Classification of Optimization Problems (1)

- Variables
  - Continuous, discrete or mixed
- Objective
  - Function of a single variable
  - Function of many variables
  - Linear
  - Sum of squares
  - Nonlinear
  - Smooth / non-smooth
  - Convex / non-convex
  - 1st derivatives are available
  - 2nd derivatives are available
- Constraints
  - None
  - Simple bounds
  - Linear
  - Non-linear
  - Equality / inequality
  - Smooth / non-smooth
  - 1st derivatives are available
  - 2nd derivatives are available
- Optimum
  - Local
  - Global

Classification of Optimization Problems (2)

- The choice of solution method is very dependent on
  - the class of the problem
  - the size of the problem
  - the structure of the problem
  - the cost of function and gradient evaluation
  - etc.

Classification of Optimization Problems (3)

- Some more jargon
  - Gradient
  - Hessian
  - Sensitivity analysis
  - Scaling
  - Normalization
  - Mathematical Programming
  - LP
  - NLP
- Optimization software
  - In
    - EXCEL GRG2
    - MATLAB OPTIM toolbox
    - NAG
    - NETLIB
  - Specialized packages
    - NPSOL
    - IDESIGN
    - LANCELOT
  - Plus hundreds of others

Structural Optimization (1)

- Rational establishment of a structural design that is the best of all possible designs within a prescribed objective and a given set of geometrical and/or behavioral limitations
- Mathematics and mechanics with engineering
- Broad multidisciplinary field
  - Aeronautical, civil, mechanical, nuclear, off-shore engineering, space technology
- Motivation
  - Limited energy resources, shortage of economic and some material resources, strong technological competition, environmental problem

Structural Optimization (2)

- Minimum cost or weight of the structure for given performance / Maximum performance for a bound on cost
  - Decreasing the weight of space, aero, or land-borne structures
  - Cost reduction of load-carrying structures for given capacity, strength, and/or stiffness requirements
  - Increasing the efficiency of fibers in composite materials by optimizing their distribution and orientation
  - Minimizing dynamic response of rotating machinery or structures subjected to external excitation
- Research in optimal structural design
  - Fundamental aspects of structural optimization
  - Development of effective numerical solution procedures for optimization of complex practical structures

Analysis Problem

- Completely specified in deterministic problems / Given in terms of probabilities in probabilistic problems
  - Structural design, properties of materials, support/loading conditions
- Determine the structural response
  - Equilibrium (or state) / constitutive equations, compatibility / boundary conditions
  - Stress, strain, deflection, natural vibration frequencies, load factors for elastic instability

Redesign (or Sensitivity Analysis)

- Design, material, or support parameters are changed (or varied) and the corresponding changes (or variations) of the structural response are determined via repeated (or special) analysis
- Conventional design procedure
  - A series of repeated changes of the structural parameters followed by analysis
  - A series of redesign analyses until a structure fulfills the behavioral requirements and is reasonable in cost
  - Changes decided by guesswork based on information obtained from the previous analysis

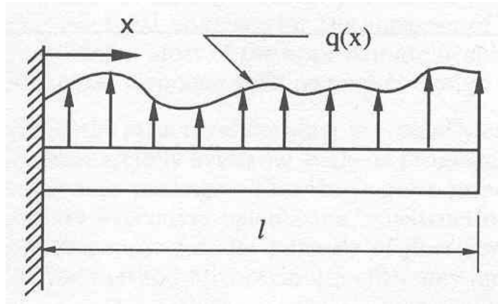
Optimization of Structures

- Set of structural parameters is subdivided into preassigned parameters and design variables
- Problem consists in determining optimal values of the design variables such that they maximize or minimize a specific function termed the objective (or criterion, or cost) function while satisfying a set of geometrical and/or behavioral requirements, which are specified prior to design, and are called constraints.

Beam Design

- Structural analysis

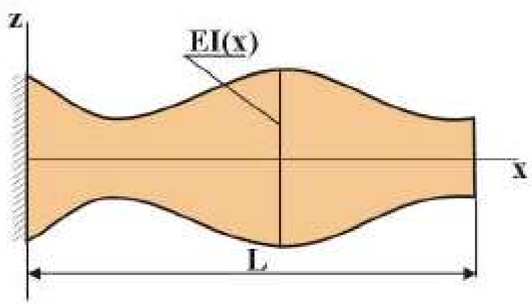
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)=q(x)$$

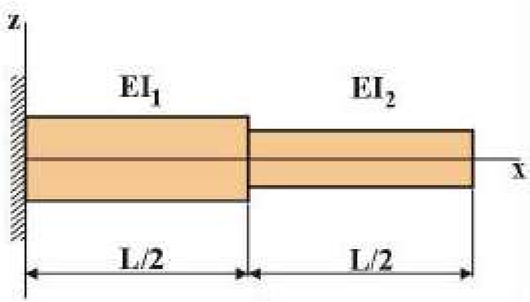


- Structural designer
  - Optimal distribution of the moment of inertia *I(x)* of the beam along its length
  - Objective function : mass $m=c\int_0^l I^p(x)\,dx$
  - Constraints : displacement $w_{\max}=\max\limits_{0\le x\le l} w(x)\le w_0$
  - Optimality condition : in the form of a differential equation in *I(x)* and *w(x)* $\rightarrow$ "calculus of variations"

Function vs. Parameter Optimization

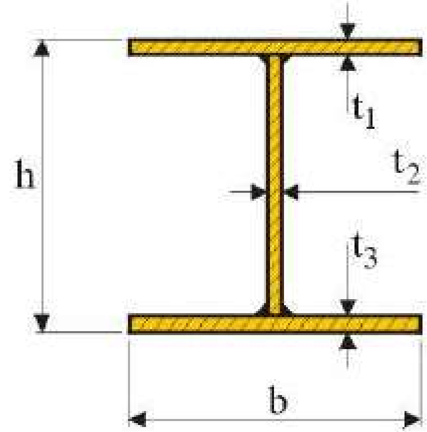
	Before 60s	After 60s
analysis solution	analytic solutions (e.g., by using infinite series)	computer implementation (e.g., finite element method)
unknown	function	discrete value
equation	differential	algebraic
discipline	calculus of variations	mathematical programming





Elements of Problem Formulation (1)

- Design variables: $\mathbf{x} = \left(x_1, x_2, \ldots, x_n \right)$
  - Parameters controlling the geometry of the structure
    - Cross-sectional dimensions
    - Member sizes
  - Material properties
  - Continuous
    - Range of variation
  - Discrete
    - Isolated values
    - Manufacturing considerations
  - Critical to the success of the optimization process



Elements of Problem Formulation (2)

- Objective function : $\mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_p(\mathbf{x}) \right]$
  - Measure of effectiveness of the design
    - Weight, displacements, stresses, vibration frequencies, buckling loads, cost
  - Multicriteria(multiobjective) optimization
    - Generate a composite objective function
    - Select the most important as the only objective function and impose limits on the others
    - Edgeworth-Pareto optimization
- Constraints
  - limits on the design variables : side constraints
  - Impose upper or lower limits on quantities : inequality constraints
  - Equality constraints $\rightarrow$ inequality constraints (some solution strategies)

Design Variables (1)

- Cross-sectional DVs: Properties of structural elements
  - Cross-sectional areas (of a bar, rod or beam)
  - Second area moment (of a beam, column or arch)
  - Thickness (of a plate)
  - Continuous (function of the spatial coordinates) / Discrete (distinct, standardized sizes)
- DVs describing the layout of a structure
  - Topological DVs: number, spatial sequence, and mutual connectivity of members and joints (integer)
  - Configurational (or geometrical) DVs: coordinates of joints, centerlines or midsurfaces of structural members (bar, beam, arch, shell)

Design Variables (2)

- Shape DVs
  - Shape of external boundaries or interface of a structure
  - Cross-sectional shape of a rod, column, or beam
  - Boundary shape of a disk, plate, or shell
- Material DVs
  - Material properties (discrete)
  - Fiber composite materials: concentration and direction of the fibers (continuous)
- Support or loading DVs
  - Support (or boundary) conditions or the distribution of loading on a structure
  - Location, number, and type of support or the external forces

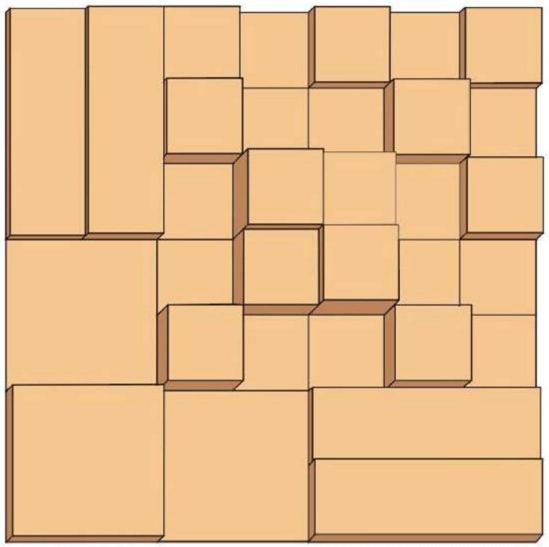
Continuous vs. Discrete

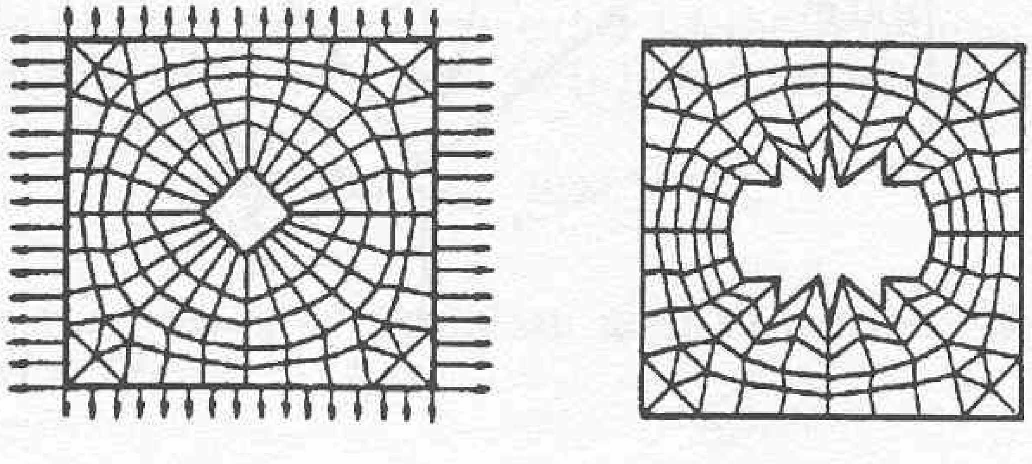
- Continuous (or distributed parameter) optimization problem
  - DVs are considered to vary continuously over the length or domain of the element
  - Rod, beam, arch, plate
- Discrete (or parameter) optimization problem
  - Inherently discrete structure
  - Truss, frame, complex practical structures

- The governing equations of both types of problem (as well as mixed types) can be derived by variational analysis

Design Variables

- Finite element model
  - Distribution of DVs should be much coarser
  - Optimal thickness distribution of a plate
    - Thickness of the FE model, 7X7?
  - Optimized shape of a hole in a plate
    - Coordinates of nodes of the FE model





(a) (b)

Objective Function

  - Cost or criterion function
  - Function whose value is to be minimized or maximized by the optimal set of values of DVs within the feasible design space
- Structural weight or cost
- Local or global measure of the structural performance
  - Stress, displacement, stress intensity factor, stiffness, plastic collapse load, fatigue life, buckling load, natural vibration frequency, aeroelastic divergence, flutter speed, etc.
- Single-criterion / Multicriteria

Problems with Multiple Objectives (1)

$$\left.\begin{array}{l} Min\;\; f_1(\boldsymbol{x}) \\ \quad\vdots \\ Min\;\; f_M(\boldsymbol{x}) \end{array}\right\} \rightarrow F(\boldsymbol{x}) = f\left[f_1(\boldsymbol{x}), \ldots, f_M(\boldsymbol{x})\right]$$

- Individual objectives are usually in contradiction with one another, hence
- If $x_1^*, \ldots, x_M^*$ are the solutions to individual objectives, then $x_1^* \neq \ldots \neq x_M^*$
- If the individual objectives are controlled by different sets of variables, then the optimum of *f* can be obtained by optimizing the individual f_i's.

$$F(\boldsymbol{x}) = f_1(\boldsymbol{x}_1) + \cdots + f_M(\boldsymbol{x}_M) = \sum_{i=1}^{M} f_i(\boldsymbol{x}_i) \quad where \quad \boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M)$$

Problems with Multiple Objectives (2)

- All objectives are controlled by the same set of variables:
  - Composite objective function

$$F(\boldsymbol{x}) = \alpha_1 f_1(\boldsymbol{x}) + \cdots + \alpha_M f_M(\boldsymbol{x}) = \sum_{i=1}^{M} \alpha_i f_i(\boldsymbol{x}_i)$$

  - Choose the most important to Max(Min), and put limits on the others.

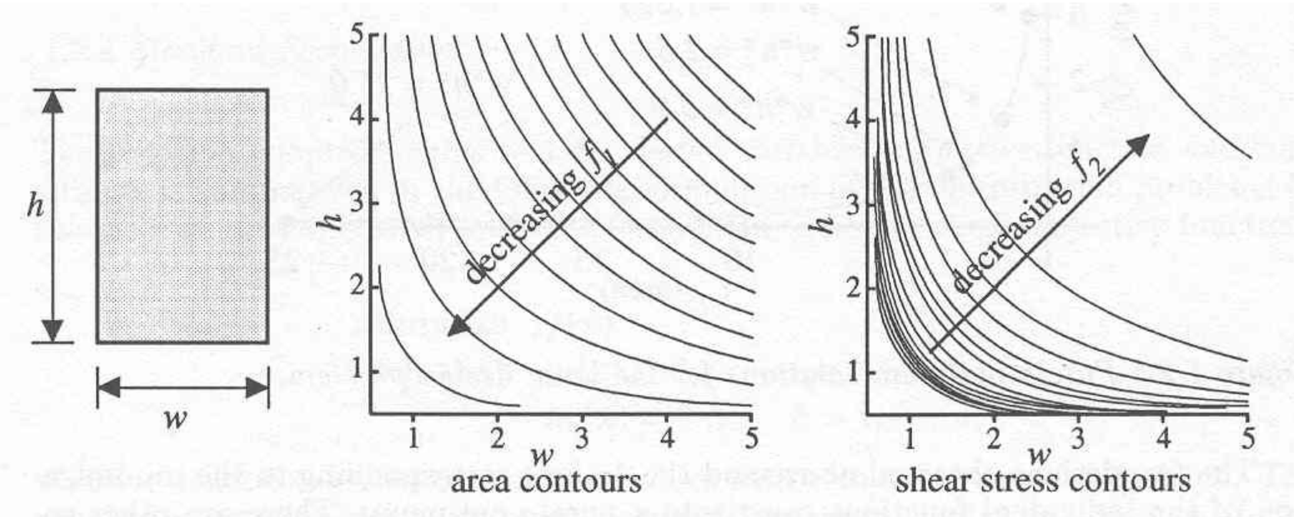
$$Min(Max) \quad f(\boldsymbol{x}) = f_2(\boldsymbol{x}) \;\; such\, that \quad f_1(\boldsymbol{x}) \geq A_1 \quad \cdots \quad f_M(\boldsymbol{x}) \geq A_M$$

  - Optimize each of the objectives w.r.t. $\boldsymbol{x}$ individually to find $f_i^{\,*}$ and the corresponding $\boldsymbol{x}_i^{*}$.

$$Min \;\; \underset{i=1,\ldots,M}{Max}\left[d_i(\boldsymbol{x})\right] \;\; or \;\; Min \; \sum_{i=1}^{M} d_i^{\,2}(\boldsymbol{x}) \;\; where \;\; d_i(\boldsymbol{x}) = \frac{f_i(\boldsymbol{x}) - f_i^{*}}{f_i^{*}}$$

Rectangular Beam (1)

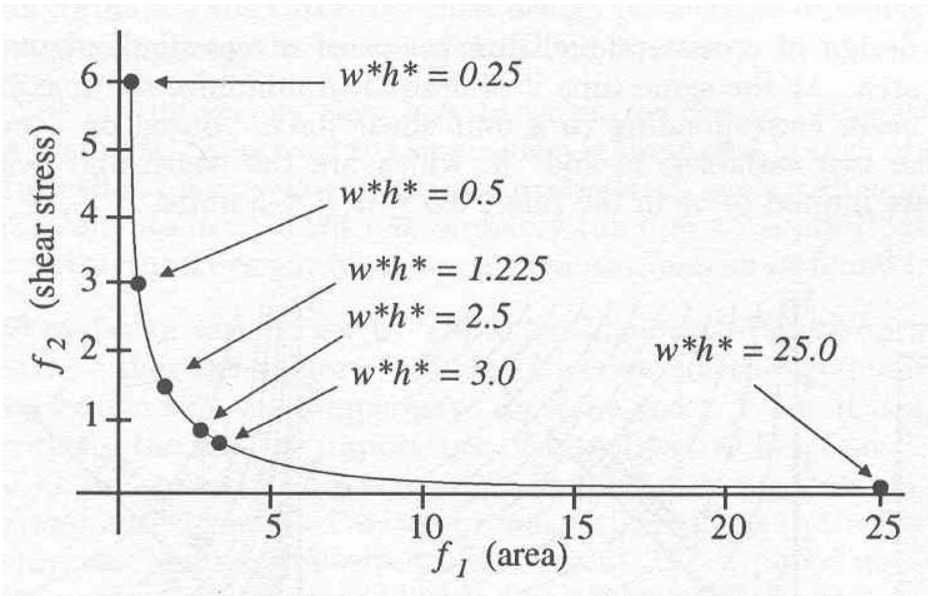
- Design variables
  - Width and height of the cross-section
- Objective functions
  - Minimize the area: $f_1 = A = wh$
  - Minimize the maximum shear stress : $f_2 = \tau_{\max} = 1.5\dfrac{V}{A} = \dfrac{3}{2wh}$
- Constraints: $0.5 \le w, h \le 5$



Rectangular Beam (2)

- Weighted sum: $f = w_1 f_1 + w_2 f_2$
- Euclidean norm of the distance from the individual minima:

$$f = \frac{f_1 - f_1^*}{f_1^*} + \frac{f_2 - f_2^*}{f_2^*}$$



Constraints

- Directly or indirectly impose limits on the range of variations of DVs
- Design space / hypersurfaces $\rightarrow$ feasible or admissible designs

- Geometrical (or side) constraints
  - Explicit restrictions on DVs
  - Manufacturing limitations, physical practicability, aesthetics
  - Typically inequality constraints: lower and upper bounds
- Behavioral constraints
  - Generally nonlinear and implicit
  - Equality: state and compatibility equations governing the structural response associated with the loading conditions
  - Inequality: restrictions on those quantities that characterize the response of the structure
    - Local (stress, deflection) / global (compliance, natural vibration frequencies)

Solution Process

- Selection of the active constraint set
- Calculation of a search direction
  - Based on the objective function and the active constraint set
- Determination of a travel distance
  - One dimensional line search
- Termination criteria
  - No improvement of the objective function w/o violating constraints
  - Check for optimality (Kuhn-Tucker conditions)

Numerical Search Techniques (1)

- Procedure
  - Selection of an initial design in the *n*-dimensional space
  - Evaluation of the function (objective and constraints) at a given point in the design space
  - Comparison of the current design with all of the preceding designs
  - A rational way to select a new design and repeat the process
- Questions
  - How is the initial design selected and what effect will it have on the outcome of the search?
  - What is a rational way to select the new designs and how does it affect the final outcome?
  - Where to stop the search?

Numerical Search Techniques (2)

- 50s: simplex method and its variations
  - LP: transportation, scheduling, chemical processes, etc.
- 60s: gradient projection, feasible directions, penalty function methods
- 70s: implementation and serious applications
- 80s: refinement of the algorithms proposed in the 60s and 70s
  - (U.S.) Vanderplaat's implementation of the feasible directions
  - CONMIN, ADS, MICRODOT
  - (Europe) Fleury's CONLIN, Schittkowski's implementation of SQP (NLPL) → IMSL library

Issues in Practical Design Optimization

- Robustness
  - theoretically guaranteed to converge to a local minimum point
- Potential constraint strategy
- Attributes of a good optimization algorithm
  - Reliability
  - Generality
  - Ease of use
  - Efficiency
    - rate of convergence / iteration(minimum number of calculations)
    - Search direction: potential constraint strategy
    - Step size determination: inexact line search, polynomial interpolation

Industrial applications

Struct Multidisc Optim 27, 460–468 (2004)
DOI 10.1007/s00158-004-0407-y

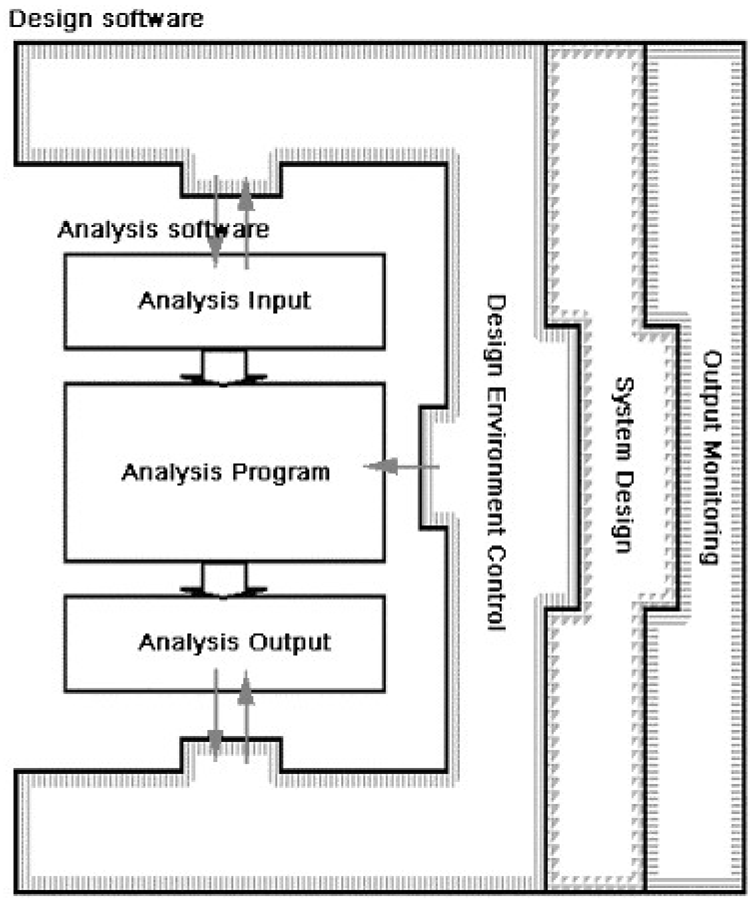
A comparative study of software systems from the optimization viewpoint

U.P. Hong, K.H. Hwang and G.J. Park

Software	Version	Developer
VisualDOC	1.2	VR&D
iSIGHT	5.0	Engineous
OPTIMUS	2.2	LMS International
ModelCenter	2.0	Phoenix Integration

No.	VisualDOC	iSIGHT	OPTIMUS	ModelCenter
1	○	○	○	○
2	×	○	○	○
3	○	○	○	○
4	×	○	○	×
5	○	×	×	×
6	×	○	×	○
7	×	○	○	○

1: GUI to define design problem
2: Multiple input/output files
3: Definition of user supplied equation
4: Script to define user defined problem
5: Compile process to integrate an analysis software
6: Real time output monitoring
7: Remote access to the analysis software



Optimization Algorithms

No.	VisualDOC	iSIGHT	OPTIMUS	ModelCenter
1	BFGS	MFD	SQP	VMM
2	FR	MMFD	GRG	CGM
3	MMFD	SLP		MFD
4	SLP	SQP		SLP
5	SQP	SAM		SQP
6		DHS		
7		EP		
8		GA		
9		HJ		
10		SA		
11		GRG		

BFGS: Broyden–Fletcher–Goldfarb–Shannon algorithm
CGM: Conjugate gradient method
DHS: Direct heuristic search
EP: Exterior penalty
FR: Fletcher–Reeves algorithm
GRG: Generalized reduced gradient
GA: Genetic algorithm
HJ: Hooke–Jeeves algorithm
MFD: Method of feasible directions
MMFD: Modified method of feasible directions
SLP: Sequential linear programming
SQP: Sequential quadratic programming
SA: Simulated aAnnealing
SAM: Successive approximation method
VMM: Variable metric method

1. VisualDOC has a powerful optimization module. Some unstable functions exist in the design environment specification, especially for interface functions with input/output files. Therefore, some improvements are needed for those functions.
2. iSIGHT is excellent in the design environment specification. The designer can easily specify the design environment. It has various capabilities in the DOE module compared to others. However, performances of the optimization algorithms are not as efficient as those of others.
3. OPTIMUS has different characteristics from others. The GUI displays the relationship between input/output variables and analysis software with a diagram. However, there are some unstable functions in the interface functions between the input/output files.
4. ModelCenter separates the design environment specification and system design software with an analysis server and ModelCenter. Therefore, ModelCenter can be the most powerful in the network environment.

1. Rosenbrock's valley problem (Kroo *et al.* 1994)

Find x_1, x_2
To minimize

$$f(x) = 100\left(x_2 - x_1^2\right)^2 + \left(1 - x_1\right)^2$$

2. Spring design problem (Haug 1979)

Find d, D, N
To minimize $f = (N+2)Dd^2$
Subject to

$$g_1 = 1.0 - \frac{D^3 N}{71\,875 d^4} \leq 0.0$$

$$g_2 = \frac{D(4D-d)}{12\,566 d^3 (D-d)} + \frac{2.46}{12\,566 d^2} - 1.0 \leq 0.0$$

$$g_3 = 1.0 - \frac{140.54 d}{D^2 N} \leq 0.0$$

$$g_4 = \frac{D+d}{1.5} - 1.0 \leq 0.0$$

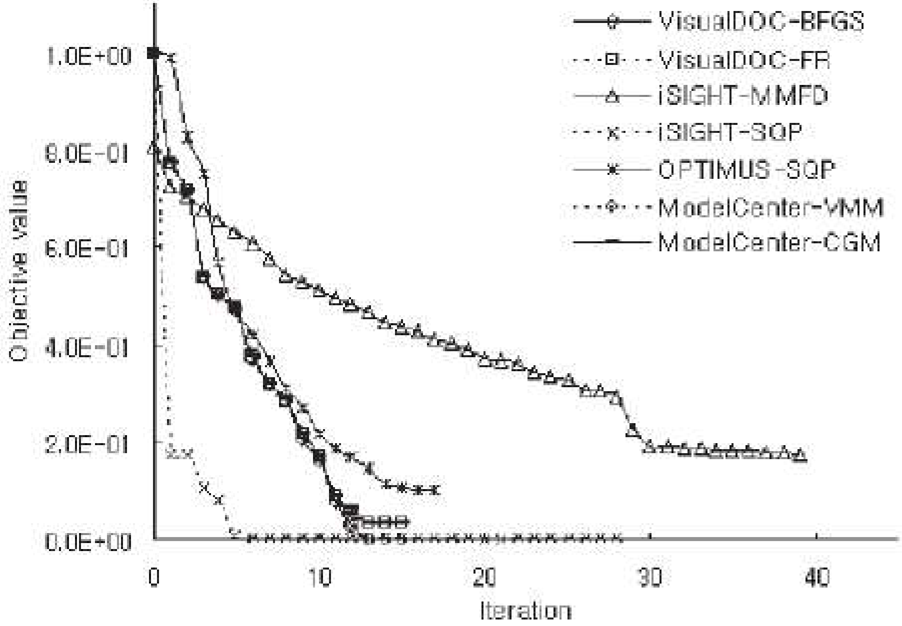


Fig. 2 Optimization results of the Rosenbrock's valley problem

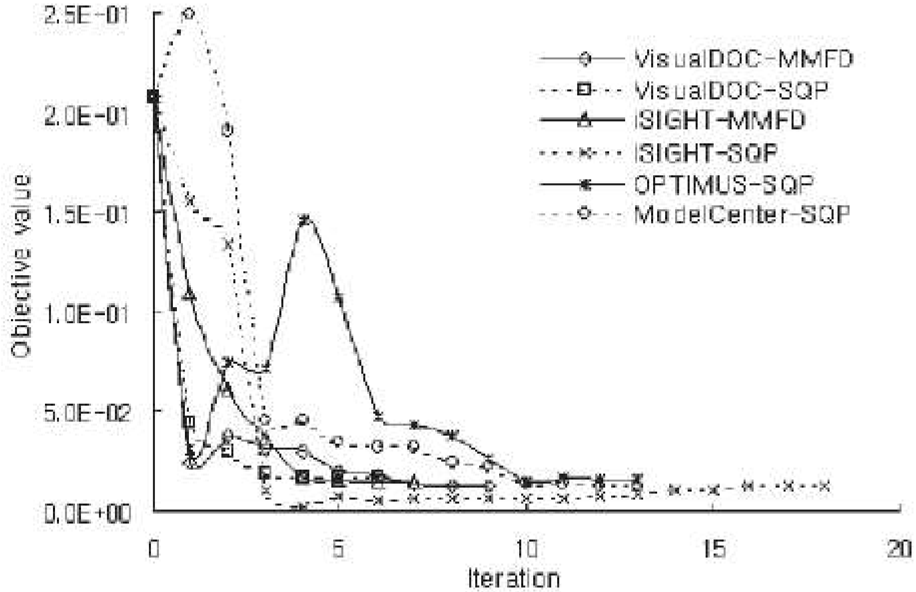


Fig. 3 Optimization results of the spring design problem

History (1)

- Galileo's problem
  - Strongest cantilever beam in bending and constant shear for minimum weight under a uniform stress constraint
- Introduction of calculus by Newton and/or Leibniz
  - Development of mathematical optimization
  - Min-max conditions: necessary conditions for optimal solutions
  - Only the unconstrained optimization problems
- Augmented Lagrangian function
  - Extension of simple min-max conditions to constrained optimization problems
  - Lagrangian multipliers: dual variables
    - Weighting factors in establishing the importance of the various constraints at different regions of design space
    - Link between the objective and the constraint functions

History (2)

- Calculus of variation (attributed to Bernoulli, Euler, Lagrange)
  - Brachistochrone problem
  - Generalization of the elementary theory of minima and maxima
  - Dealing with extremum of a function of functions
  - Solution? One or more functions represented by differential equations
  - Solution of D.E. → optimal path, or all the optimal points
  - Euler-Lagrangian equations → most of field equations of mechanics
  - Principles of least action: originally derived by Euler
    - Hamilton's principle → most of dynamic system equations based on Newton's Laws
    - Lagrange's equation → basis for an elegant description of Newtonian dynamics
  - Numerical difficulties in practical applications

History (3)

- The Euler-Lagrange equations: extreme conditions
  - Yield one or more nonlinear differential equations for solution
  - Variational approach: difficult to solve, restricted continuity and differentiability
  - Numerical approach: approximation of derivatives by differences and of integrals by sum
    - Differential equation → algebraic equation
    - Reliable? accuracy, time steps, convergence

History (4)

- Separation of the analysis and design as different problems
  - Analysis: determination of the state of the system as a function of time and spatial coordinates
  - Differential equations of analysis are obtained by minimization or maximization of one or more functions
    - e.g., in solid mechanics, potential energy in the system
    - Dependable variable: state variables → define the state of the system
    - Independent variable: spatial coordinates and time
  - Design: minimization or maximization of a predefined performance function subject to a set of constraint conditions
    - Variables: physical parameters that define the configuration of the system, sizes and/or geometrical quantities of the structural elements

Well-Established Areas

- Single-criterion optimization problems
- Optimal plastic design
  - Design against plastic collapse (limit load)
  - Uniform energy dissipation
- Elastic optimal design under static loading
  - Elastic design under strength, stiffness, or stability requirement
- Optimal layout of trusses
- Optimal design under dynamic loading
  - Natural frequency / forced steady state / transient response requirements

Basic Structural Optimization Setting

- Two sets of variables:
  - Design variables ρ
  - State variables U
- which are coupled via a state equation that (for given fixed values of the design variables) gives values of the state variables

- Terminology in applied mathematics
  - Mathematical programs with equilibrium constraints
  - PDE constrained optimization problems

Structural Optimization Formulation (1)

$$\min_{\boldsymbol{\rho},\boldsymbol{U}} \ \Phi\left(\boldsymbol{\rho},\boldsymbol{U}\right)$$

Objective function

$$\text{subject to} \ \ V\left(\boldsymbol{\rho}\right) \le V^{*}$$

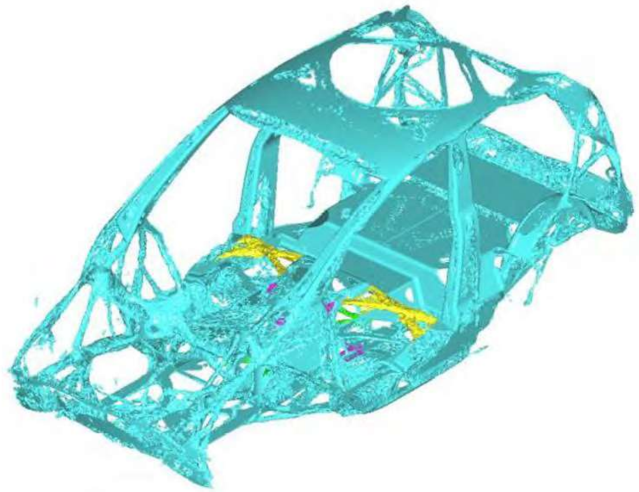
$$g_i\left(\boldsymbol{\rho},\boldsymbol{U}\right) \le g_i^{*} \quad i = 1,\ldots,M$$

$$\rho_{\min} \le \boldsymbol{\rho} \le \rho_{\max}$$

Constraints

$$\boldsymbol{K}\left(\boldsymbol{\rho}\right)\boldsymbol{U} = \boldsymbol{F}$$

State equation (FE)



Structural Optimization Formulation (2)

$$
\boxed{
\begin{aligned}
&\min_{\boldsymbol{\rho},\boldsymbol{U}} \ \Phi(\boldsymbol{\rho},\boldsymbol{U}) \\
&\text{subject to} \ \ V(\boldsymbol{\rho}) \le V^* \\
&\qquad\qquad\quad g_i(\boldsymbol{\rho},\boldsymbol{U}) \le g_i^* \quad i = 1,\ldots,M \\
&\qquad\qquad\quad \rho_{\min} \le \boldsymbol{\rho} \le \rho_{\max} \\
&\qquad\qquad\quad \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}
\end{aligned}
}
$$

Simultaneous Analysis And Design (SAND)

$$
\boxed{
\begin{aligned}
&\min_{\boldsymbol{\rho}} \ \Phi(\boldsymbol{\rho},\boldsymbol{U}) \\
&\text{subject to} \ \ V(\boldsymbol{\rho}) \le V^* \\
&\qquad\qquad\quad g_i(\boldsymbol{\rho},\boldsymbol{U}) \le g_i^* \quad i = 1,\ldots,M \\
&\qquad\qquad\quad \rho_{\min} \le \boldsymbol{\rho} \le \rho_{\max}
\end{aligned}
}
$$

$$
\boxed{\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}}
$$

Nested Analysis And Design (NAND)

Choices to Make

- Analysis setting: state equation

$$div\left(E_{ijkl}(\rho)\varepsilon_{kl}\right)+f_i=0 \quad \text{in } \Omega \text{ or } div\left(E_{ijkl}\varepsilon_{kl}\right)+f_i=0 \quad \text{in } \Omega(\rho)$$

$$\sigma_{ij,j}+f_i=0 \quad \text{in } \Omega$$

$$\int_\Omega E_{ijkl}(\rho)\varepsilon_{kl}(u)\varepsilon_{kl}(v)d\Omega=\int_\Omega fv d\Omega,\ \forall v\in U$$

$$\Rightarrow \quad \boxed{K(\boldsymbol{\rho})\boldsymbol{U}=\boldsymbol{F}}$$

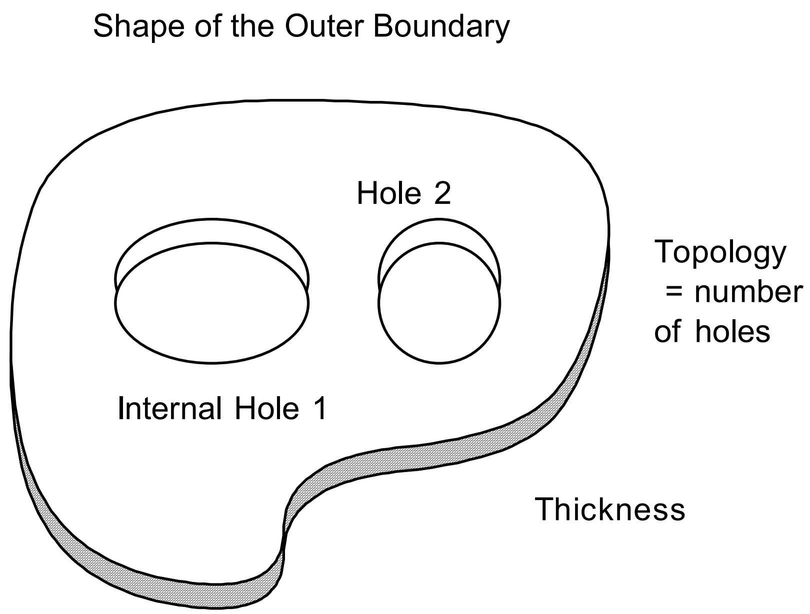
- Design space: choice of design variables
  - Sizing variables
  - Shape variables
  - Topology variables: 0-1 in each point
- Cost and constraint functions

Structural Optimization

- Lay-out of the structure
  - Information on the topology, shape and sizing of the structure
- Sizing optimization
  - Optimal thickness distribution of a linearly elastic plate
  - Optimal member areas in a truss structure
- Shape optimization
  - Optimum shape of the domain which is the design variable
- Topology optimization
  - Determination of features such as the number and location and shape of holes and the connectivity of the domain

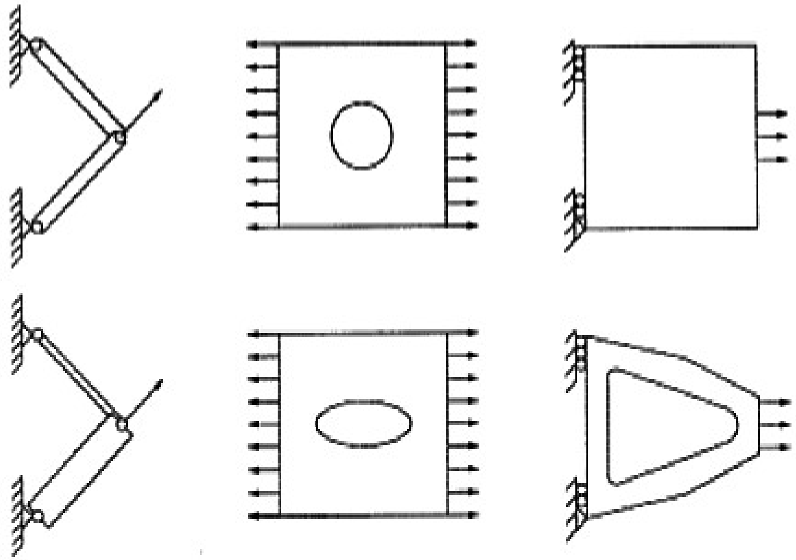
Three Major Design Problems

- Sizing Optimization (1960)
  - How thick it is?
  - Thickness
  - cross sectional properties
  - Finite element model is fixed
- Shape Optimization (1973)
  - What are the boundaries?
  - Location and/or radii of holes/arcs
  - Control points of splines
  - Element shapes change during optimization
- Topology Optimization (1988)
  - Where are the holes?
  - Number of holes
  - Shape of holes
  - Finite element topology possible not defined

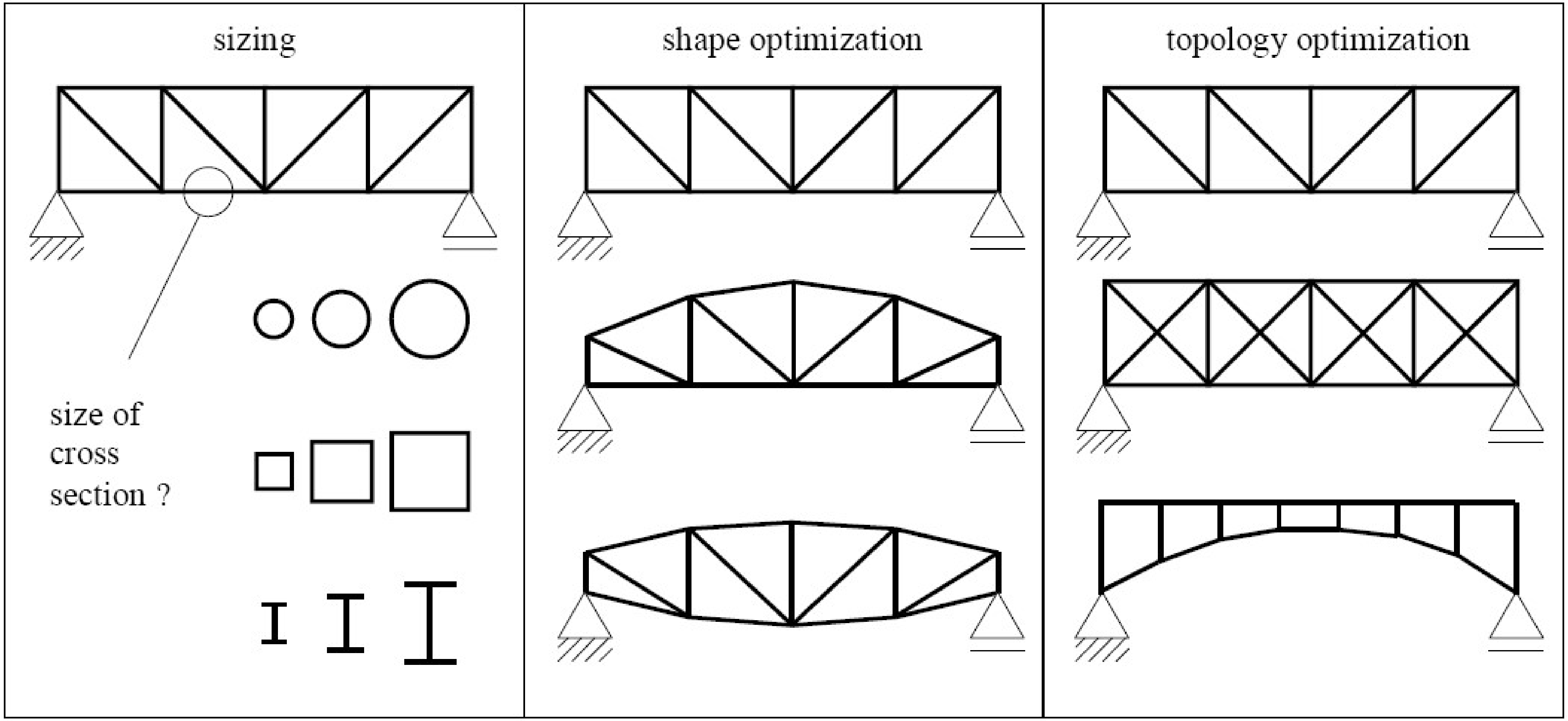


Concept of Structural Optimization

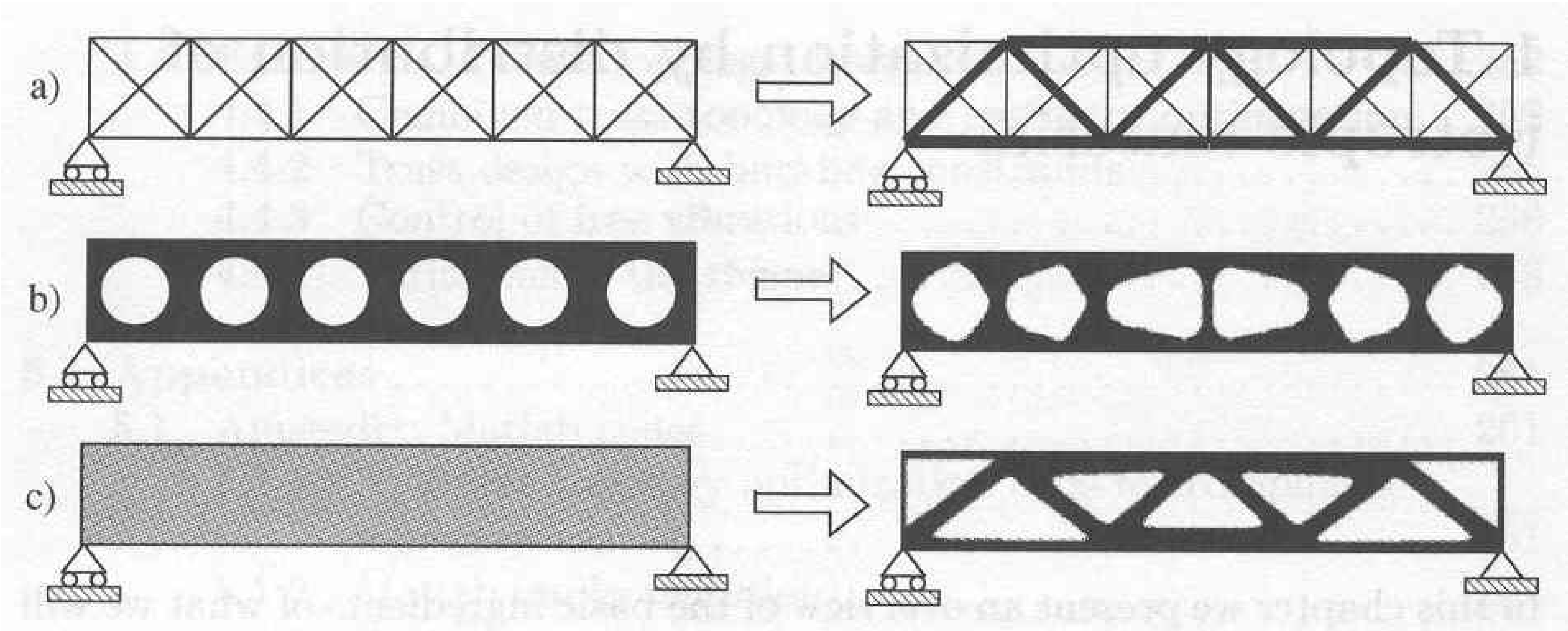
- Size Optimization (1960)
  - thickness, cross section
  - restricted to beam/frame-like structures
- Shape Optimization (1973)
  - location of holes/arcs, radii of holes/arcs
  - control points of splines
- Topology Optimization (1988)
  - number of holes, shape of holes, connectivity



Levels of Structural Optimization

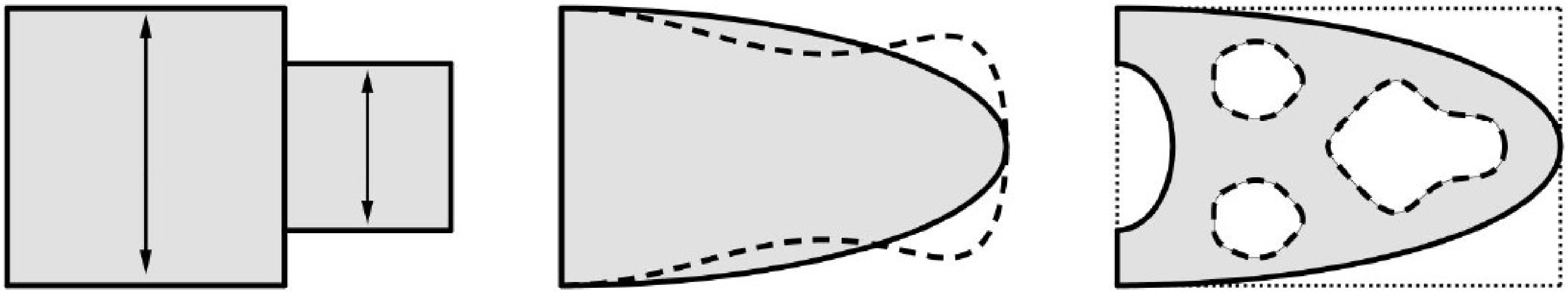


Categories of Structural Optimization



M.P.Bendsoe and O. Sigmund, Topology Optimization: Theory, Methods and Applications, Springer, 2003

Three categories of optimal design problems



1) Parametric optimization 2) Shape optimization 3) Topology optimization