

EXERCISE 16.5

- (a) From the definition of element geometry in the second row of (E16.3):

$$x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e = \ell \left[\frac{1}{2} \xi (\xi + 1) + \left(\frac{1}{2} + \alpha \right) (1 - \xi^2) \right], \quad (\text{E16.12})$$

we get

$$J = \frac{\partial x}{\partial \xi} = \left(\frac{1}{2} - 2\xi\alpha \right) \ell. \quad (\text{E16.13})$$

(In 1D, the Jacobian is a scalar and is the same as its determinant.) J is linear in ξ . Consequently its max/min values over the element occur at the end nodes:

$$J_1 = J|_{\xi=-1} = \left(\frac{1}{2} - 2\alpha \right) \ell, \quad J_2 = J|_{\xi=1} = \left(\frac{1}{2} + 2\alpha \right) \ell \quad (\text{E16.14})$$

Both J_1 and J_2 are obviously positive if $-\frac{1}{4} < \alpha < \frac{1}{4}$, and if so $J > 0$ inside the element. If $\alpha = 0$, $J = \frac{1}{2}\ell$ is constant over the element.

- (b) The strain displacement matrix linking
- $e = \mathbf{B} \mathbf{u}^e$
- is given by

$$\mathbf{B} = \frac{d\mathbf{N}^e}{dx} = \frac{d\mathbf{N}^e}{d\xi} \frac{d\xi}{dx} = J^{-1} \left[\frac{\partial N_1^e}{\partial \xi} \quad \frac{\partial N_2^e}{\partial \xi} \quad \frac{\partial N_3^e}{\partial \xi} \right] = \frac{1}{\ell(\frac{1}{2} - 2\alpha\xi)} \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi \right] \quad (\text{E16.15})$$

- (c) From Chapter 12 the element strain energy is (12.18). Replacing the
- $e = \mathbf{B} \mathbf{u}^e$
- of item (b) gives
- $U^e = \frac{1}{2} (\mathbf{u}^e)^T \int_0^\ell \mathbf{B}^T \mathbf{E} \mathbf{B} dx \mathbf{u}^e \stackrel{\text{def}}{=} \frac{1}{2} (\mathbf{u}^e)^T \mathbf{K}^e \mathbf{u}^e$
- , whence

$$\mathbf{K}^e = \int_0^\ell \mathbf{B}^T \mathbf{E} \mathbf{B} dx = \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} \frac{dx}{d\xi} d\xi = \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} J d\xi. \quad (\text{E16.16})$$

For the two-point Gauss rule the *Mathematica* script of Figure E16.4 gives

$$\mathbf{K}^e = \frac{EA}{\ell(3 - 16\alpha^2)} \begin{bmatrix} 7 - 16\alpha & 1 & -8(1 - 2\alpha) \\ 1 & 7 + 16\alpha & -8(1 + 2\alpha) \\ -8(1 - 2\alpha) & -8(1 + 2\alpha) & 16 \end{bmatrix} \quad (\text{E16.17})$$

If $\alpha = 0$ this reduces to

$$\mathbf{K}^e = \frac{EA}{3\ell} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \quad (\text{E16.18})$$

- (d) If
- $\alpha = 0$
- , from item (a)
- $J = \frac{1}{2}\ell$
- is constant whereas
- $\mathbf{B}$
- is linear in
- ξ
- . Because
- EA
- is constant,
- $EA \mathbf{B}^T \mathbf{B} J$
- is a
- quadratic*
- polynomial in
- ξ
- . This is exactly integrated by a Gauss rule with 2 or more points.

EXERCISE 16.7 The *Mathematica* script of Figure E16.7 gives the complicated result:

$$\mathbf{f}^e = \frac{qL}{24} \begin{bmatrix} 3\xi_L^2 - 2(1+4\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(-3 + 2(1+\alpha(4-3\xi_R))\xi_R) \\ -3\xi_L^2 + (-2+8\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(3 - 2\xi_R(-1+\alpha(4+3\xi_R))) \\ -4(\xi_L(3-\xi_L^2+3\alpha\xi_L(-2+\xi_L^2)) - 3\xi_R + 6\alpha\xi_R^2 + \xi_R^3 - 3\alpha\xi_R^4) \end{bmatrix} \quad (\text{E16.19})$$

If $\alpha = 0$, $\xi_L = -1$ and $\xi_R = 1$ so the load extends over the whole bar, this expression simplifies to

$$\mathbf{f}^e = \frac{qL}{6} [1 \quad 1 \quad 4]^T. \quad (\text{E16.20})$$

This is nothing but Simpson's rule for integration.

EXERCISE 17.1 This exercise can be done using the *Mathematica* script listed in Figure E17.5. Running that code produces the results shown in Figure E17.6.

```

ClearAll[Em,v,a,b,h]; Em=48; h=1; a=4; b=2; v=0;
ncoor={{0,0},{a,0},{a,b},{0,b}};
Emat=Em/(1-v^2)*{{1,v,0},{v,1,0},{0,0,(1-v)/2}};
For [p=1, p<=4, p++,
  Ke=Quad4IsoPMembraneStiffness[ncoor,Emat,h,{True,p}];
  Print["Gauss integration rule: ",p," x ",p];
  Print["Ke=",Chop[Ke]//MatrixForm];
  Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]]]]
];

```

FIGURE E17.5. Script to do Exercise 17.1.

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Gauss integration rule: 1 x 1
Ke =  $\begin{pmatrix} 18. & 6. & 6. & -6. & -18. & -6. & -6. & 6. \\ 6. & 27. & 6. & 21. & -6. & -27. & -6. & -21. \\ 6. & 6. & 18. & -6. & -6. & -6. & -18. & 6. \\ -6. & 21. & -6. & 27. & 6. & -21. & 6. & -27. \\ -18. & -6. & -6. & 6. & 18. & 6. & 6. & -6. \\ -6. & -27. & -6. & -21. & 6. & 27. & 6. & 21. \\ -6. & -6. & -18. & 6. & 6. & 6. & 18. & -6. \\ 6. & -21. & 6. & -27. & -6. & 21. & -6. & 27. \end{pmatrix}$ 
Eigenvalues of Ke= {896., 60., 24., 0, 0, 0, 0, 0}
Gauss integration rule: 2 x 2
Ke =  $\begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$ 
Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}
Gauss integration rule: 3 x 3
Ke =  $\begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$ 
Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}
Gauss integration rule: 4 x 4
Ke =  $\begin{pmatrix} 24. & 6. & 0 & -6. & -12. & -6. & -12. & 6. \\ 6. & 36. & 6. & 12. & -6. & -18. & -6. & -30. \\ 0 & 6. & 24. & -6. & -12. & -6. & -12. & 6. \\ -6. & 12. & -6. & 36. & 6. & -30. & 6. & -18. \\ -12. & -6. & -12. & 6. & 24. & 6. & 0 & -6. \\ -6. & -18. & -6. & -30. & 6. & 36. & 6. & 12. \\ -12. & -6. & -12. & 6. & 0 & 6. & 24. & -6. \\ 6. & -30. & 6. & -18. & -6. & 12. & -6. & 36. \end{pmatrix}$ 
Eigenvalues of Ke= {896., 60., 36., 24., 24., 0, 0, 0}

```

FIGURE E17.6. Results from running the script of Figure E17.5.

The K^e given by the 1×1 Gauss rule has a rank deficiency of two because it has five zero eigenvalues instead of three. This behavior is explained in Chapter 19.

The 2×2 , 3×3 and 4×4 rules produce the same stiffness matrix. This matrix has three zero eigenvalues, which correspond to the three independent rigid body modes in two dimensions.

The reason for the repeating stiffness matrices is that the integrand $h\mathbf{B}^T\mathbf{E}\mathbf{B}J$ is at most quadratic in ξ and η because h and $\mathbf{E}$ are constant, $\mathbf{B}$ is linear in ξ and η , and for a rectangle J is constant over the element. A 2-point product Gauss rule is exact for up to cubic polynomials in the ξ and η directions, so it does quadratics exactly.

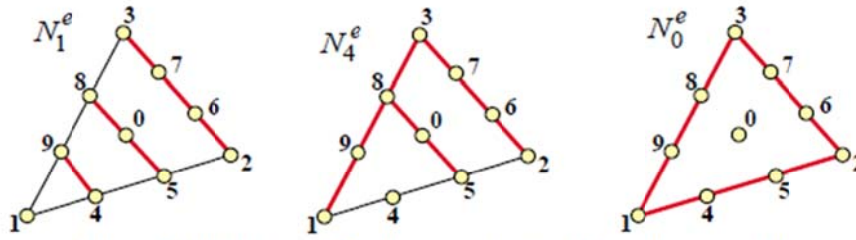


FIGURE E18.6. Lines used for direct construction of shape functions of the 10-node cubic triangle.

EXERCISE 18.1

- (a) The shape functions for nodes 1, 4 and 0 of the 10-node cubic triangle can be obtained using the lines marked in red in Figure E18.6. The process is as follows.

$$N_1^e(\zeta_1, \zeta_2, \zeta_3) = c_1 \zeta_1 (\zeta_1 - \frac{1}{3}) (\zeta_1 - \frac{2}{3})$$

$$N_1^e(1, 0, 0) = c_1 \times 1 \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} c_1 = 1, \quad \text{hence } c_1 = \frac{9}{2}, \quad (\text{E18.4})$$

$$N_1^e = \frac{9}{2} \zeta_1 (\zeta_1 - \frac{1}{3}) (\zeta_1 - \frac{2}{3}) = \frac{1}{2} \zeta_1 (3\zeta_1 - 1)(3\zeta_1 - 2)$$

$$N_4^e(\zeta_1, \zeta_2, \zeta_3) = c_4 \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3})$$

$$N_4^e(\frac{2}{3}, \frac{1}{3}, 0) = c_4 \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27} c_4 = 1, \quad \text{hence } c_4 = \frac{27}{2}, \quad (\text{E18.5})$$

$$N_4^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3}) = \frac{9}{2} \zeta_1 \zeta_2 (3\zeta_1 - 1)$$

$$N_0^e(\zeta_1, \zeta_2, \zeta_3) = c_0 \zeta_1 \zeta_2 \zeta_3$$

$$N_0^e(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = c_0 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27} c_0 = 1, \quad \text{hence } c_0 = 27, \quad (\text{E18.6})$$

$$N_0^e = 27 \zeta_1 \zeta_2 \zeta_3.$$

The remaining shape functions can be obtained by appropriate index substitutions:

$$N_2^e = \frac{9}{2} \zeta_2 (\zeta_2 - \frac{1}{3}) (\zeta_2 - \frac{2}{3}), \quad N_3^e = \frac{9}{2} \zeta_3 (\zeta_3 - \frac{1}{3}) (\zeta_3 - \frac{2}{3}),$$

$$N_4^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_1 - \frac{1}{3}), \quad N_5^e = \frac{27}{2} \zeta_1 \zeta_2 (\zeta_2 - \frac{1}{3}),$$

$$N_6^e = \frac{27}{2} \zeta_2 \zeta_3 (\zeta_2 - \frac{1}{3}), \quad N_7^e = \frac{27}{2} \zeta_2 \zeta_3 (\zeta_3 - \frac{1}{3}),$$

$$N_8^e = \frac{27}{2} \zeta_3 \zeta_1 (\zeta_3 - \frac{1}{3}), \quad N_9^e = \frac{27}{2} \zeta_3 \zeta_1 (\zeta_1 - \frac{1}{3}). \quad (\text{E18.7})$$

Here factors $9/2$ and $27/2$ may be partially absorbed to in the factors to make $\zeta_1 - \frac{1}{3} \rightarrow 3\zeta_1 - 1$, etc.

- (b) The verification that the sum is unity is done with the *Mathematica* script shown in Figure E18.7. Notice the substitution enforcing $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

```
Nf={ζ1*(3*ζ1-1)*(3*ζ1-2)/2, ζ2*(3*ζ2-1)*(3*ζ2-2)/2,
  ζ3*(3*ζ3-1)*(3*ζ3-2)/2, 9*ζ1*ζ2*(3*ζ1-1)/2, 9*ζ1*ζ2*(3*ζ2-1)/2,
  9*ζ2*ζ3*(3*ζ2-1)/2, 9*ζ2*ζ3*(3*ζ3-1)/2, 9*ζ3*ζ1*(3*ζ3-1)/2,
  9*ζ3*ζ1*(3*ζ1-1)/2, 27*ζ1*ζ2*ζ3};
S=Simplify[Sum[Nf[[i]],{i,10}]/.{ζ3->1-ζ1-ζ2}]; Print[S];
```

FIGURE E18.7. Unit sum verification for 10-node cubic triangle.

EXERCISE 18.6 The shape function for the internal node 5 is the bubble function $N_5 = (1 - \xi^2)(1 - \eta^2)$, which also appears in Eq. (15.17) as N_9 of the 9-node quad. Take $N_i = \bar{N}_i + c_5 N_5$, where $\bar{N}_i$, $i = 1, 2, 3, 4$ are the shape functions (16.15) of the four-node bilinear quadrilateral. These functions are not zero at node 5, hence the correction by a multiple of the bubble function. Evaluation at node 5 ($\xi = \eta = 0$) shows that $c_5 = -1/4$. The corrected corner shape functions are $N_i = \bar{N}_i - N_5/4$, $i = 1, 2, 3, 4$. For example,

$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2). \quad (\text{E18.10})$$

These four corner shape functions vary linearly over all sides because the bubble function N_5 vanishes over the 4 sides. Hence interelement continuity is maintained.

Unit sum check: $N_1 + N_2 + N_3 + N_4 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 - 4 \times \frac{1}{4} N_5 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 = 1$, since it is known that the shape functions $\bar{N}_i$ of the 4-node quad add up to one.