) \\
\text{center: } N_9^e(\xi,\eta) = c_9 L_{1-2} L_{2-3} L_{3-4} L_{4-1} = c_9(\xi-1)(\eta-1)(\xi+1)(\eta+1) = c_9\left(1-\xi^2\right)\left(1-\eta^2\right)
\end{cases}
$$

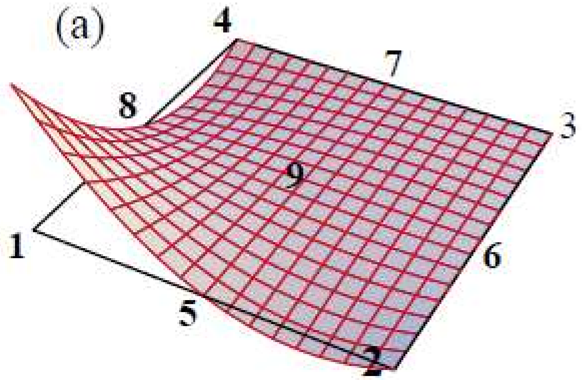
$$
\text{(B)}\ \begin{cases}
\text{corner: } N_1^e(-1,-1) = c_1(2)(2)(-1)(-1) \rightarrow c_1 = \dfrac{1}{4} \rightarrow N_1^e = \dfrac{1}{4}(\xi-1)(\eta-1)\xi\eta \\
\text{midside: } N_5^e(0,-1) = c_5(1)(-1)(2) \rightarrow c_5 = -\dfrac{1}{2} \rightarrow N_5^e = -\dfrac{1}{2}\left(1-\xi^2\right)\eta(1-\eta) \\
\text{center: } N_9^e(0,0) = c_9\left(1-\xi^2\right)\left(1-\eta^2\right) = c_9(1)(1) \rightarrow c_9 = 1 \rightarrow N_9^e = \left(1-\xi^2\right)\left(1-\eta^2\right)
\end{cases}
$$

(C) The polynomial order of N_i^e over any side that belongs to node i is two.

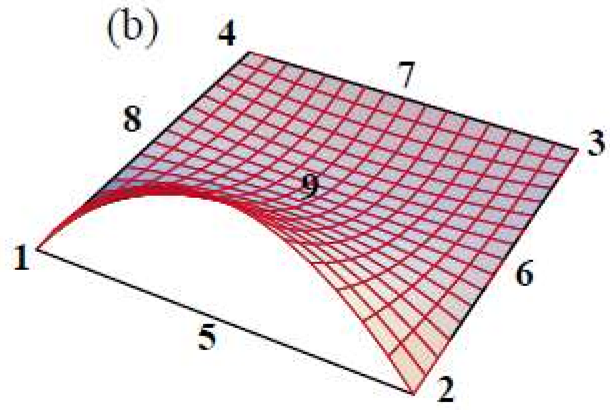
Because there are three nodes on each side, the compatibility condition is verified.

(D) $\sum_{i=1}^{9} N_i^e = 1 \leftarrow$ Exercise 18.3

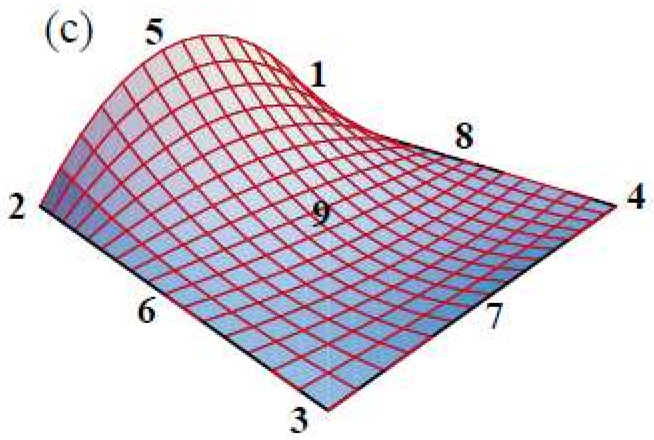
Nine-Node Biquadratic Quadrilateral (3)



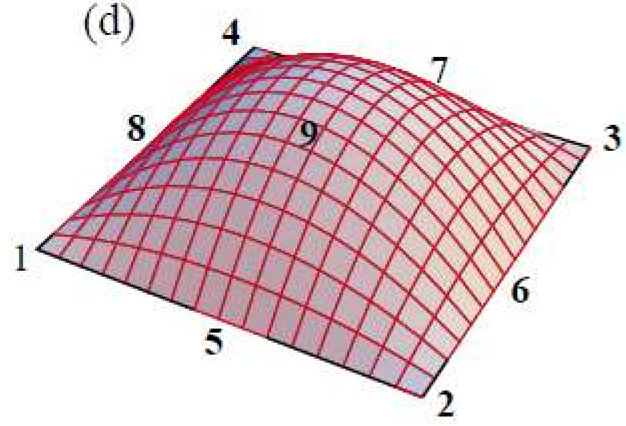
$$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta$$



$$N_5^e = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1)$$

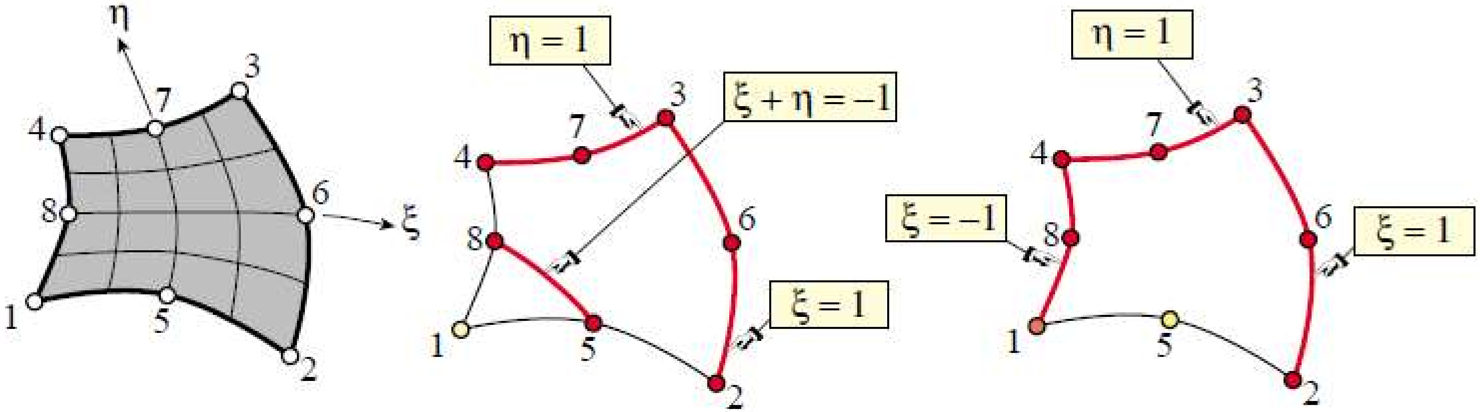


$N_5^e = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1)$ (back view)



$$N_9^e = (1 - \xi^2)(1 - \eta^2)$$

Eight-Node "Serendipity" Quadrilateral



(A) $\begin{cases} \text{corner: } N_1^e(\xi,\eta) = c_1 L_{2-3} L_{3-4} L_{5-8} = c_1(\xi-1)(\eta-1)(1+\xi+\eta) = c_1(1-\xi)(1-\eta)(1+\xi+\eta) \\ \text{midside: } N_5^e(\xi,\eta) = c_5 L_{2-3} L_{3-4} L_{4-1} = c_5(\xi-1)(\xi+1)(\eta-1) = c_5(1-\xi^2)(1-\eta) \end{cases}$

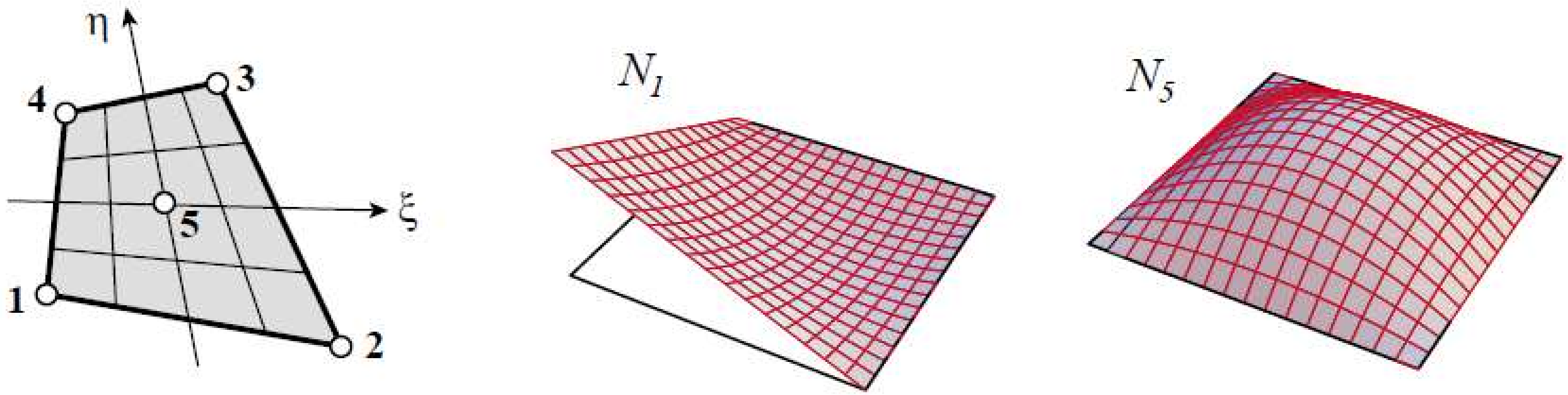
(B) $\begin{cases} \text{corner: } N_1^e(-1,-1) = c_1(2)(2)(-1) \to c_1 = -\dfrac{1}{4} \to N_1^e = -\dfrac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \\ \text{midside: } N_5^e(0,-1) = c_5(1)(2) \to c_5 = \dfrac{1}{2} \to N_5^e = \dfrac{1}{2}(1-\xi^2)(1-\eta) \end{cases}$

(C) The polynomial order of N_i^e over any side that belongs to node i is two.

Because there are three nodes on each side, the compatibility condition is verified.

(D) $\sum_{i=1}^{8} N_i^e = 1 \leftarrow$ Exercise 18.4

Can the Magic Wand Fail? Yes



(A) $N_1^e(\xi,\eta) = c_1 L_{2-3} L_{3-4} L_{2-5-4} = c_1(\xi-1)(\eta-1)(\xi+\eta) = c_1(1-\xi)(1-\eta)(\xi+\eta)$

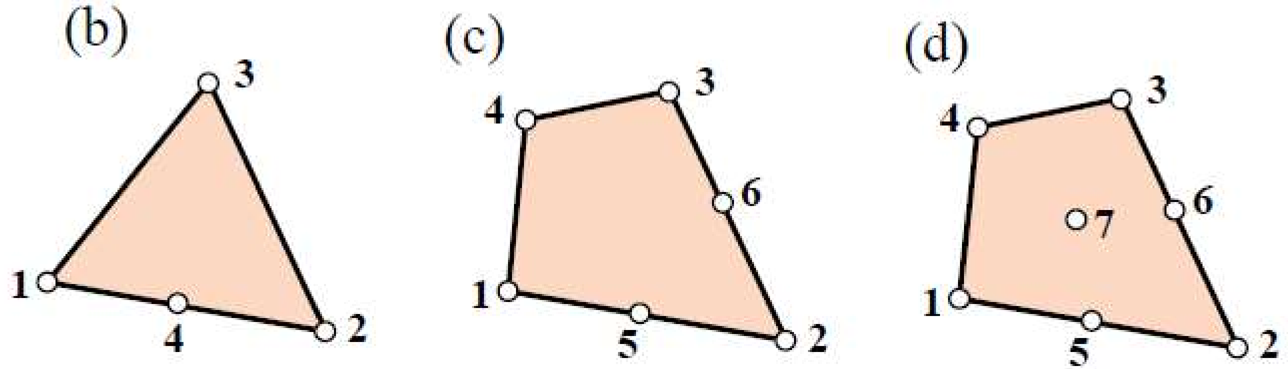
(B) $N_1^e(-1,-1) = c_1(2)(2)(-8) = 1 \rightarrow c_1 = -\dfrac{1}{8} \rightarrow N_1^e = -\dfrac{1}{8}(1-\xi)(1-\eta)(\xi+\eta)$

It violates (C) along sides 1-2 and 4-1, because it varies quadratically over them with only two nodes per side

$\Rightarrow$ hierarchical correction approach, which employs a combination of terms

$$N_i^e = \underbrace{c_i L_1^c L_2^c \cdots L_m^c}_{\text{lower order element parent element}} + \underbrace{d_i L_1^d L_2^d \cdots L_n^d}_{\text{corrective shape function that vanishes at the nodes of the parentelement}} \quad (\text{superposition?})$$

Transition Element Example



corner nodes but midnodes only over certain sides

(A) $N_1^e = c_1 L_{2-3} L_{3-4} = c_1 \zeta_1 (\zeta_1 - \zeta_2)$

(B) $N_1^e (1,0,0) = c_1 (1)(1) = 1 \rightarrow c_1 = 1 \rightarrow N_1^e = \zeta_1 (\zeta_1 - \zeta_2)$

(C) Over side 1-3$(\zeta_2 = 0) \rightarrow N_1^e = \zeta_1^{\ 2}$, This varies quadratically but there are only 2 nodes on that side. $\rightarrow$ Fail $\rightarrow$ [Hierarchical Corrections]

start from the shape function for the 3-node linear triangle: $N_1^e = \zeta_1$

apply a correction that vanishes at all nodes but 4 : $N_1^e = \zeta_1 + c_1 \zeta_1 \zeta_2$ (superposition)

$N_1^e \left(\frac{1}{2}, \frac{1}{2}, 0 \right) = \frac{1}{2} + c_1 \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = 0 \rightarrow c_1 = -2 \rightarrow N_1^e = \zeta_1 - 2\zeta_1 \zeta_2$ (D) $\sum_{i=1}^{4} N_i^e = 1 \leftarrow$ Exercise 18.8

Convergence Requirements for Finite Element Discretization

- Convergence
  – discrete (FEM) solution approaches the analytical (math model) solution in some sense
  – **Consistency (Completeness, Compatibility) + Stability**

- Consistency (discrete vs. mathematical model)
  – Completeness: individual elements
  – Compatibility (continuity?): element patches
- Stability
  – Rank Sufficiency: individual elements
  – Positive Jacobian: individual elements

Variational Index (m)

- Highest derivative of unknown variable (displacement)

- Bar: m = 1

$$\Pi\left[u\right]=\int_0^L\left(\frac{1}{2}u'EAu'-qu\right)dx$$

- Beam: m = 2

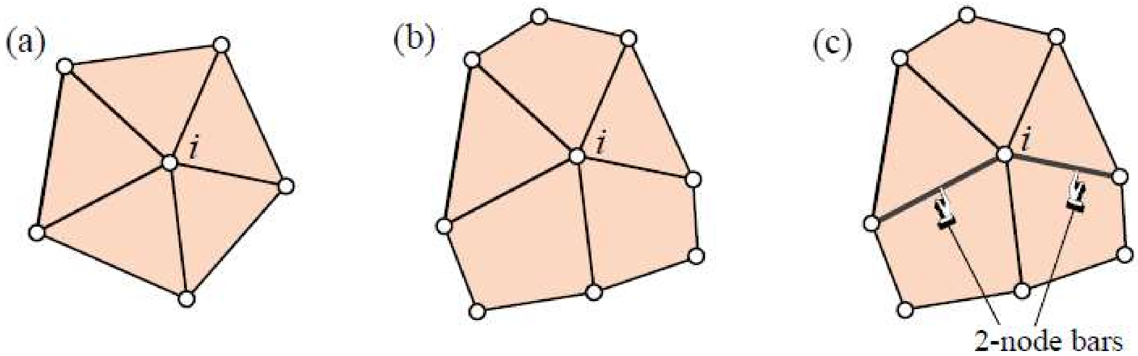
$$\Pi\left[u\right]=\int_0^L\left(\frac{1}{2}v''EIv''-qv\right)dx$$

Completeness & Compatibility in Terms of m

- Completeness
  - The element shape functions must represent exactly all polynomial terms of order ≤ m in the Cartesian coordinates. A set of shape functions that satisfies this condition is call m-complete
  - Applies *at the element level* and involves *all* shape functions of the element
  - Satisfied if the sum of the shape functions is unity and the element is compatible
- Compatibility (complete finite element mesh)
  - The patch trial functions must be $C^{(m-1)}$ continuous between elements [“conforming”], and C^m piecewise differentiable inside each element [“finite energy”]
  - Applies at two levels: individual element, and element patch
  - Element with conforming shape function: conforming
  - Conforming element that satisfies finite energy requirement: compatible

Element Patches

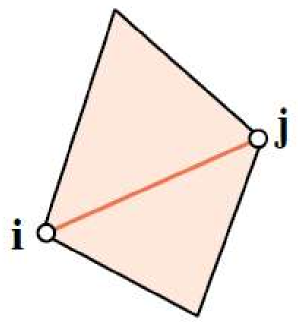
- A patch is the set of all elements attached to a given node:



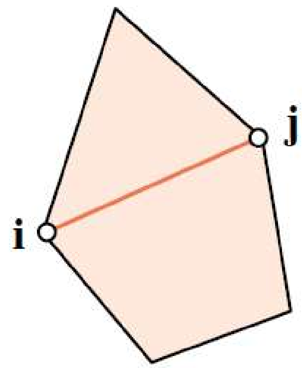
- A finite element patch trial function
  – union of shape functions activated by setting a degree of freedom at that node to unity, while all other freedoms are zero
  – "propagates" only over the patch, and is zero beyond it
- Ex. plane stress: m = 1 in two dimensions
  – any linear polynomial in x, y with a constant as special case
  – patch trial functions must be C^0 continuous between elements, and C^1 inside elements

Interelement Continuity is the Toughest to Meet

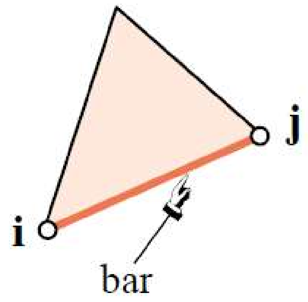
- Simplification: for *matching meshes* it is enough to check compatibility between a *pair of adjacent elements*
  - one in which adjacent elements share sides, nodes and degrees of freedom
  - restrict consideration first to a pair of adjacent elements, and then to the side shared by these elements



Two 3-node linear triangles



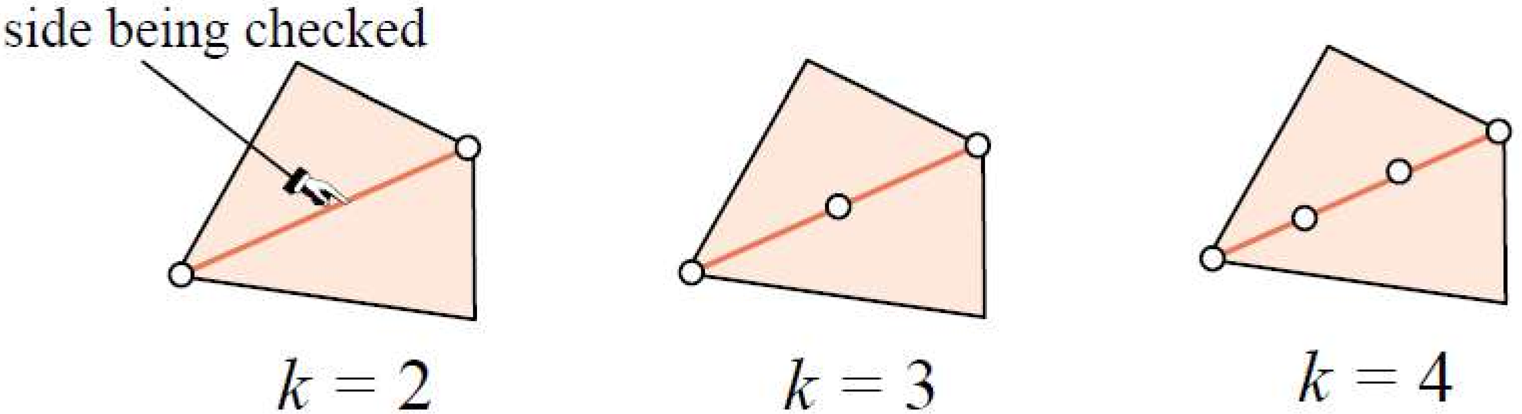
One 3-node linear triangle and one 4-node bilinear quad



One 3-node linear triangle and one 2-node bar

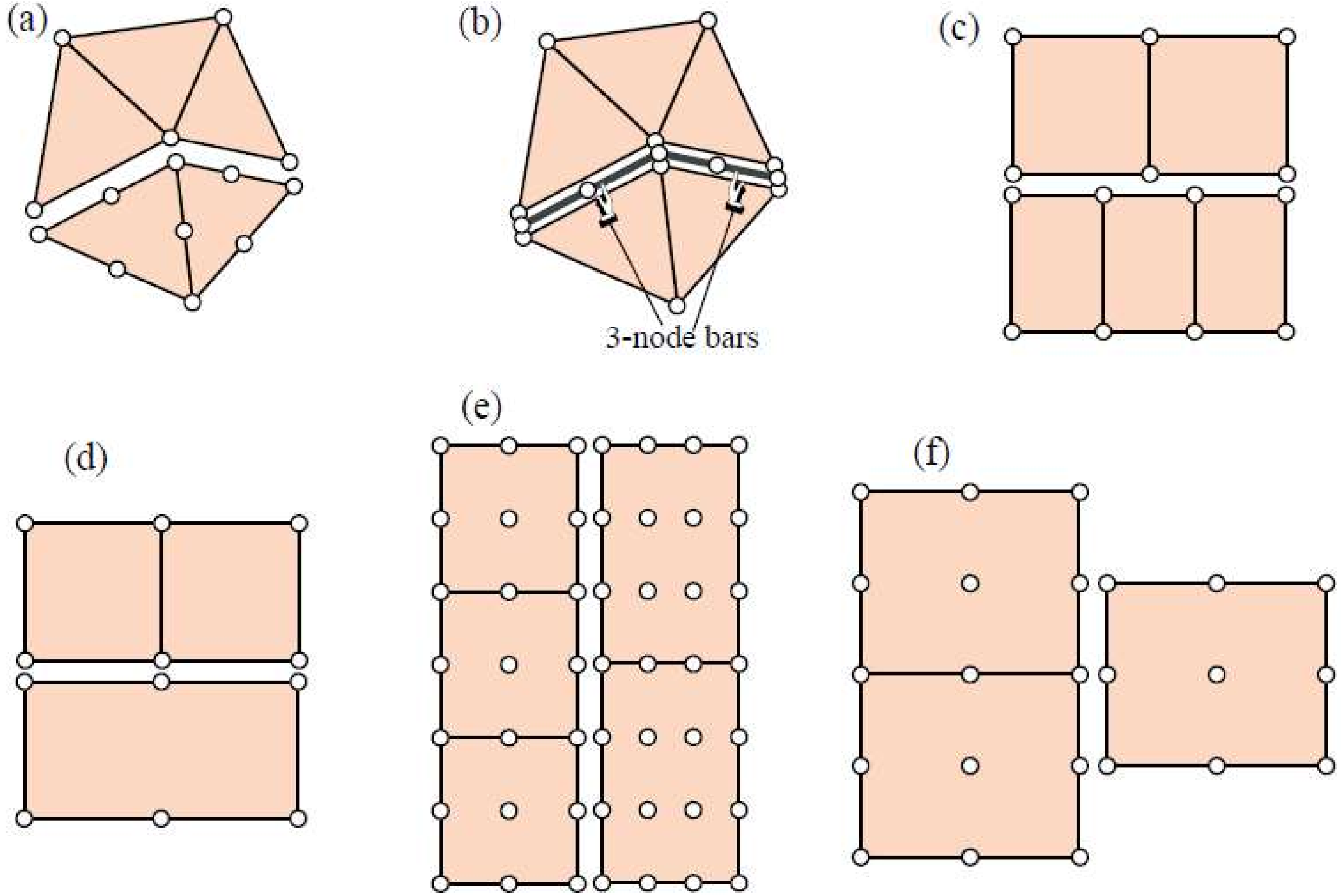
Side Continuity Check for Plane Stress Elements

- Polynomial Shape Functions in Natural Coordinates
- Let k be the number of nodes on a side:



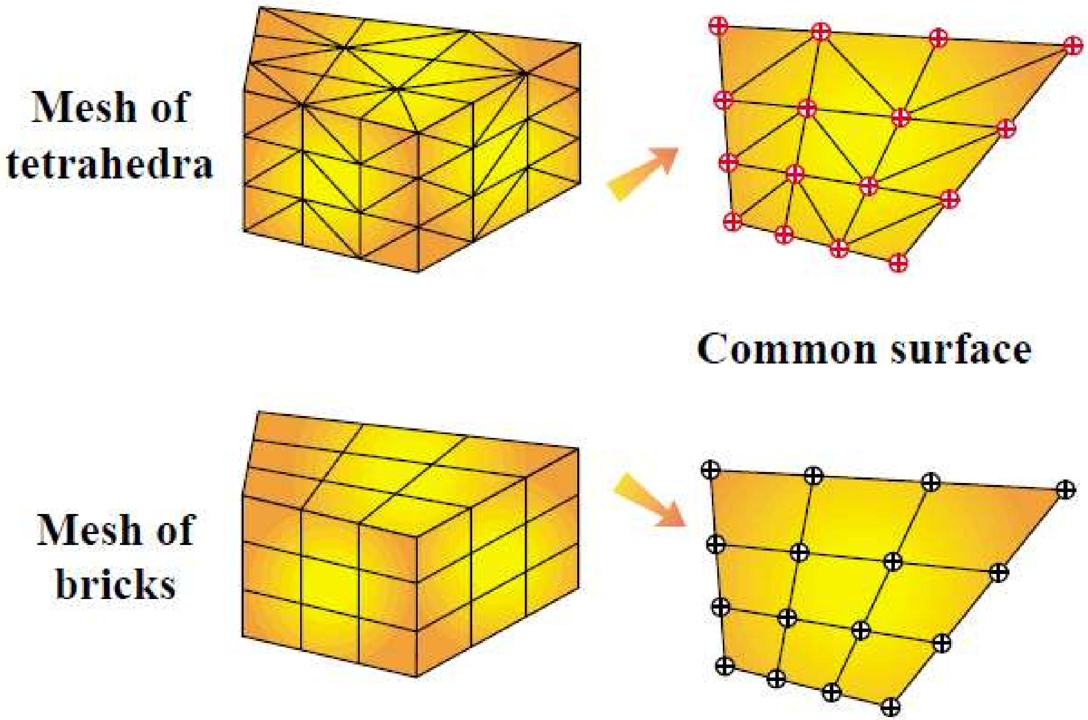
- The variation of each element shape function along the side must be of polynomial order $k - 1$
  - If ***more, continuity is violated***
  - If ***less, nodal configuration is wrong*** (too many nodes)

2D Nonmatching Mesh Examples



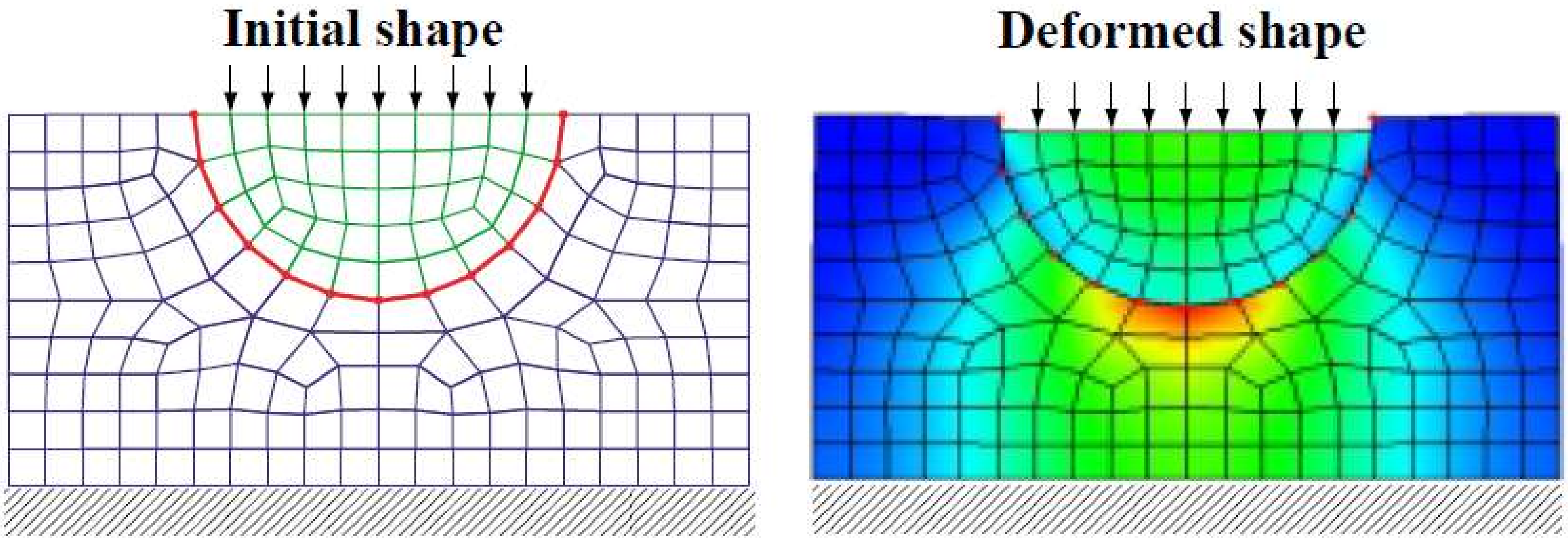
nonmatching nodes, matching nodes but different element types

3D Nonmatching Mesh Examples



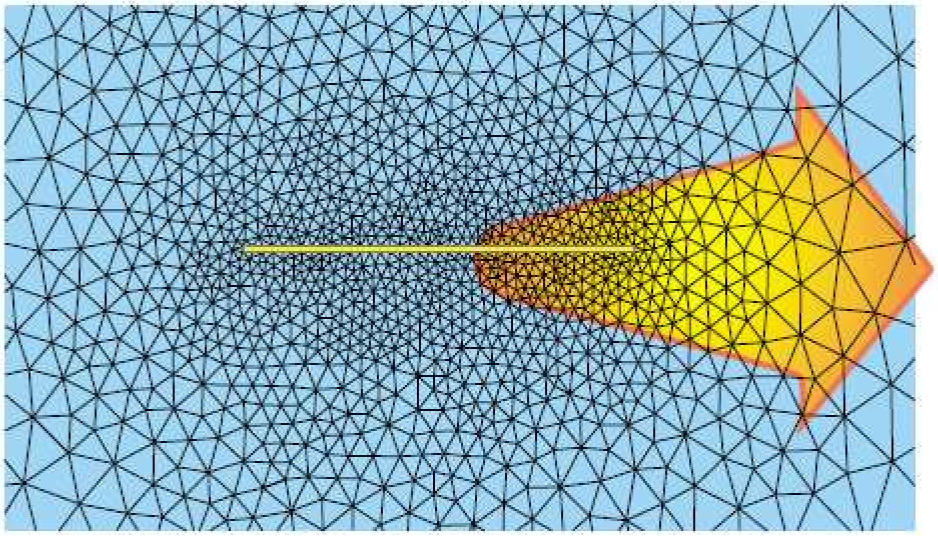
Nodes and boundary-quad edges and DOFs match, but element types are different, leading to violation of C^0 continuity

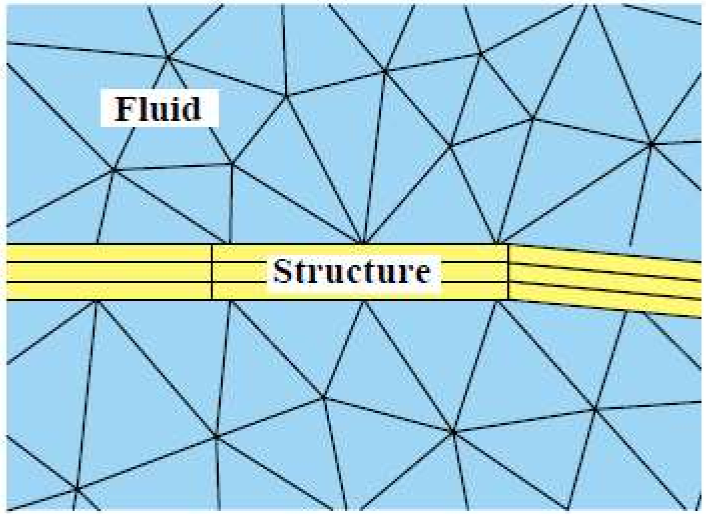
Nonmatching Meshes in Contact-With-Slip (a Geometrically Nonlinear Problem)



In contact and impact problems, matching meshes are the exception rather than the rule. Even if the meshes match at initial contact, slipping may produce a nonmatching mesh in the deformed configuration.

Nonmatching Meshes in FSI Problem





Two-dimensional model to simulate flow around a thin plate.
If the meshes are independently prepared, node locations will not generally match.

Stability

- Rank Sufficiency
  - Discrete model must possess the same solution
  - Uniqueness attributes of the mathematical model
  - For displacement finite elements:
    - rigid body modes (RBMs) must be preserved
    - no zero-energy modes other than RBMs
  - Can be tested by looking at the rank of the stiffness matrix
- Jacobian Positiveness: positive Jacobian determinant
  - The determinant of the Jacobian matrix that relates Cartesian and natural coordinates must be everywhere positive within the element

Rank Sufficiency

- The element stiffness matrix must not possess any zero-energy kinematic modes other than rigid body modes
- This can be checked by verifying that the element stiffness matrix has the correct rank:
  - **correct rank = # of element DOF − # of RBMs**
- A stiffness matrix that has correct rank (a.k.a. proper rank) is called rank sufficient

Rank Analysis of Element Stiffness

n_F : number of element DOF

n_R : number of independent rigid body modes

n_G : number of Gauss points in integration rule for $\mathbf{K}$

n_E : order of $\mathbf{E}$ (stress-strain) matrix

r_C : correct (proper) rank $\left(n_F - n_R\right)$

r : actual rank of stiffness matrix

d : rank deficiency $\left(r_C - r\right)$

Rank Sufficiency for Numerically Integrated Finite Elements

$$\begin{cases} \text{rank of } \mathbf{K}\text{: } r = \min\left(n_F - n_R, n_E n_G\right) \\ \text{rank deficiency: } d = \left(n_F - n_R\right) - r \end{cases}$$

$$\rightarrow n_E n_G \geq n_F - n_R$$

plane stress (n nodes): $n_F = 2n,\ n_R = 3,\ n_E = 3$

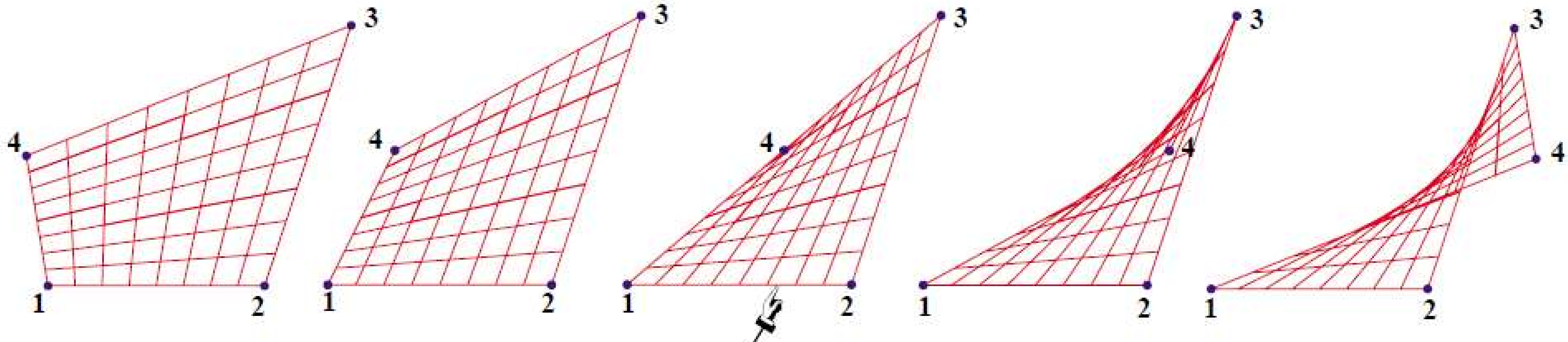
$$\rightarrow r = \min\left(2n - 3, 3n_G\right)$$

Rank Sufficiency for Some Plane Stress iso-P Elements

Element	n	n_F	$n_F - 3$	Min n_G	Recommended rule
3-node triangle	3	6	3	1	centroid*
6-node triangle	6	12	9	3	3-midpoint rule*
10-node triangle	10	20	17	6	7-point rule*
4-node quadrilateral	4	8	5	2	2 x 2
8-node quadrilateral	8	16	13	5	3 x 3
9-node quadrilateral	9	18	15	5	3 x 3
16-node quadrilateral	16	32	29	10	4 x 4

Positive Jacobian Requirement (1)

- Three-node triangle
  - J = 2A (constant): J > 0 → corner nodes must be positioned and numbered so that A > 0 (convexity condition)
- 2D elements with more than 3 nodes
  - proper location of corner nodes



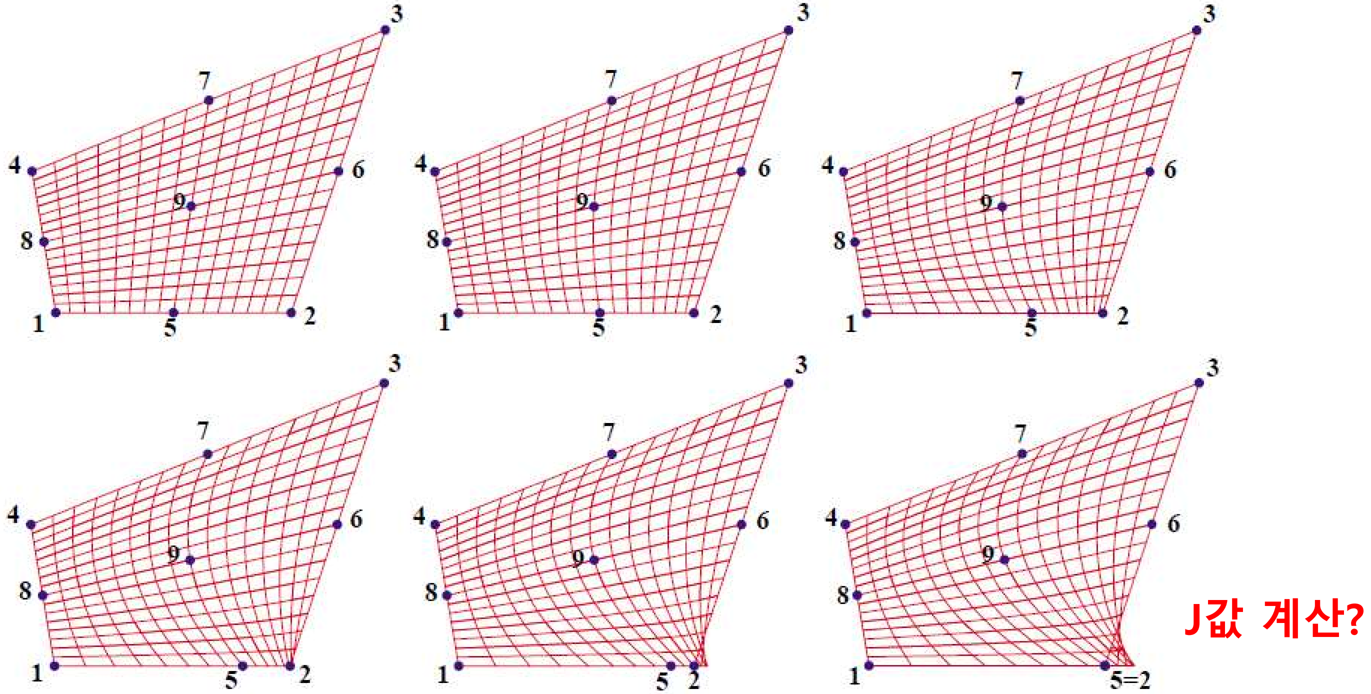
Displacing a Corner Node 4 of 4-Node Quad

"Triangle"

J값 계산?

Positive Jacobian Requirement (2)

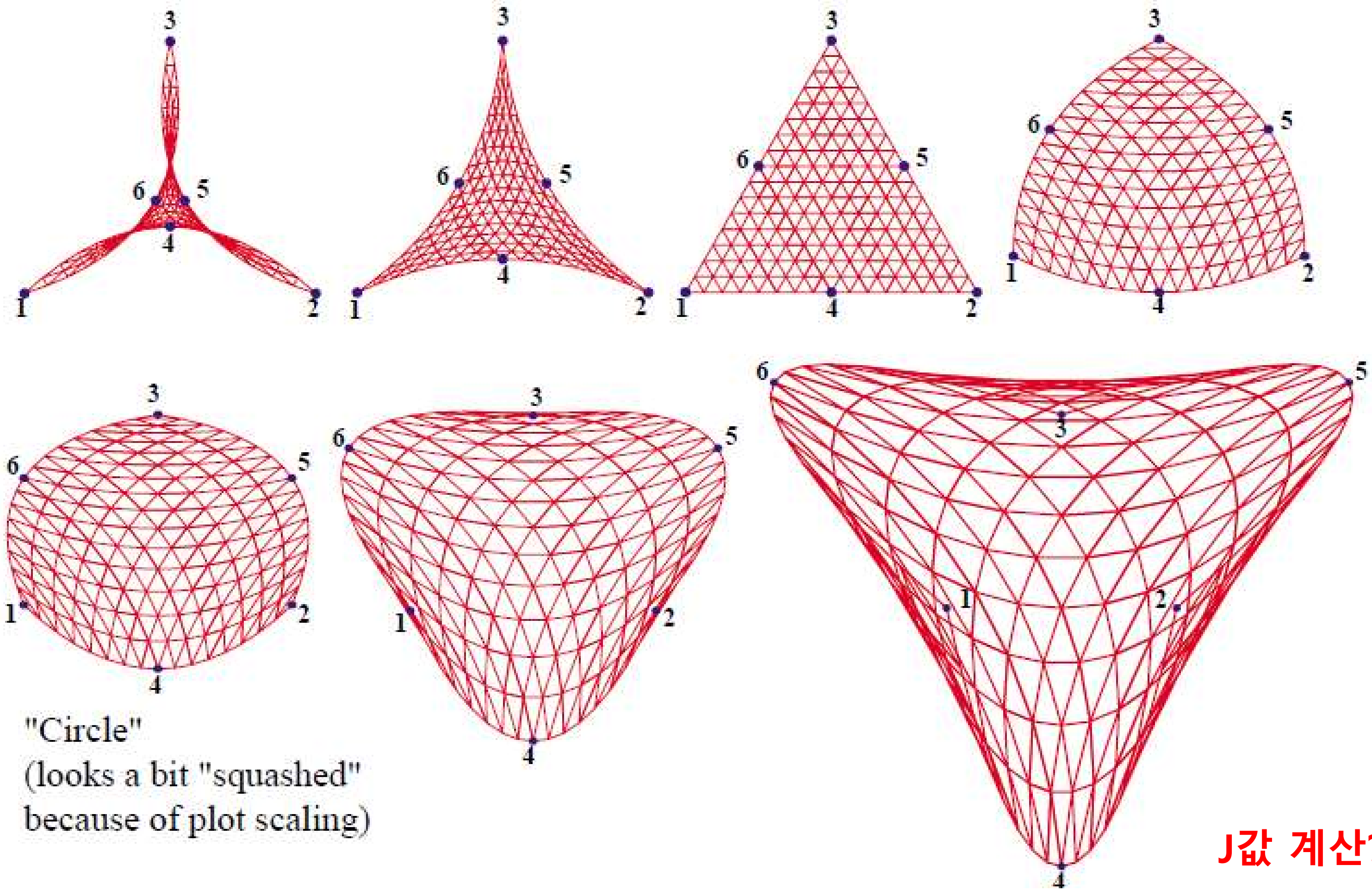
- For higher order elements
  - proper location of corner nodes is not enough
  - non-corner nodes (midside, interior, etc.) must be placed sufficiently close to their natural locations (midpoints, centroids, etc.) to avoid violent local distortions



Displacing a Midside Node 5 of 9-Node Quad

Positive Jacobian Requirement (3)

Displacing Midside Nodes of 6-Node Equilateral Triangle along midpoint normals



J값 계산? OK