EXERCISE 7.1 Trouble spots from recipe are: B, F, J, M (entrant corners), N, D, I (concentrated forces). See Figure E7.7 for a physical justification.

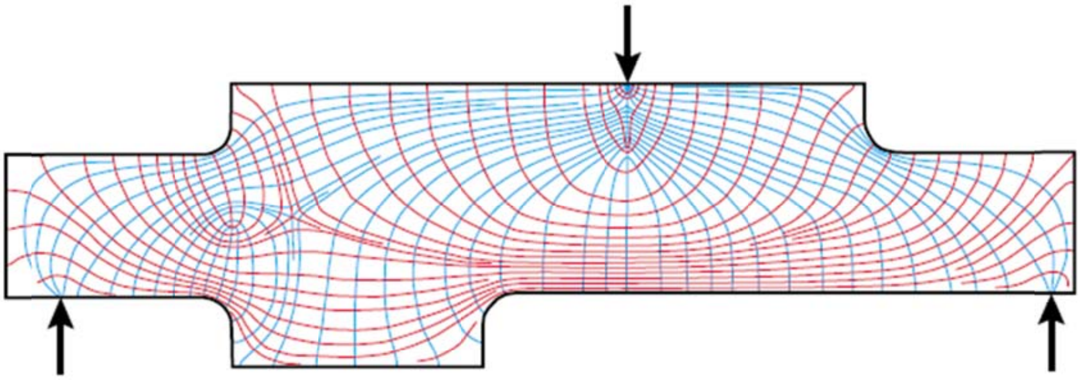

FIGURE E7.7. Figure shows principal stress trajectories or "isostatics." They are drawn in red for tension and blue for compression. (Determined from experimental data gathered in 1964 photoelasticity project at UC Berkeley.) Trajectories "bunch up" in regions of high stress gradients near loads and entrant corners.

EXERCISE 7.2 What perhaps are the two simplest solutions for the transition zone are pictured in Figure E7.8.

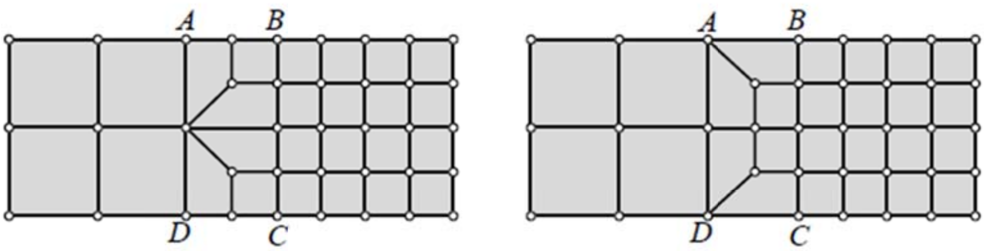

FIGURE E7.8. Two simple transition-mesh solutions for Exercise 7.2.

Several quadrilateral-based "transition meshes" are illustrated in Figure E7.9 for completeness; redrawn from Irons and Ahmad [148]. The solutions of Figure E7.8 are essentially variations of (e) and (f). Solution (d) is interesting in that it produces somewhat better looking element shapes.

EXERCISE 11.3

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g A \ell \int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} d\zeta = \rho g A \ell \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \tfrac{1}{2}\rho g A \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad \text{(E11.6)}$$

This is the same answer given by element-by-element load lumping.

EXERCISE 11.4

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g \ell \int_0^1 \begin{bmatrix} \big[A_i(1-\zeta)+A_j\zeta\big](1-\zeta) \\ \big[A_i(1-\zeta)+A_j\zeta\big]\zeta \end{bmatrix} d\zeta = \rho g \ell \begin{bmatrix} \frac{1}{3}A_i + \frac{1}{6}A_j \\ \frac{1}{6}A_i + \frac{1}{3}A_j \end{bmatrix}. \qquad \text{(E11.7)}$$

Obviously if $A_i = A_j = A$ we recover (E11.6).

EXERCISE 12.1 A *Mathematica* script for $\mathbf{K}^e$ by analytical integration is shown in Figure E12.1.

```
ClearAll[EI,EIi,EIj,Le,ξ,ℓ]; Le=ℓ;
Be={{6*ξ,(3*ξ-1)*Le,-6*ξ,(3*ξ+1)*Le}};
EI=EI1*(1-ξ)/2+EI2*(1+ξ)/2;
Ke=1/(2*Le^3)*Integrate[EI*Transpose[Be].Be,{ξ,-1,1}];
Ke=Simplify[Ke];
Print["Ke for variable xsec beam:\n", Ke//MatrixForm];
ClearAll[EI];  Ke=Simplify[Ke/.{EI2->EI,EI2->EI}];
Print["Ke for EI1=EI2=EI is ", Ke//MatrixForm];
```

Ke for variable xsec beam:

$$\begin{pmatrix} \frac{6\,(EI1+EI2)}{\ell^3} & \frac{2\,(2\,EI1+EI2)}{\ell^2} & -\frac{6\,(EI1+EI2)}{\ell^3} & \frac{2\,(EI1+2\,EI2)}{\ell^2} \\ \frac{2\,(2\,EI1+EI2)}{\ell^2} & \frac{3\,EI1+EI2}{\ell} & -\frac{2\,(2\,EI1+EI2)}{\ell^2} & \frac{EI1+EI2}{\ell} \\ -\frac{6\,(EI1+EI2)}{\ell^3} & -\frac{2\,(2\,EI1+EI2)}{\ell^2} & \frac{6\,(EI1+EI2)}{\ell^3} & -\frac{2\,(EI1+2\,EI2)}{\ell^2} \\ \frac{2\,(EI1+2\,EI2)}{\ell^2} & \frac{EI1+EI2}{\ell} & -\frac{2\,(EI1+2\,EI2)}{\ell^2} & \frac{EI1+3\,EI2}{\ell} \end{pmatrix}$$

FIGURE E12.1. Script to solve Exercise 12.1

Transcribing the result:

$$\mathbf{K}^e = \frac{1}{\ell^3}\begin{bmatrix} 6(EI_1+EI_2) & 2(2EI_1+EI_2)\ell & -6(EI_1+EI_2) & 2(EI_1+EI_2)\ell \\ & (3EI_1+EI_2)\ell^2 & -2(2EI_1+EI_2)\ell & (EI_1+EI_2)\ell^2 \\ & & 6(EI_1+EI_2) & -2(2EI_1+2EI_2)\ell \\ symm & & & (EI_1+2EI_2)\ell^2 \end{bmatrix}. \qquad \text{(E12.10)}$$

The output of the check $EI_1 = EI_2 = EI$ is omitted to save space, but it does reproduce the matrix (12.20).

EXERCISE 12.2 A *Mathematica* script for $\mathbf{f}^e$ by analytical integration is shown in Figure E12.2.

```
ClearAll[q,q1,q2,ξ,ℓ]; Le=ℓ;
Ne={2*(1-ξ)^2*(2+ξ), (1-ξ)^2*(1+ξ)*Le,
    2*(1+ξ)^2*(2-ξ),-(1+ξ)^2*(1-ξ)*Le}/8;
q=q1*(1-ξ)/2+q2*(1+ξ)/2;
fe=Simplify[ (Le/2)*Integrate[q*Ne,{ξ,-1,1}] ];
Print["fe^T for lin varying load q:\n",fe];
ClearAll[q]; fe=Simplify[fe/.{q1->q,q2->q}];
Print["check for q1=q2=q: ",fe];
```

fe^T for lin varying load q:

$\{\frac{1}{20}(7\,q1+3\,q2)\,\ell \quad \frac{1}{60}(3\,q1+2\,q2)\,\ell^2 \quad \frac{1}{20}(3\,q1+7\,q2)\,\ell \quad -\frac{1}{60}(2\,q1+3\,q2)\,\ell^2\}$

FIGURE E12.2. Script to solve Exercise 12.2

Transcribing the result:

$$\mathbf{f}^e = \frac{\ell}{60}\left[\,3(7q_1+3q_2) \quad \ell(3q_1+2q_2) \quad 3(3q_1+7q_2) \quad \ell(2q_1+3q_2)\,\right]^T . \tag{E12.11}$$

Table 12.1. Results From Exercise 12.6

	Monomial degree						
	0	1	2	3	4	5	6
Exact	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{2}{7}$
One-point rule (E12.6)	2	0	0				
Two-point rule (E12.7)	2	0	$\frac{2}{3}$	0	$\frac{2}{9}$		
Three-point rule (E12.8)	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{6}{25}$

The output of the check $q_i = q_j = q$ is omitted to save space, but it does reproduce (12.21).