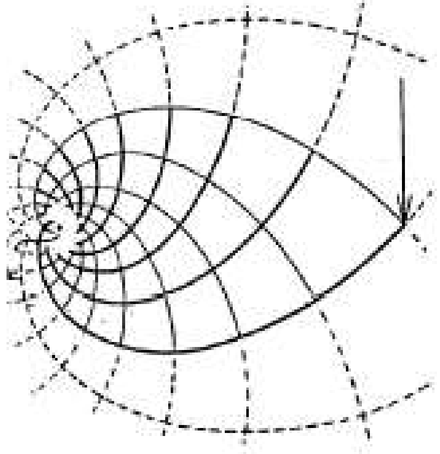
Topology Optimization: Concept and Theory

- History
- Concept: paradigm shift
  - extended design domain
  - Characteristic function
- Approach
  - Microstructure or Homogenization
  - Density or SIMP
- Problem formulation
  - Optimality conditions
  - Procedure
- Mean compliance vs. stress

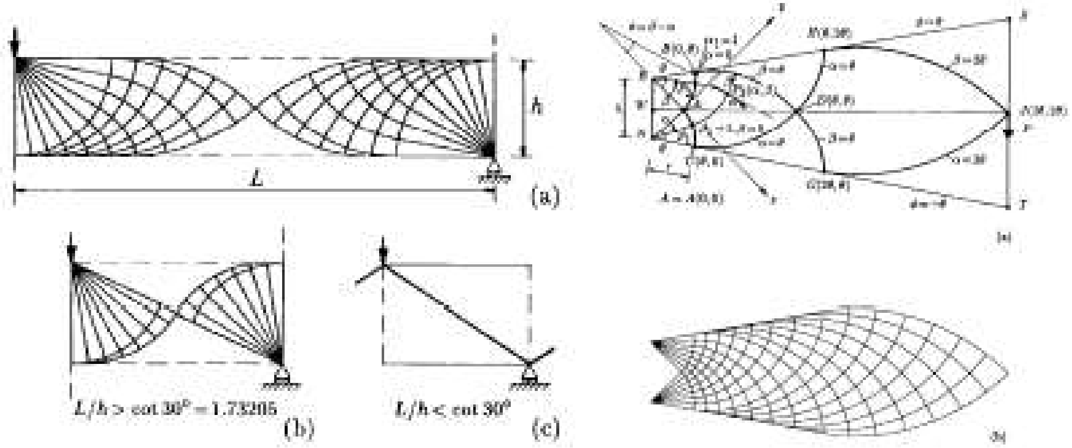
Structure's Topology

– Topology: a branch of mathematics concerned with those properties of geometric configurations (as point sets) which are unaltered by elastic deformations (as a stretching or a twisting) that are homeomorphisms
– Basic geometric form that remains invariant through stretching or twisting
– Altered only when two material point in a body join (close a hole) or when two are created from one (form a hole)
– Topology Optimization: best geometric configuration (i.e., number of holes, connectivity of bars, etc.) of a structure
  - Help determine the placement and number of holes and/or stiffening members
  - Provide a good starting point for further structural refinement via sizing and shape optimization
  - Beneficial early in the design cycle when least is known about the design and when design changes are easily accommodated

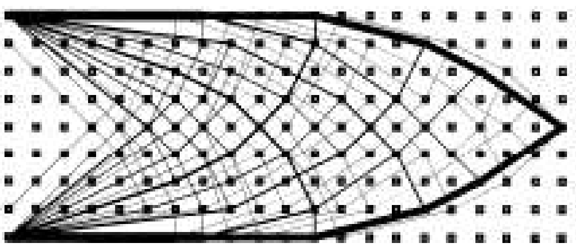
History of Topology Optimization (1)



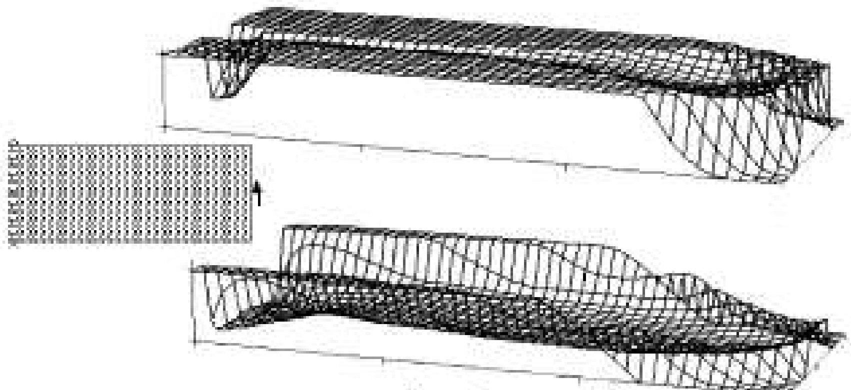
Michell ~1900



Prager/Hemp/Rozvany ~1960-2011

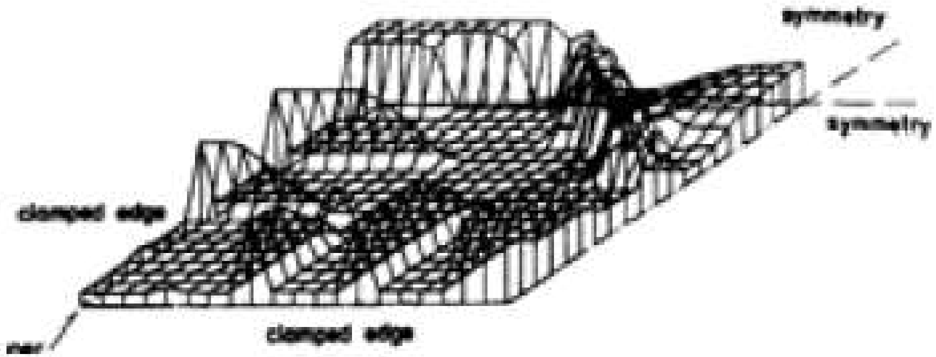


Dorn+Gomory/Fleron (1964)
Pedersen (1970)

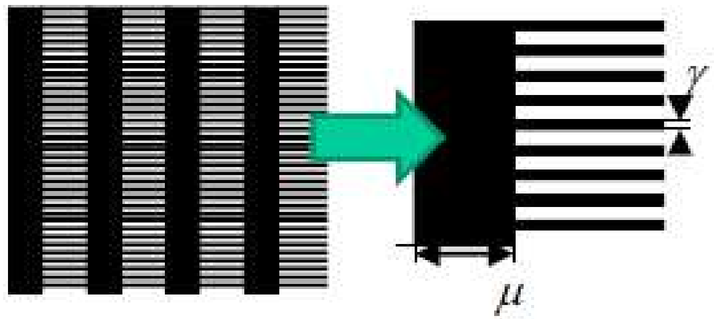


Cea and Malanowski, 1970
Armand 1980

History of Topology Optimization (2)

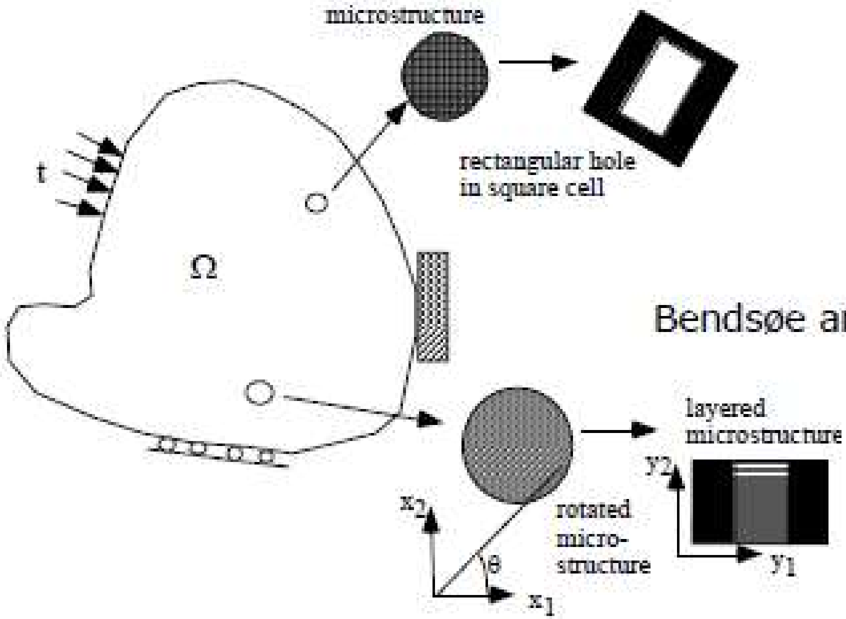


Cheng and Olhoff (1981)



$$E_{ijkl}^{H} = E_{ijkl}^{H}(\mu, \gamma)$$

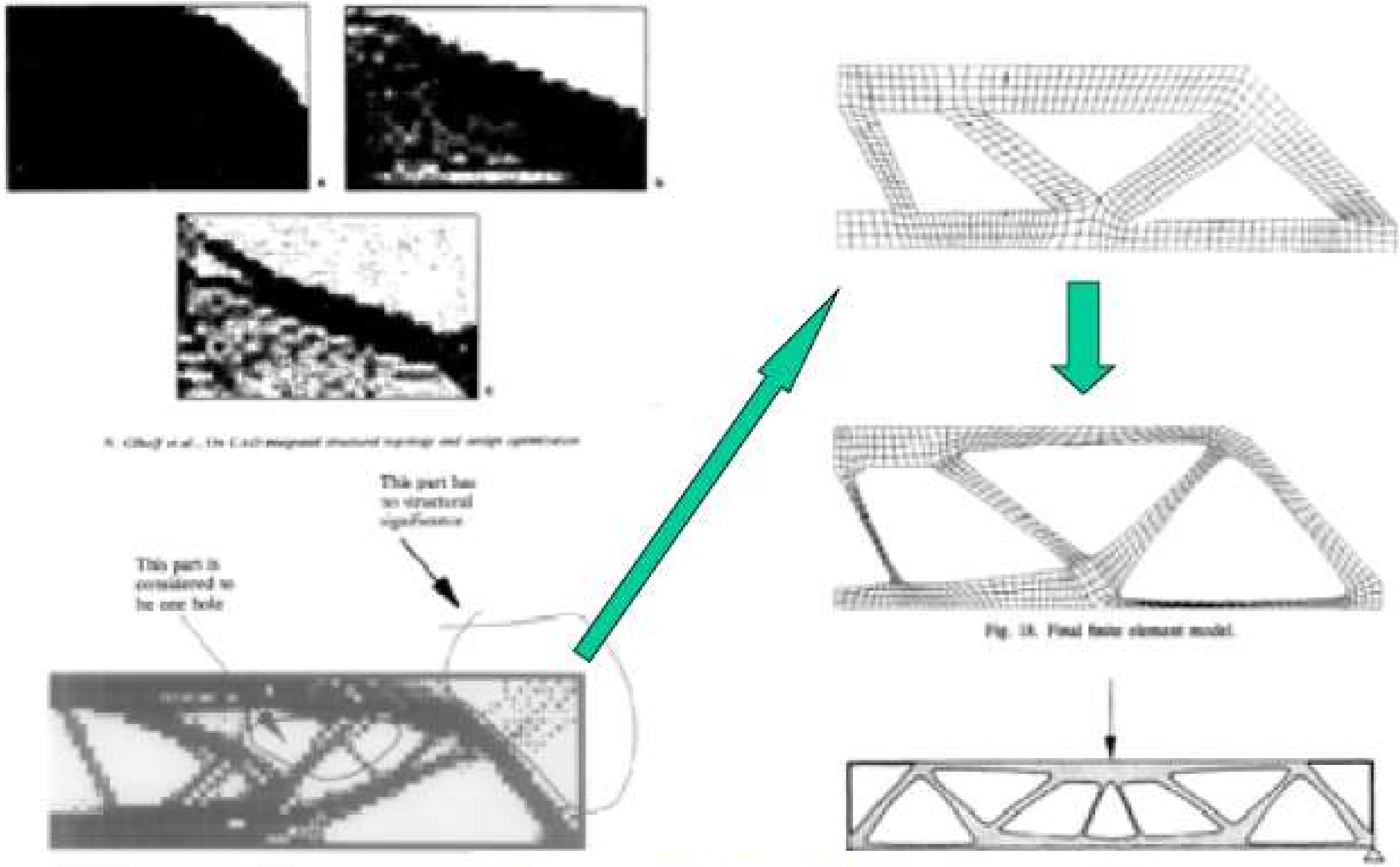
Tartar/Lurie/Cherkaev ~1970



Bendsøe and Kikuchi (1988)

History of Topology Optimization (3)

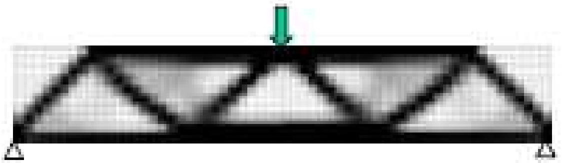
Homogenization-based approach by Bendsøe and Kikuchi (1988)



Olhoff, Bendsøe and Rasmussen (1991)

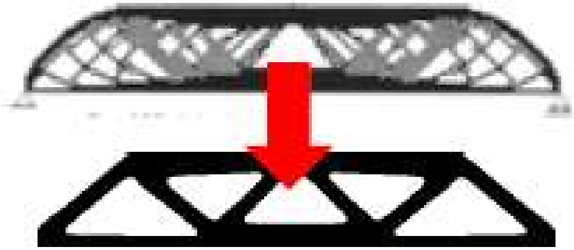
History of Topology Optimization (4)

"Density method"
"SIMP"



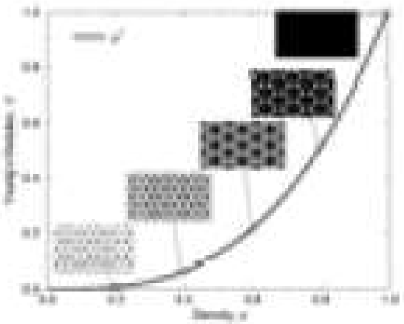
Bendsøe (1989)
Zhou and Rozvany (1991)
Meljnek (1992)

Mesh indep.
filtering



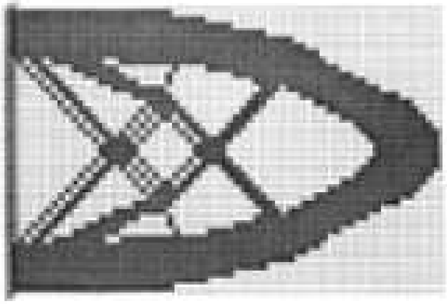
Sigmund (1994/1997)

Physical verification
of SIMP approach

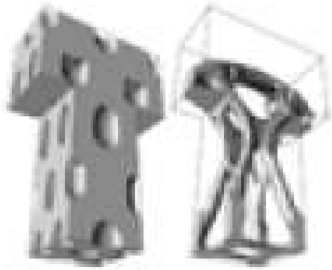


Bendsøe and Sigmund
(1999)

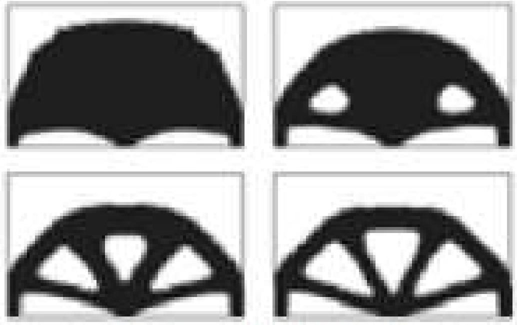
History of Topology Optimization (5)



Evolutionary structural optimization
Xie and Steven (1993)



Level-set approaches
Osher/Wang (2003)
Allaire (2002/2004)

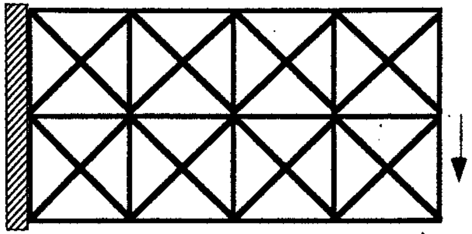


Topological derivatives
Allaire *et al.* (2005)

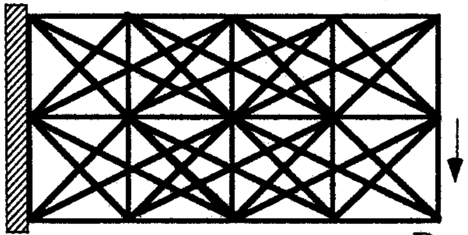
Discrete level set + top. deriv. approach
(Challis *et al.*, 2007 a.o.)

Ground Structures

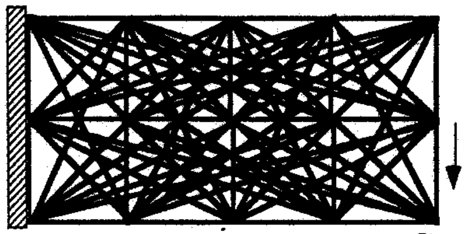
Truss ground structures of variable complexity



A



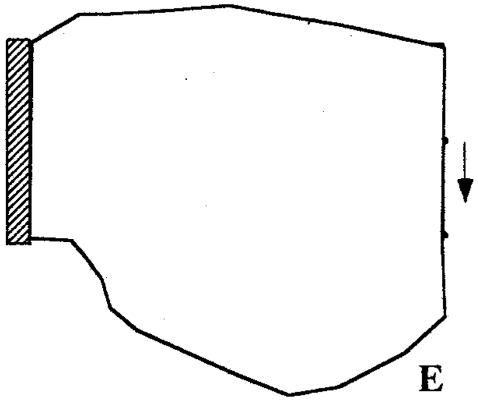
B



C

Continuum model design domains

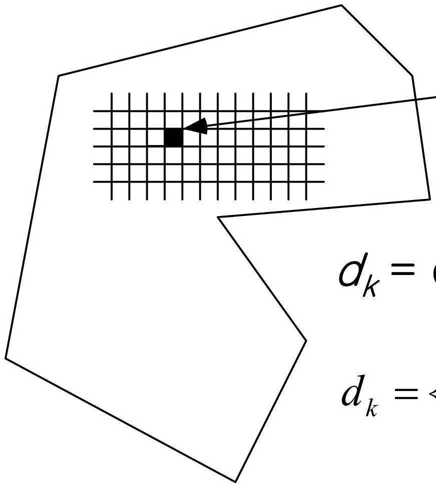
D

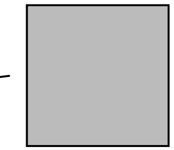


E

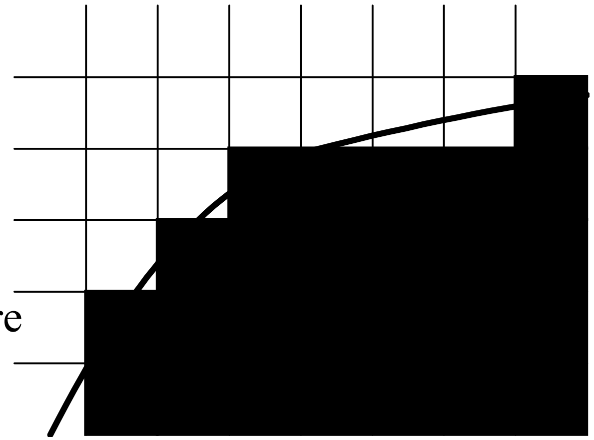
Paradigm Change

– Shape is recognized by a set of pixels





Gray Scale = Density



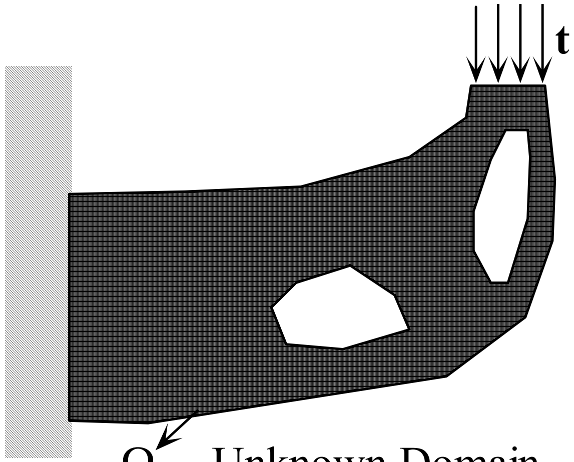
d_k = density of k–th pixel/voxel

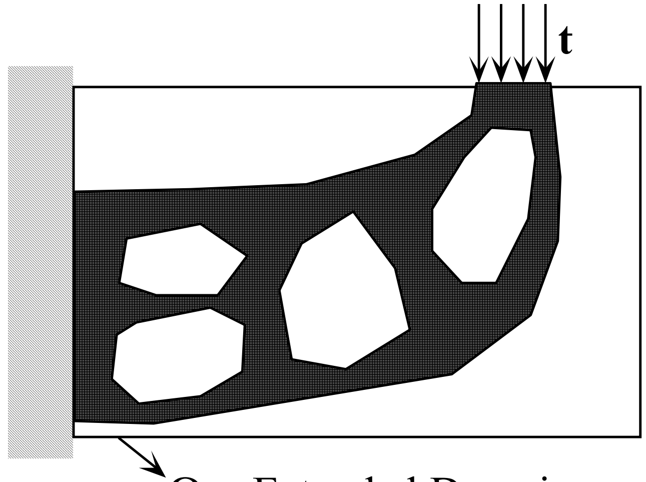
$$d_k = \begin{cases} 1 & \text{if occupied by solid structure} \\ \alpha & \text{if structure is perforated} \\ 0 & \text{if no structure is placed} \end{cases}$$

– Design optimization is completely decoupled with any sort of mesh adaptation

– Shape and topology design variables are transformed into the density of material or elasticity matrix of material which is assigned in each finite element of a fixed FE model, at least a fixed FE mesh generated at the initial time.

Extended Fixed Domain (1)



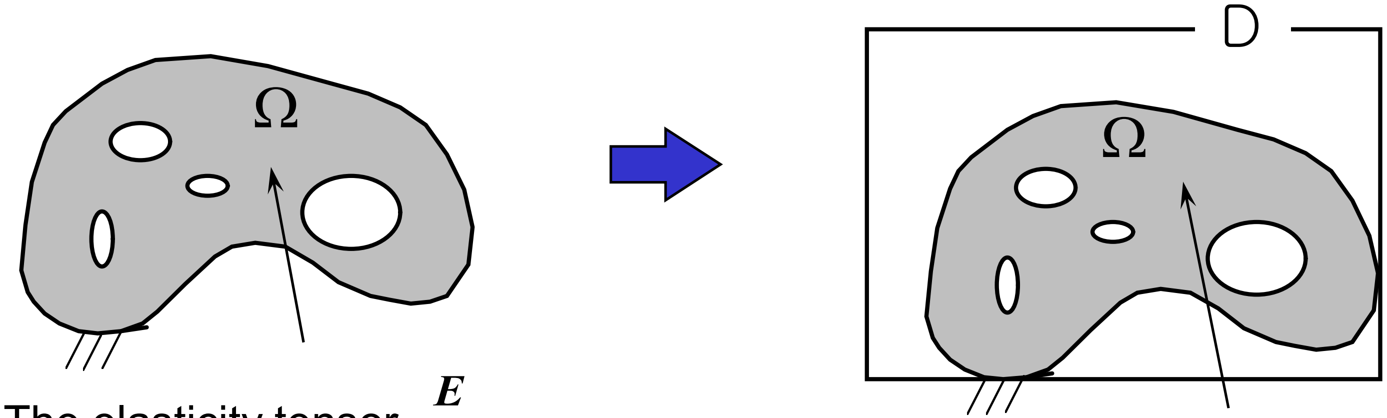


Old approach: Find the boundaries of the unknown structure
(Zienkiewicz and Campbell, 1973)

New approach: Find the material distribution in the extended fixed domain
(Bendsøe and Kikuchi, 1988)

Extended Fixed Domain (2)

- Extension Ω to the fixed domain D
  - $D \supset \Omega$ is the extended design domain that is fixed and known a priori
  - Structural optimization → optimal material distribution



The elasticity tensor $\boldsymbol{E}$

$\chi_\Omega \boldsymbol{E} \in L^\infty(\Omega)$
very discontinuous

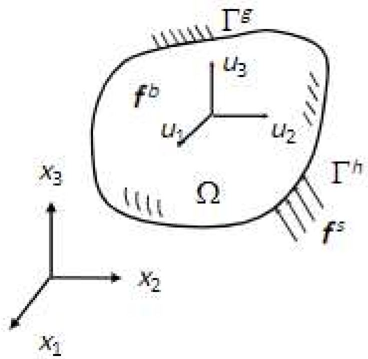
- Characteristic function

$$\chi_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

  - Internal virtual work: $\int_\Omega \varepsilon(v)^T E \varepsilon(u) d\Omega = \int_D \varepsilon(v)^T \underbrace{\chi_\Omega E}_{\text{new material constants}} \varepsilon(u) d\Omega$

Linear Elasticity (←Theory of Elasticity ←Continuum Mechanics)

– mathematical model of how solid objects deform and become internally stressed due to prescribed loading conditions
– Strong form



https://www.continuummechanics.org/

Equation/Form	Cartesian Coordinate	Tensor (coordinate independent)
Compatibility (Strain-Displacement)	$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$	$\boldsymbol{\varepsilon} = \frac{\mathbf{1}}{\mathbf{2}}[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\boldsymbol{T}}]$
Constitutive (Hooke's Law)	$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$	$\boldsymbol{\sigma} = \boldsymbol{E}:\boldsymbol{\varepsilon}$
Equilibrium (motion)	$\sigma_{ij,i} + f_j = \rho\partial_{tt}u_j$	$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{\rho\ddot{u}}$

Principle of Minimum Potential Energy

– For all displacements that satisfy the boundary conditions, known as kinematically admissible displacements, those which satisfy the boundary-valued problem make the total potential energy stationary on solution space

– Stationary condition ←→ first variation = 0

$$\boldsymbol{\Pi}(\boldsymbol{u}) = \boldsymbol{U}(\boldsymbol{u}) - \boldsymbol{W}(\boldsymbol{u}) = \frac{\mathbf{1}}{\mathbf{2}}\iint_{\boldsymbol{\Omega}} \boldsymbol{\sigma}:\boldsymbol{\varepsilon}\, d\boldsymbol{\Omega} - \iint_{\boldsymbol{\Omega}} \boldsymbol{u}\cdot\boldsymbol{f}\, d\boldsymbol{\Omega} - \int_{\boldsymbol{\Gamma}} \boldsymbol{u}\cdot\boldsymbol{t}\, d\boldsymbol{\Gamma}$$

$$\rightarrow \boldsymbol{\delta\Pi}(\boldsymbol{u},\boldsymbol{v}) = \mathbf{0}$$

– Virtual displacement: small arbitrary perturbation (variation) of real displacement

$$\boldsymbol{\delta u} = \lim_{\tau\to 0}\frac{\mathbf{1}}{\boldsymbol{\tau}}[(\boldsymbol{u}+\boldsymbol{\tau\eta}) - \boldsymbol{u}] = \frac{\boldsymbol{d}}{\boldsymbol{d\tau}}(\boldsymbol{u}+\boldsymbol{\tau\eta})\Big|_{\boldsymbol{\tau}=\mathbf{0}} = \boldsymbol{\eta} \equiv \boldsymbol{v} \in \boldsymbol{H}^{\mathbf{1}}(\boldsymbol{\Omega})$$

$$H^1(\Omega) = \left\{\boldsymbol{v}: \Omega \rightarrow R \,\middle|\, \int_\Omega \boldsymbol{v}\cdot\boldsymbol{v}\, d\Omega < +\infty, \int_\Omega \boldsymbol{\partial v}\cdot\boldsymbol{\partial v}\, d\Omega < +\infty\right\}$$

$$\boldsymbol{\delta}\left(\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}}\right) = \frac{\boldsymbol{\partial}(\boldsymbol{\delta u})}{\boldsymbol{\partial x}} = \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}} \rightarrow \boldsymbol{\delta\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\delta\sigma} = \boldsymbol{E}:\boldsymbol{\varepsilon}(\boldsymbol{v})$$

– Variation of strain energy: energy bilinear form

$$\boldsymbol{\delta U(u,v) = \frac{1}{2}\iint_{\Omega} [\delta\sigma : \varepsilon + \sigma : \delta\varepsilon] d\Omega = \frac{1}{2}\iint_{\Omega} [\varepsilon(v) : E : \varepsilon(u) + \varepsilon(u) : E : \varepsilon(v)] d\Omega}$$

$$\boldsymbol{= \iint_{\Omega} \varepsilon(v) : E : \varepsilon(u) d\Omega = a(u,v)}$$

– Variation of work done by applied loads: load linear form

$$\boldsymbol{\delta W(u,v) = \iint_{\Omega} \delta u \cdot f d\Omega - \int_{\Gamma} \delta u \cdot t d\Gamma = \iint_{\Omega} v \cdot f d\Omega - \int_{\Gamma} v \cdot t d\Gamma = l(v)}$$

– Thus, principle of minimum potential energy becomes

$$\boldsymbol{\delta\Pi(u,v) = \delta U(u,v) - \delta W(u,v) = a(u,v) - l(v) = 0 \rightarrow a(u,v) = l(v)}$$

Principle of Virtual Work

- Instead of solving the strong form directly, we want to solve the equation with relaxed requirement (weak form)
- When a system is in equilibrium, the forces applied to the system will not produce any virtual work for arbitrary virtual displacements
- If the virtual work becomes zero for arbitrary virtual displacement, then it satisfied the original equilibrium equation in a weak sense

$$\iint_{\Omega} \left(\sigma_{ij,i} + f_j\right) v_j d\Omega = 0$$

→ Integration by parts

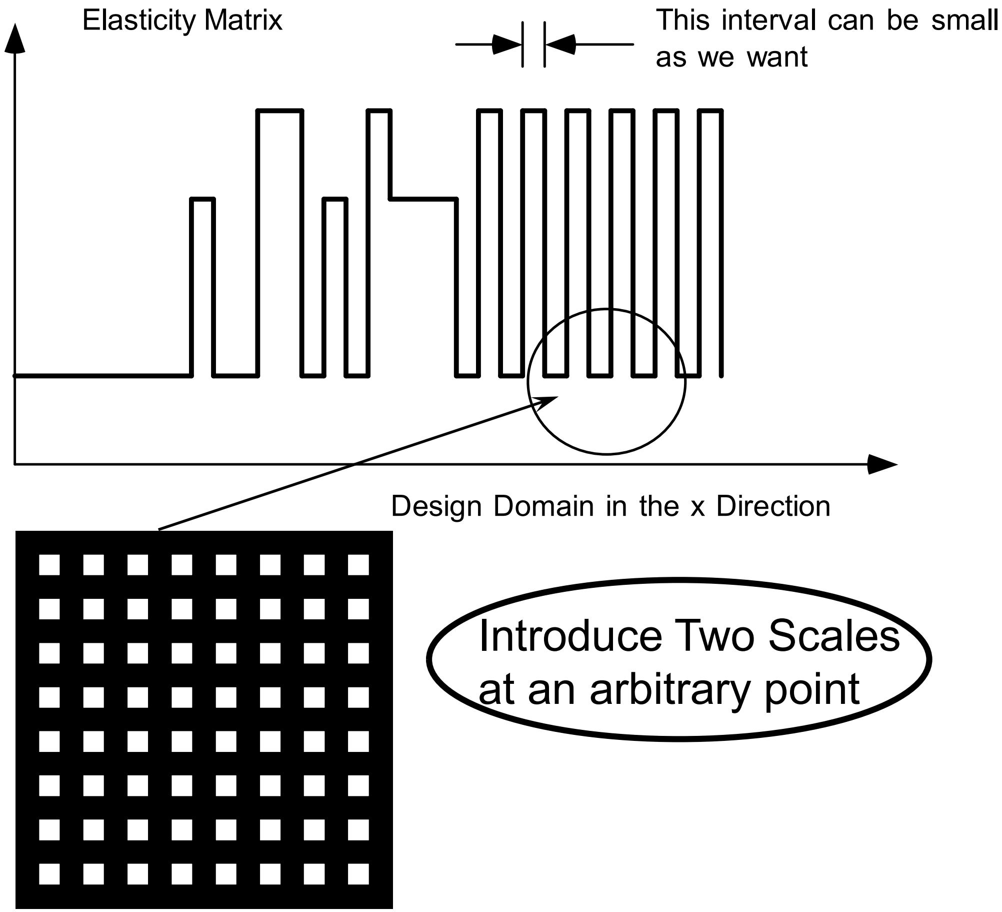
→ Divergence theorem

$$\iint_{\Omega} \sigma_{ij} v_{j,i} d\Omega = \iint_{\Omega} f_j v_j d\Omega + \int_{\Gamma} t_j v_j d\Gamma$$

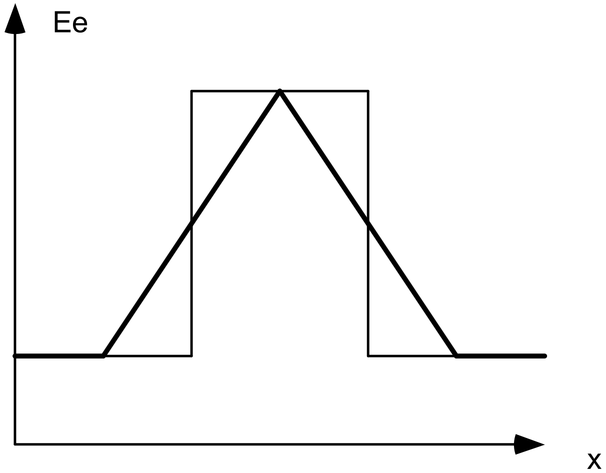
$$\boldsymbol{a(u, v) = l(v)}$$

Characteristic Function

– Very Rapidly Varying Function $\chi_\Omega E$ cannot be approximated by a differentiable function of position *x* in the standard way



possibly rapidly varying



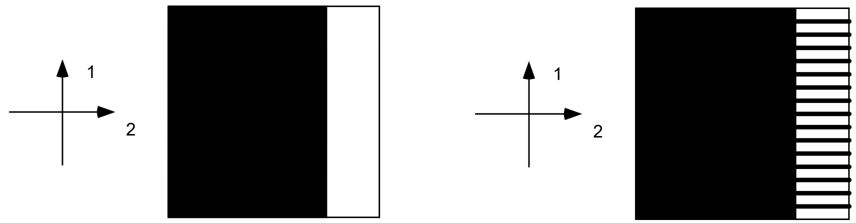
No way to approximate by a standard way with a distribution function

Many Choices

- The key feature is an approximation of the extended elasticity matrix $\chi_{\Omega}E \in L^{\infty}(D)$

$$L^{\infty}(\Omega) = \{f: \Omega \rightarrow R \mid \exists M > 0, |f| \leq M \ alomost\ everywhere\ in\ \Omega\}$$

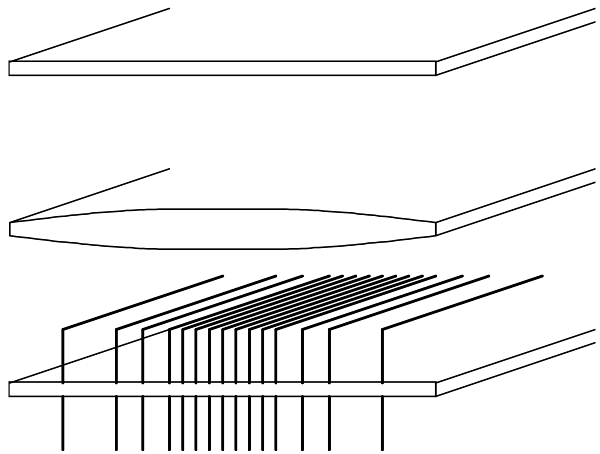
- There are infinitely many ways to approximate this by using
  - Generalized porous media constitutive equations (bio-mechanics, Geo-mechanics)
  - Power law of density / elasticity constants: $\chi_{\Omega}E \approx \rho^{p}E$
  - Rank 1 & Rank 2 orthotropic materials: $\chi_{\Omega}E \approx E^{H}$



  - Others

Relaxation

- Possible Approximation: Homogenization
  - Introduce the two scales $\left(x, y = \frac{x}{\varepsilon}\right)$ and the micro-scale perforation, and then $\chi_\Omega E$ is approximated by the homogenized average elasticity matrix E^H
- Heaviside function: $H(x) = \begin{cases} 1 \ (x \geq 0) \\ 0 \ (x < 0) \end{cases} \rightarrow H_\xi(x) = \begin{cases} 1 \ (x > \xi) \\ \frac{x}{2\xi} + \frac{1}{2} \ (|x| \leq \xi) \\ 0 \ (x < -\xi) \end{cases}$
- G. Cheng and N. Olhoff (1981)
  - in plate thickness optimization
  - smoothly varying thickness is not optimum
  - optimum involves rapidly changing ribs
  - Homogenization is required

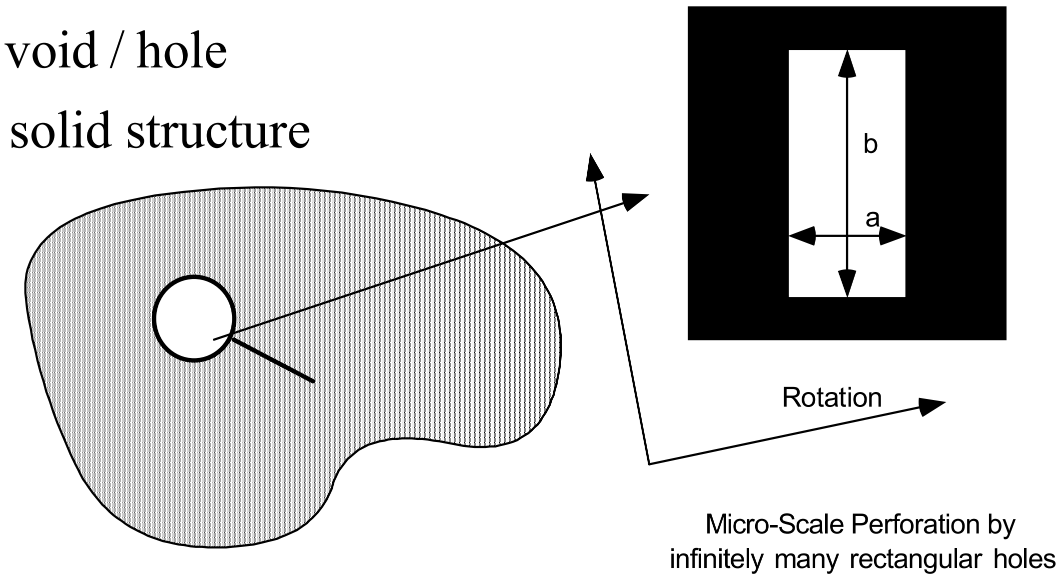


Mathematical Background

- Lurier, Cherkaev, Fedrov (1981): G-convergence (average sense)
- Kohn and Strang (1984): microscale perforation and specialized variational principle
- Murat and Tartar (1983): homogenization theory

$$a = b = 1 \quad \Leftrightarrow \quad \text{void / hole}$$

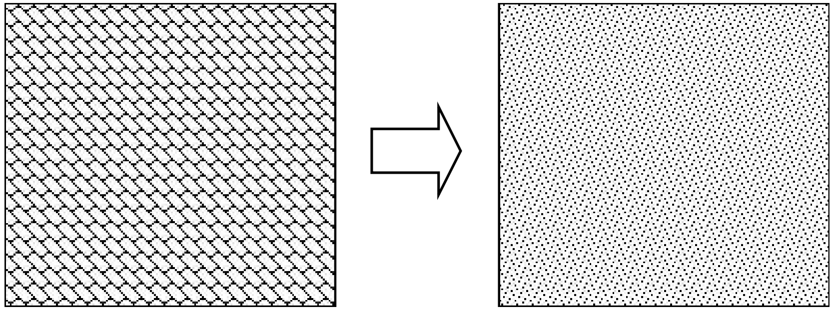
$$a = b = 0 \quad \Leftrightarrow \quad \text{solid structure}$$



Bendsøe and Kikuchi: Design Variables are (a, b, θ) at every where

Why Homogenization ?

– Shape design can be transformed into design of material constants (material distribution over a fixed design domain)

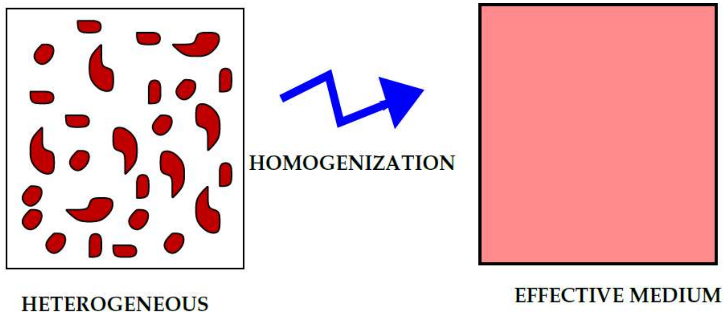


– If the exact heterogeneity is used in mechanics, we must introduce so fine finite elements to represent all the detail. This is a difficult task.

– This idealization is regarded as the homogenization in theoretical mechanics.

Homogenization Method → Optimal Design?

– A kind of averaging methods for partial differential equations
– Commonly used to determine the averaged (or effective or homogenized, or equivalent, or macroscopic) parameters of a heterogeneous medium



– Based on the concept of relaxation
  - makes ill-posed problems well-posed by enlarging the space of admissible “shapes”
  - think of generalized shapes as limits of minimizing sequences of classical shapes
  - allows, as admissible shapes, composite materials obtained by micro-perforation of the original material (fine mixtures of material and void)

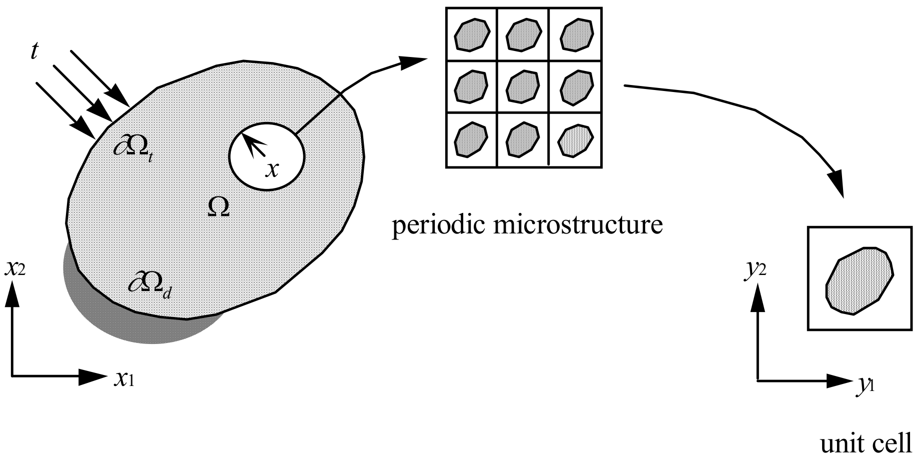
Theory of Homogenization (1): Lions (1981)

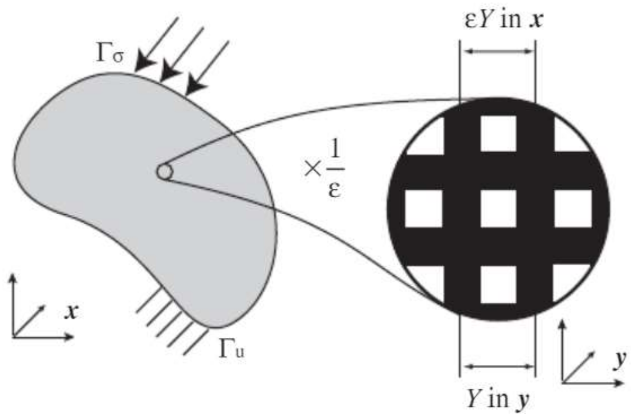
– Microstructure and asymptotic expansion of the displacement field of a composite

$$u^{\varepsilon}(x) = u(x,y) = u^{0}(x) + \varepsilon u^{1}(x,y) + \varepsilon^{2} u^{2}(x,y) + \cdots, \qquad y = \frac{x}{\varepsilon}$$

$$\xrightarrow{\frac{\partial}{\partial x_i}\left(\varphi(x,y)\big|_{y=\frac{x}{\varepsilon}}\right) = \frac{\partial \varphi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial y_i}} \varepsilon(u^{\varepsilon}) \approx \underbrace{\varepsilon_x(u^{0})}_{\text{strains due to the average displacement over the unit cell}} + \underbrace{\varepsilon_y(u^{1})}_{\text{strains due to the first order variation of the displacement from the averge one}} + \varepsilon\varepsilon_x(u^{1})$$

$$\varepsilon(u^{\varepsilon})^{T} = \left[\frac{\partial u_1^{\varepsilon}}{\partial x_1} \quad \frac{\partial u_2^{\varepsilon}}{\partial x_2} \quad \frac{\partial u_3^{\varepsilon}}{\partial x_3} \quad \frac{\partial u_2^{\varepsilon}}{\partial x_3} + \frac{\partial u_3^{\varepsilon}}{\partial x_2} \quad \frac{\partial u_3^{\varepsilon}}{\partial x_1} + \frac{\partial u_1^{\varepsilon}}{\partial x_3} \quad \frac{\partial u_1^{\varepsilon}}{\partial x_2} + \frac{\partial u_2^{\varepsilon}}{\partial x_1}\right]^{T}$$





Theory of Homogenization (2)

– Total potential energy

$$F\left(v^{\varepsilon}\right)=\frac{1}{2}\int_{\Omega}\varepsilon_x\left(v^{\varepsilon}\right)^T E^{\varepsilon}\varepsilon_x\left(u^{\varepsilon}\right)d\Omega-\int_{\Omega}\left[\varepsilon_x\left(v^{\varepsilon}\right)^T\sigma_0+\left(v^{\varepsilon}\right)^T\rho f\right]d\Omega-\int_{\partial\Omega_t}\left(v^{\varepsilon}\right)^T t d\partial\Omega$$

$$\approx\frac{1}{2}\int_{\Omega}\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)+\varepsilon\varepsilon_x\left(v^1\right)\right]^T E^{\varepsilon}\left[\varepsilon_x\left(u^0\right)+\varepsilon_y\left(u^1\right)+\varepsilon\varepsilon_x\left(u^1\right)\right]d\Omega$$

$$-\int_{\Omega}\left\{\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)+\varepsilon\varepsilon_x\left(v^1\right)\right]^T\sigma_0+\left(v^0+\varepsilon v^1\right)^T\rho f\right\}d\Omega-\int_{\partial\Omega_t}\left(v^0+\varepsilon v^1\right)^T t d\partial\Omega$$

$$=\frac{1}{2}\int_{\Omega}\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)\right]^T E^{\varepsilon}\left[\varepsilon_x\left(u^0\right)+\varepsilon_y\left(u^1\right)\right]d\Omega$$

$$-\int_{\Omega}\left\{\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)\right]^T\sigma_0+\left(v^0\right)^T\rho f\right\}d\Omega-\int_{\partial\Omega_t}\left(v^0\right)^T t d\partial\Omega+O(\varepsilon)$$

$$\text{for a periodic function: } \lim_{\varepsilon\to 0}\int_{\Omega}\Phi^{\varepsilon}(x)d\Omega=\int_{\Omega}\frac{1}{|Y|}\int_Y\Phi(x,y)dYd\Omega,\qquad \Phi^{\varepsilon}(x)=\Phi\left(x,\frac{x}{\varepsilon}\right)$$

$$F\left(v^0,v^1\right)=\frac{1}{2}\int_{\Omega}\frac{1}{|Y|}\int_Y\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)\right]^T E(x,y)\left[\varepsilon_x\left(u^0\right)+\varepsilon_y\left(u^1\right)\right]dYd\Omega$$

$$-\int_{\Omega}\frac{1}{|Y|}\int_Y\left\{\left[\varepsilon_x\left(v^0\right)+\varepsilon_y\left(v^1\right)\right]^T\sigma_0(x,y)+\left(v^0\right)^T\rho(x,y)f\right\}dYd\Omega-\int_{\partial\Omega_t}\left(v^0\right)^T t d\partial\Omega$$

Theory of Homogenization (3)

– Equilibrium equation of the macrostructure

$$\int_\Omega \frac{1}{|Y|}\int_Y \left[\varepsilon_x\left(v^0\right)\right]^T E(x,y)\left[\varepsilon_x\left(u^0\right)+\varepsilon_y\left(u^1\right)\right] dY d\Omega$$

$$= \int_\Omega \frac{1}{|Y|}\int_Y \left\{\left[\varepsilon_x\left(v^0\right)\right]^T \sigma_0(x,y) + \left(v^0\right)^T \rho(x,y) f\right\} dY d\Omega - \int_{\partial\Omega_t} \left(v^0\right)^T t\, d\partial\Omega \qquad \forall v^0$$

– Equilibrium equation of the microstructure

$$\int_\Omega \frac{1}{|Y|}\int_Y \left[\varepsilon_y\left(v^1\right)\right]^T \left\{E(x,y)\left[\varepsilon_x\left(u^0\right)+\varepsilon_y\left(u^1\right)\right] - \sigma_0(x,y)\right\} dY d\Omega = 0 \qquad \forall v^1$$

- Relationship between first order variation and macroscopic strains

$$u^1(x,y) = -\chi(x,y)\varepsilon_x\left(u^0(x)\right) + \Psi(x,y)$$

- χ_1 is the displacement field of the unit cell when it is subject to the unit macroscopic strain $\varepsilon_{x_1}\left(u^{(0)}\right) = 1$ while other components are zero.
- Characteristic displacements of the unit cell

Theory of Homogenization (4)

– Equilibrium equation of the microstructure

$$\int_{\Omega}\frac{1}{|Y|}\int_{Y}\left[\varepsilon_{y}\left(v^{1}\right)\right]^{T}\left\{E(x,y)\left[I-\varepsilon_{y}\left(\chi(x,y)\right)\right]-\sigma_{0}(x,y)\right\}dY\varepsilon_{x}\left(u^{0}\right)d\Omega$$

$$+\int_{\Omega}\frac{1}{|Y|}\int_{Y}\left[\varepsilon_{y}\left(v^{1}\right)\right]^{T}\left\{E(x,y)_{y}\left(\Psi(x,y)\right)-\sigma_{0}(x,y)\right\}dYd\Omega=0 \quad \forall v^{1}$$

$$E^{H}(x)=\frac{1}{|Y|}\int_{Y}E(x,y)\left[I-\varepsilon_{y}\left(\chi(x,y)\right)\right]dY$$

$$\sigma_{0}{}^{H}(x)=\frac{1}{|Y|}\int_{Y}\left\{\sigma_{0}(x,y)-E(x,y)_{y}\left(\Psi(x,y)\right)\right\}dY$$

$$\rho^{H}(x)=\frac{1}{|Y|}\int_{Y}\rho(x,y)dY$$

– Equilibrium equation of the macrostructure

$$\int_{\Omega}\varepsilon_{x}\left(v^{0}\right)^{T}E^{H}(x)\varepsilon_{x}\left(u^{0}\right)d\Omega=\int_{\Omega}\left[\varepsilon_{x}\left(v^{0}\right)^{T}\sigma_{0}{}^{H}(x)+\left(v^{0}\right)^{T}\rho^{H}(x)f\right]d\Omega+\int_{\partial\Omega_{t}}\left(v^{\varepsilon}\right)^{T}td\partial\Omega \quad \forall v^{0}$$

Homogenization Process

- Unit cell problem for computing characteristic displacements

$$\int_Y \left[\varepsilon_y\left(v^1\right)\right]^T E(x,y)\varepsilon_y\left(\chi(x,y)\right)dY = \int_Y \left[\varepsilon_y\left(v^1\right)\right]^T E(x,y)dY \quad \forall v^1$$

$$\int_Y \left[\varepsilon_y\left(v^1\right)\right]^T E(x,y)\varepsilon_y\left(\Psi(x,y)\right)dY = \int_Y \left[\varepsilon_y\left(v^1\right)\right]^T \sigma_0(x,y)dY \quad \forall v^1$$

- Compute the homogenized elasticity matrix

$$E^H(x) = \frac{1}{|Y|}\int_Y E(x,y)\left[I - \varepsilon_y\left(\chi(x,y)\right)\right]dY, \quad \sigma_0{}^H(x) = \frac{1}{|Y|}\int_Y \left\{\sigma_0(x,y) - E(x,y)_y\left(\Psi(x,y)\right)\right\}dY$$

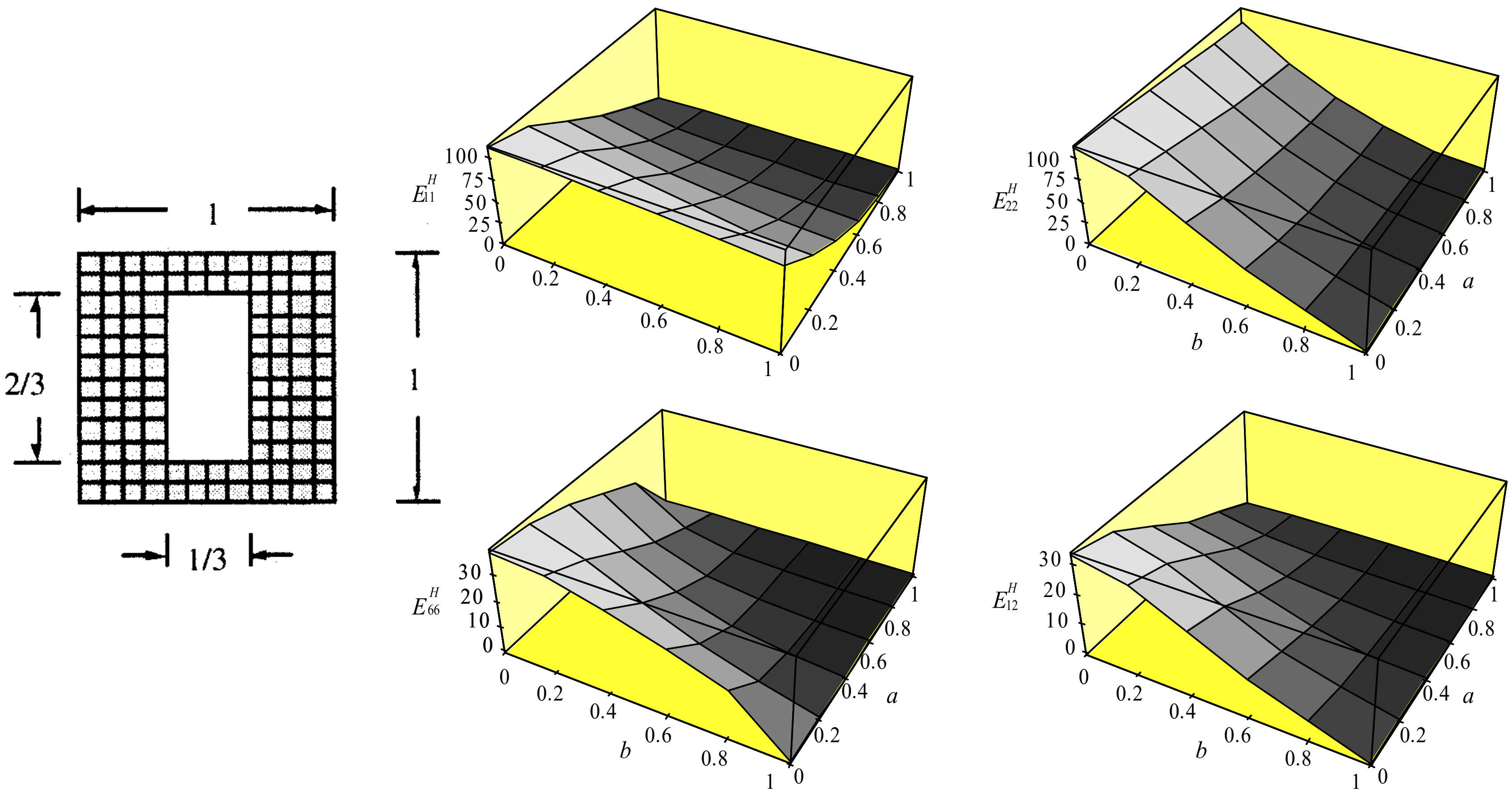
- Solve the macroscopic homogenized problem

$$\left.\int_\Omega \varepsilon_x\left(v^0\right)^T E^H(x)\varepsilon_x\left(u^0\right)d\Omega = \int_\Omega \left[\varepsilon_x\left(v^0\right)^T \sigma_0{}^H(x) + \left(v^0\right)^T \rho^H(x) f\right]d\Omega + \int_{\partial\Omega_t}\left(v^\varepsilon\right)^T t d\partial\Omega \quad \forall v^0\right\} \rightarrow u^0$$

- Compute strains and stresses in the unit cell at an arbitrary point

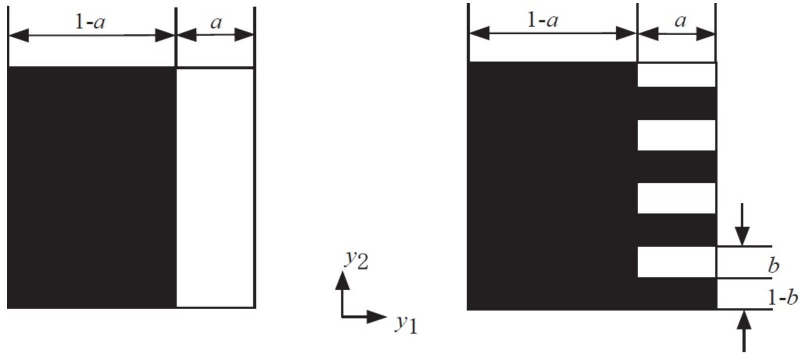
$$u^1(x,y) = -\chi(x,y)\varepsilon_x\left(u^0(x)\right) + \Psi(x,y)$$

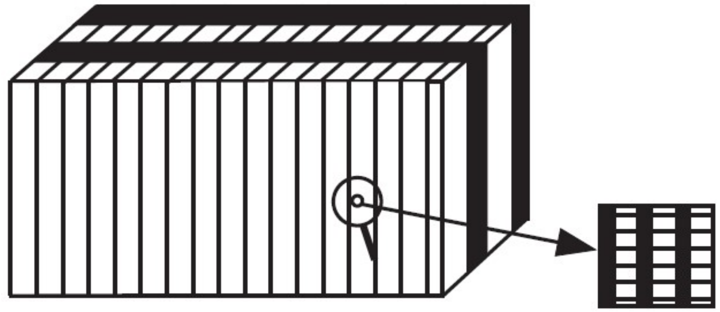
Homogenized Elasticity Tensor



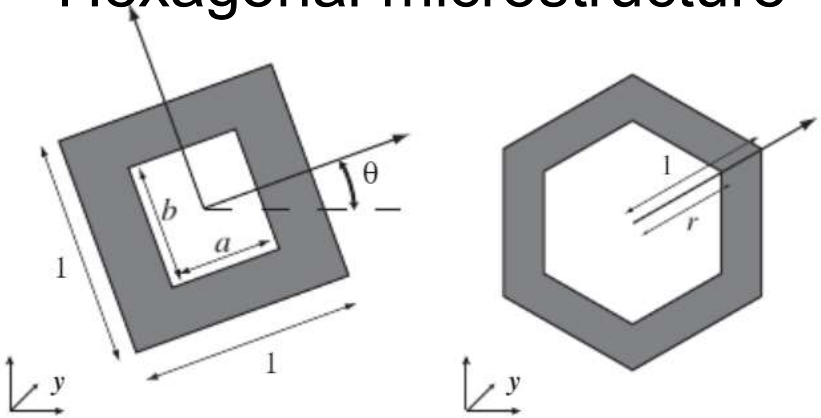
Homogenized Material Properties

- Layered microstructure





- Rectangular microstructure
  - Rectangular hole is very close to Rank 2 materials
- Hexagonal microstructure



INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2004; **59**:1925–1944 (DOI: 10.1002/nme.945)

Continuous approximation of material distribution for topology optimization

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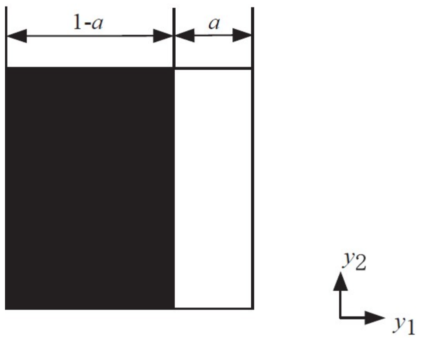
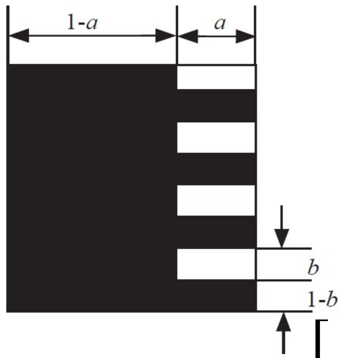
Layered Microstructures

– No resistance in shear → rotate to the principal stress direction (upper bound)

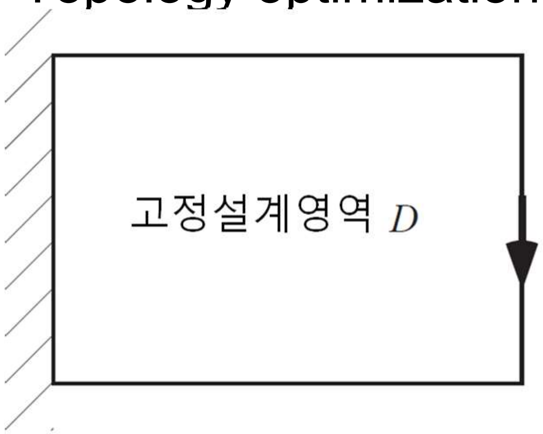


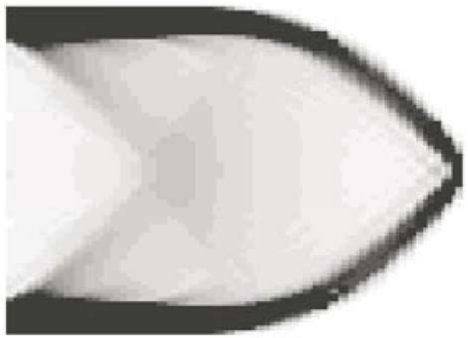


$$E^H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-a)E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

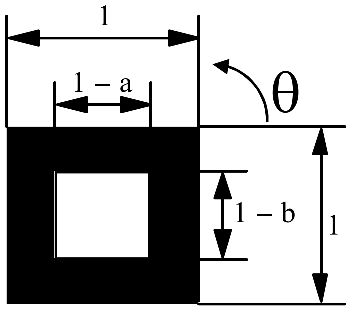
$$E^H = \frac{E}{(1-\nu^2)(1-ab)} \begin{bmatrix} (1-b) & \nu(1-b) & 0 \\ \nu(1-b) & \begin{matrix}(1-\nu^2)(1-ab)(1-a) \\ +\nu^2(1-b)\end{matrix} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

– Topology optimization → grayscale? bone structure

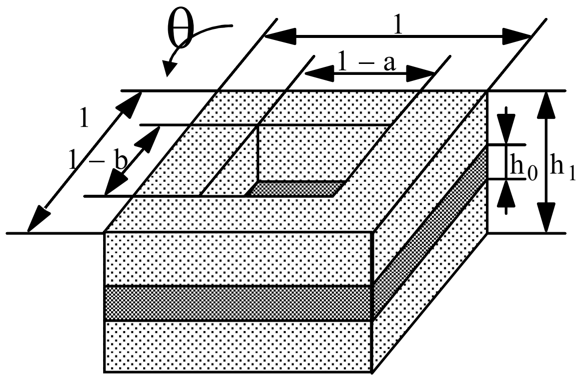




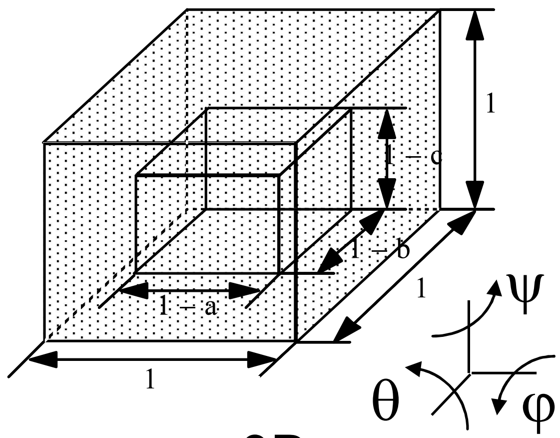
Rectangular Microstructures



2D



Plate/Shell

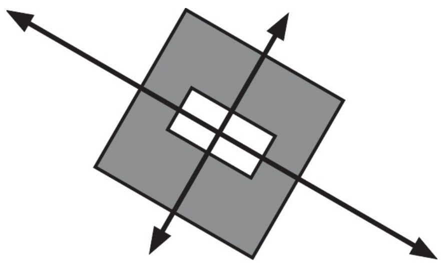


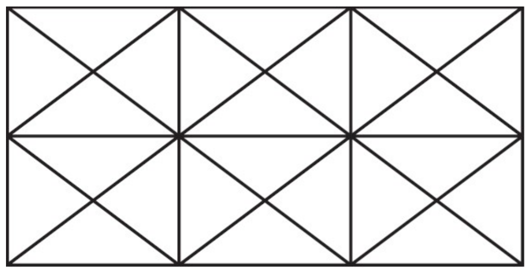
3D

$$D = \{a, b\}, \Theta = \{\theta\}$$
$$\rho = \rho_0(a + b - ab)$$

$$D = \{a, b\}, \Theta = \{\theta\}$$
$$\rho = \rho_0[h_0(a + b - ab)(h_1 - h_0)]$$

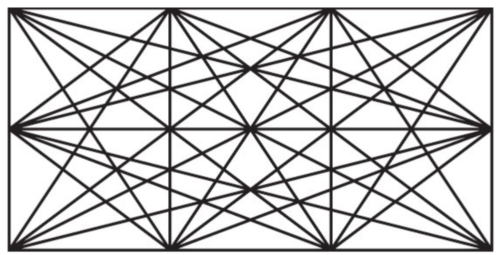
$$D = \{a, b, c\}, \Theta = \{\theta, \psi, \varphi\}$$
$$\rho = \rho_0[1 - (1 - a)(1 - b)(1 - c)]$$





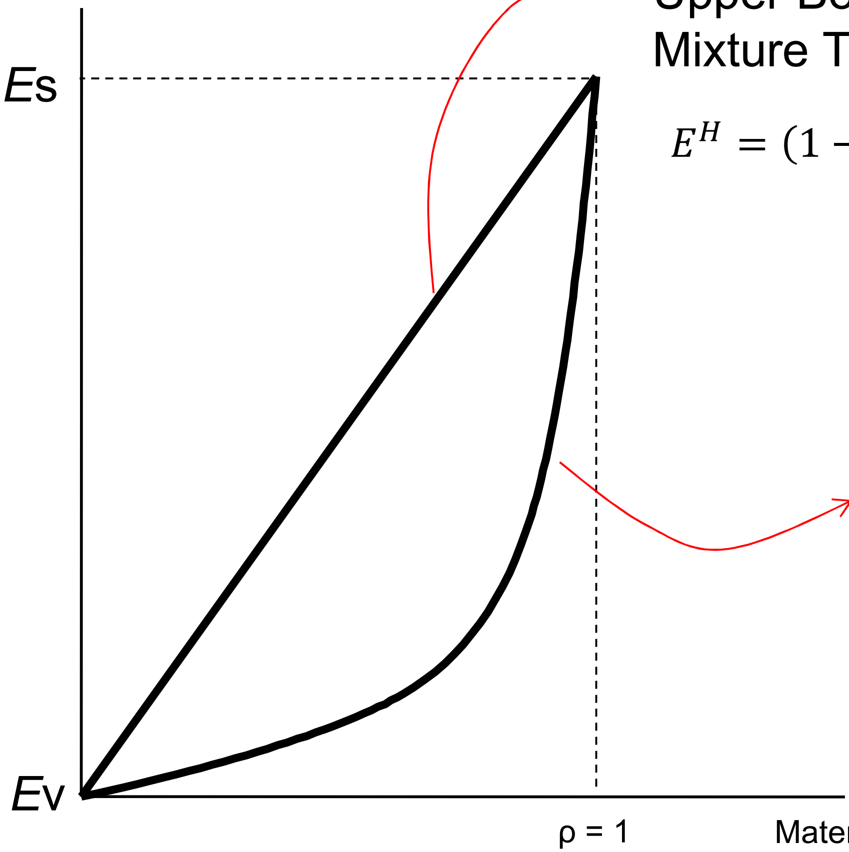
$$D = \{a, b\}$$

vs.



$$D = \{a, b\}, \Theta = \{\theta\}$$

Homogenized Average Young's Modulus



Upper Bound: uniaxial tension
Mixture Theory (1D)

$$E^H = (1-\rho)E_v + \rho E_s$$

$$\begin{aligned} \varepsilon &= \varepsilon_v = \varepsilon_s \\ \sigma &= (1-\rho)\sigma_v + \rho\sigma_s \\ E\varepsilon &= (1-\rho)E_v\varepsilon_v + \rho E_s\varepsilon_s \\ E &= (1-\rho)E_v + \rho E_s \end{aligned}$$

Lower Bound
Mixture Theory (1D)

$$E^H = \frac{1}{\dfrac{1-\rho}{E_v} + \dfrac{\rho}{E_s}}$$

$$\begin{aligned} \sigma &= \sigma_v = \sigma_s \\ \varepsilon &= (1-\rho)\varepsilon_v + \rho\varepsilon_s \\ \frac{\sigma}{E} &= (1-\rho)\frac{\sigma_v}{E_v} + \rho\frac{\sigma_s}{E_s} \\ E &= \left(\frac{1-\rho}{E_v} + \frac{\rho}{E_s}\right)^{-1} \end{aligned}$$

$$\begin{pmatrix} harmonic \\ mean \end{pmatrix} \le \begin{pmatrix} geometric \\ mean \end{pmatrix} \le \begin{pmatrix} arithmetic \\ mean \end{pmatrix}$$

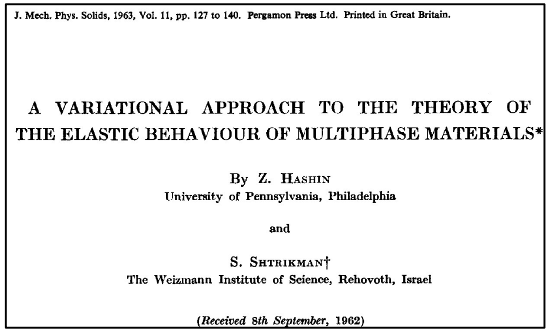
$$\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \le \sqrt{ab} \le \frac{a+b}{2}$$

Which microstructure is better?

- Hashin-Shtrikman bounds
  - Upper and lower bounds for the effective elastic moduli of quasi-isotropic and quasi-homogeneous multiphase materials

- Layered microstructure: upper bound
- Rectangular/Hexagonal microstructure: lower bound

- Material property of composite → upper bound
  - Stiff structure
- Topology optimization → lower bound
  - Separation of mixed materials



J. Mech. Phys. Solids, 1963, Vol. 11, pp. 127 to 140. Pergamon Press Ltd. Printed in Great Britain.

A VARIATIONAL APPROACH TO THE THEORY OF THE ELASTIC BEHAVIOUR OF MULTIPHASE MATERIALS*

By Z. Hashin
University of Pennsylvania, Philadelphia

and

S. Shtrikman†
The Weizmann Institute of Science, Rehovoth, Israel

(*Received 8th September*, 1962)

Without Microstructures: Density Approach

$$\boldsymbol{E}^H = f(\boldsymbol{E}, \rho) = f(\rho)\boldsymbol{E} = \begin{cases} \rho^p \boldsymbol{E} & \text{(SIMP)} \\ \dfrac{\rho}{1 + q(1-\rho)} \boldsymbol{E} & \text{(RAMP)} \end{cases}$$

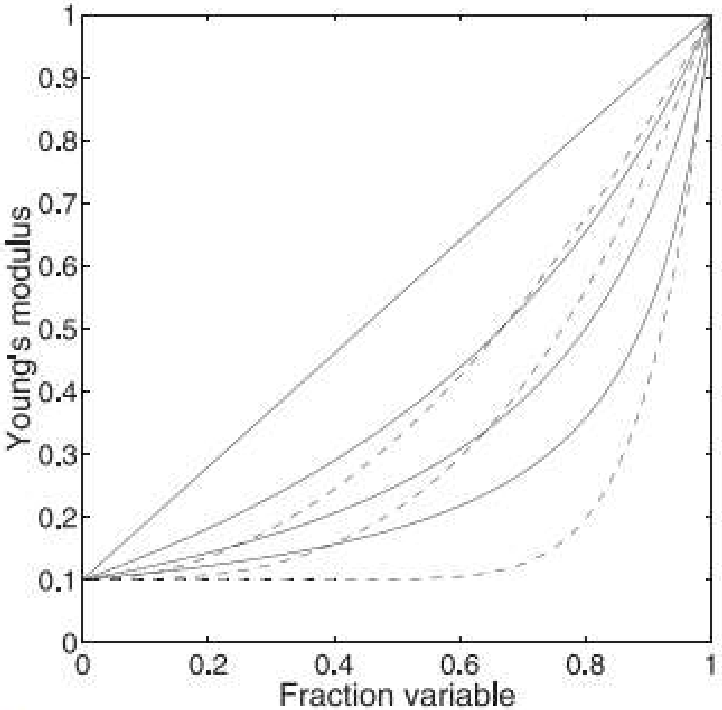
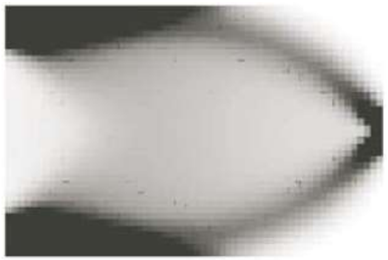


Fig. 1 A comparison of the SIMP model and the alternative model when $E_1 = 1$ and $E_0 = 0.1$. $E_p(x_j)$ for $p = 2, 3, 10$ are shown as dashed lines; $E_q(x_j)$ for $q = 0, 1.5, 4, 9$ are shown as solid lines

$$p \geq \begin{cases} max\left\{\dfrac{2}{1-\nu}, \dfrac{4}{1+\nu}\right\} & \text{(2D)} \\ max\left\{15\dfrac{1-\nu}{7-5\nu}, \dfrac{3}{2}\dfrac{1-\nu}{1-2\nu}\right\} & \text{(3D)} \end{cases}$$

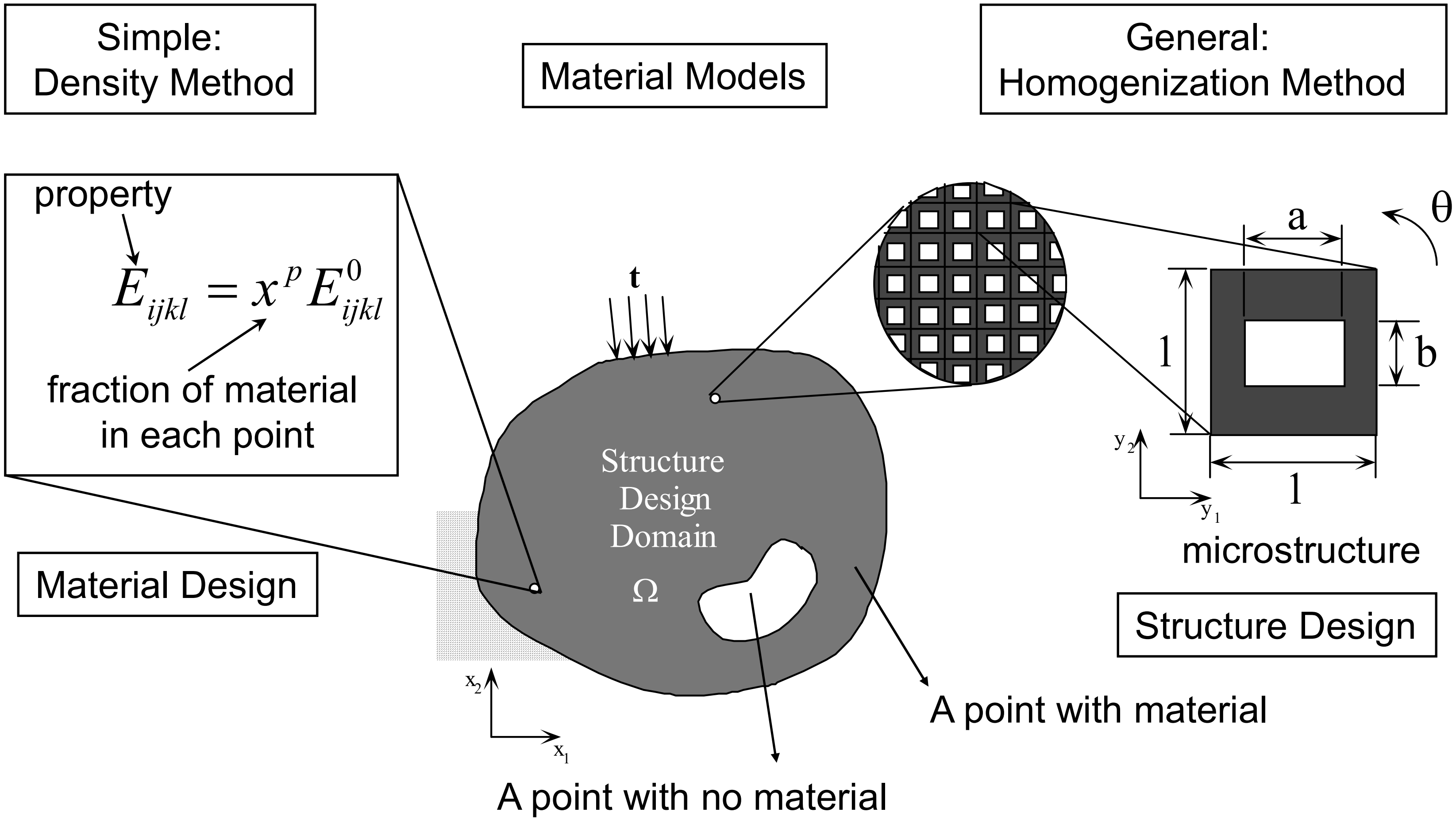
$$\nu = \frac{1}{3} \rightarrow p = 3\ (2D), p = 2\ (3D)$$



$p = 1$

$p = 3$

Summary: Relaxation of Design Domain



Microstructure or Homogenization based Approach

- M. Bendsøe and N. Kikuchi, Generating Optimal Topologies in Structural Design Using a Homogenization Method, Comp. Meth. Appl. Mech. Engrg, 71, pp.197-224, 1988
- Adjust the dimensions and orientation of a void in the material of each element
- Effective material properties are then computed by homogenizing over each element
- Provide bounds on theoretical performance ☺
- Three design variables per element for planar structures → Cumbersome determination and evaluation ☹

COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 71 (1988) 197–224
NORTH-HOLLAND

GENERATING OPTIMAL TOPOLOGIES IN STRUCTURAL DESIGN USING A HOMOGENIZATION METHOD

Martin Philip BENDSØE
Mathematical Institute, The Technical University of Denmark, DK-2800 Lyngby, Denmark

and

Noboru KIKUCHI
Applied Mechanics Department, Computational Mechanics Laboratory, University of Michigan, Ann Arbor, MI 48109, U.S.A.

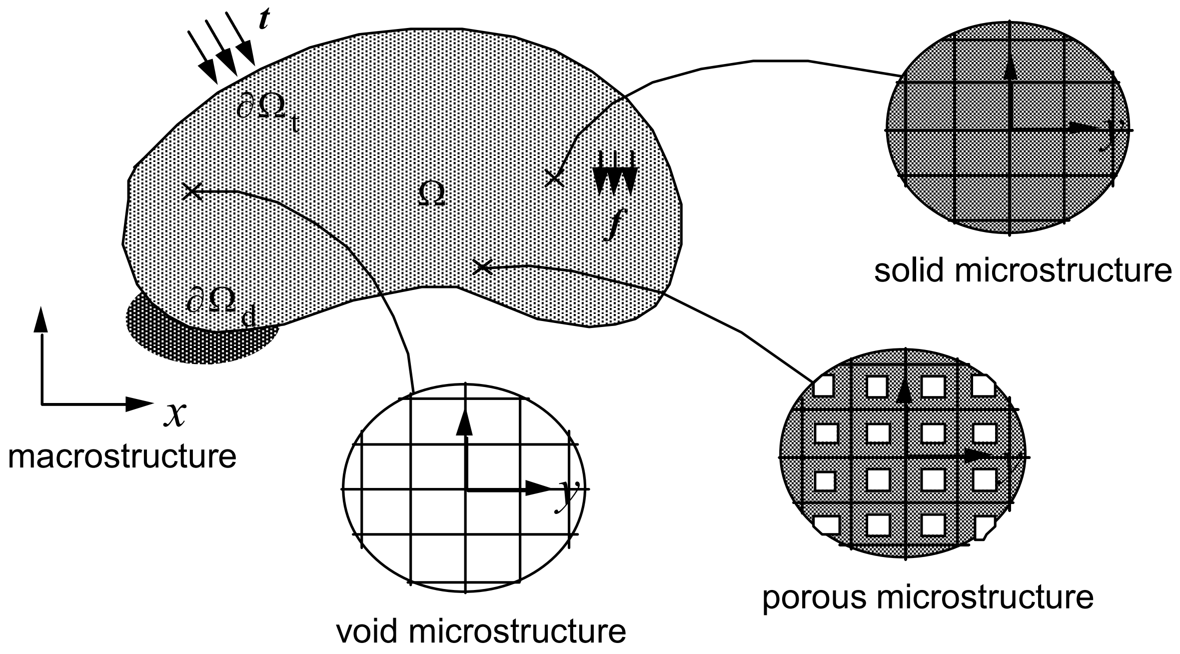
Received 28 March 1988

Homogenization Design Method

Optimal Material Distribution

Theory of Homogenization

Optimization Algorithm



* independent of finite element discretization

Density Approach (1)

- Power-law or SIMP (Solid Isotropic Material with Penalization)
- Bendsøe(1989), Zhou and Rozvany(1991), Mlejnek(1993)
- Relative material density raised to some power times, engineering approach
- No physical material ☹ → physically permissible
  - With a perimeter/gradient constraint or filtering techniques
  - Sigmund and Petersson(1998), Bendsøe and Sigmund(1999)

Structural Optimization 1, 193–202 (1989)

Structural Optimization

Optimal shape design as a material distribution problem

M.P. Bendsøe

Mathematical Institute, The Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark

Computer Methods in Applied Mechanics and Engineering 106 (1993) 1–26
North-Holland

CMA 352

An engineer's approach to optimal material distribution and shape finding

H.P. Mlejnek and R. Schirrmacher

Institute for Computer Applications, University of Stuttgart, Pfaffenwaldring 27, W-7000 Stuttgart 80, Germany

Received 30 August 1990

Density Approach (2)

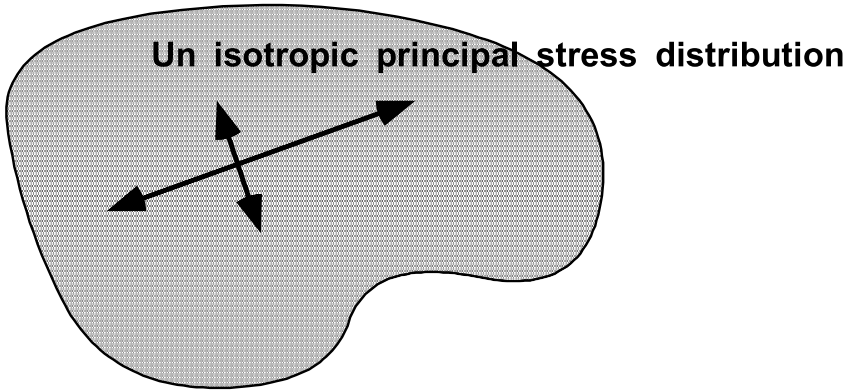
– Most popular approach at present

$$\chi_{\Omega} E \approx \rho^{p} E$$

$$\rho = \begin{cases} 1 & \text{if solid structure} \\ 0 & \text{if void/hole} \end{cases}$$

– Density approach can make only isotropic perforation

– This requires a lot of meshing to have reasonable layout



– However, easy programming and handy design variable

Standard Formulation

$$\min_{\substack{l(\boldsymbol{u}) \le l_{\max} \\ a(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{u})\ \forall \boldsymbol{v}}} \quad \text{Total Weight}$$

- Internal virtual work

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega} (\partial \mathbf{v})^{T} \underbrace{\boldsymbol{\chi}_{\boldsymbol{\Omega}} \mathbf{E}} \underbrace{(\partial \mathbf{u})} \, d\Omega - \underbrace{\omega}^{2} \int_{\Omega} v^{T} \underbrace{\boldsymbol{\chi}_{\boldsymbol{\Omega}} \rho} \underbrace{\mathbf{u}} \, d\Omega$$

ω: shifted(excited)frequency

  – Strain-Displacement relation: $\varepsilon(\boldsymbol{v}) = \partial \boldsymbol{v}$
  – Stress-Strain relation: $\boldsymbol{\sigma} = \boldsymbol{E}\varepsilon - \boldsymbol{\sigma}_0$

- External virtual work

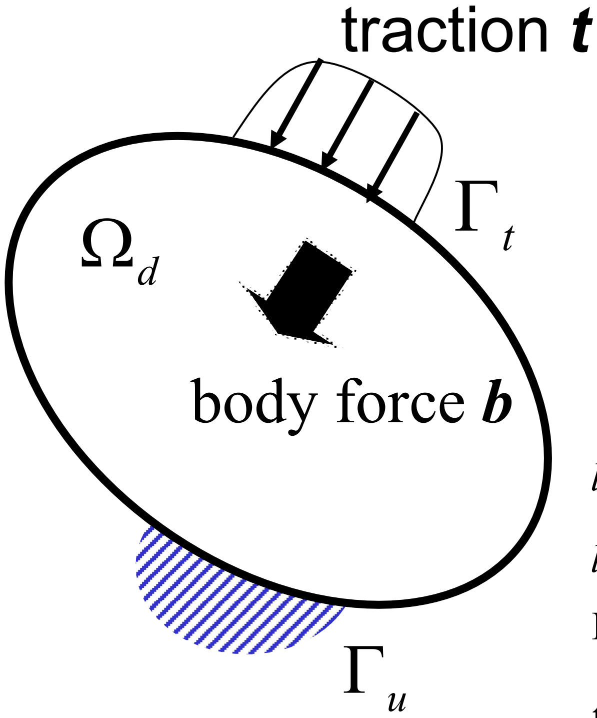
$$l(\mathbf{v}) = \underbrace{\int_{\boldsymbol{\Omega}} (\partial \mathbf{v})^{T} \boldsymbol{\chi}_{\boldsymbol{\Omega}} \boldsymbol{\sigma}_{\mathbf{0}} \boldsymbol{d\Omega}}_{\substack{\text{work done by} \\ \text{initial stress} \\ \sigma_0 = E\alpha\Delta T}} + \underbrace{\int_{\boldsymbol{\Omega}} \boldsymbol{v}^{T} \boldsymbol{\chi}_{\boldsymbol{\Omega}} \rho \mathbf{b} \boldsymbol{d\Omega}}_{\substack{\text{work done by} \\ \text{body force}}} + \underbrace{\int_{\boldsymbol{\Gamma}_{t}} \boldsymbol{v}^{T} \boldsymbol{t} \boldsymbol{d}\boldsymbol{\Gamma}}_{\substack{\text{work done by} \\ \text{traction}}}$$

$$l(\mathbf{u}) = \text{mean compliance by } \{\boldsymbol{\sigma}_0 \quad \rho\mathbf{b} \quad \mathbf{t}\}$$

Mean Compliance

- If the thermal stresses, body forces, and tractions are specified, if the displacement resulted by such applied forces is small, it means that the structure is stiff in its global response.
  - Minimization of the mean compliance = Maximization of the global stiffness
- If constrained displacement is specified on the boundary, then the resulted stress (that is traction) on the boundary must be large if the structure is stiff.
  - In this case, we have to maximize the mean compliance
- Mean compliance was introduced by Prager and Taylor to define structural optimization for continuum solids and structures.
- Weight minimization with stress and displacement constraints was introduced for trusses, beams, and other space frame type structures in aerospace and civil engineering since stresses are bounded in these frame structures.

Stiffness Maximization: Force Applied



$$\int_{\Omega_d} \boldsymbol{\varepsilon}(\mathbf{v}) : \chi_\Omega \mathbf{E} : \boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega = \int_{\Omega_d} \chi_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}\, d\Gamma \quad \mathbf{u}\in V^{\Omega_d}, \forall \mathbf{v}\in V^{\Omega_d}$$

$$\rightarrow \left\{ \begin{array}{l} a(\mathbf{u},\mathbf{v}) = \int_{\Omega_d} \boldsymbol{\varepsilon}(\mathbf{v}) : \chi_\Omega \mathbf{E} : \boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega \\ l(\mathbf{v}) = \int_{\Omega_d} \chi_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}\, d\Gamma \end{array} \right\} \rightarrow a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}) \quad \mathbf{u}\in V^{\Omega_d}, \forall \mathbf{v}\in V^{\Omega_d}$$

$$l_{m.c.} = l(\mathbf{u}) = \int_{\Omega_d} \chi_\Omega \mathbf{b}\cdot\mathbf{u}\, d\Omega + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{u}\, d\Gamma$$

$$l(\mathbf{u}) = a(\mathbf{u},\mathbf{u}) = 2\times(\text{strain energy})$$

Minimize (mean compliance) $\equiv$ Minimize (strain energy)

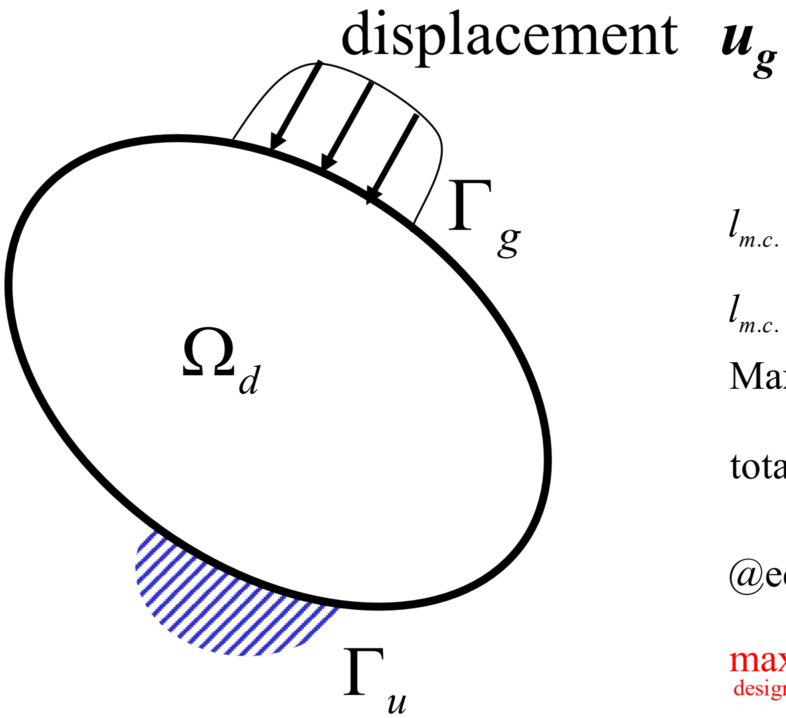
total potential energy: $F(\mathbf{v}) = \frac{1}{2} a(\mathbf{v},\mathbf{v}) - l(\mathbf{v}) \quad \forall \mathbf{v}\in V^{\Omega_d}$

@equilibrium, $F(\mathbf{u}) = \underset{\mathbf{v}\in V^{\Omega_d}}{\text{minimize}}\ F(\mathbf{v}) = \frac{1}{2} a(\mathbf{u},\mathbf{u}) - l(\mathbf{u}) = -\frac{1}{2} l(\mathbf{u})$

$$\color{red}{\underset{\text{design variables}}{\text{minimize}}\ l_{m.c.} = l(\mathbf{u})} = \underset{\text{design variables}}{\text{minimize}} \left\{-2F(\mathbf{u})\right\} = \underset{\text{design variables}}{\text{minimize}} \left\{-2\left[\underset{\mathbf{v}\in V^{\Omega_d}}{\text{minimize}}\ F(\mathbf{v})\right]\right\}$$

$$\rightarrow \color{red}{\underset{\text{design variables}}{\text{maximize}}\ F(\mathbf{u})} = \underset{\text{design variables}}{\text{maximize}}\ \underset{\mathbf{v}\in V^{\Omega_d}}{\text{minimize}}\ F(\mathbf{v})$$

Stiffness Maximization: Displacement Applied



$$l_{m.c.} = \int_{\Gamma_g} \mathbf{t}_r \cdot \mathbf{u}_g \, d\Gamma \neq l(\mathbf{u})$$

$$l_{m.c.} = a(\mathbf{u},\mathbf{u}) = 2 \times (\text{strain energy})$$

Maximize (mean compliance) $\equiv$ Maximize (strain energy)

total potential energy: $F(\mathbf{v}) = \frac{1}{2} a(\mathbf{v},\mathbf{v}) \quad \forall \mathbf{v} \in V^{\Omega_d}$

@equilibrium, $F(\mathbf{u}) = \underset{\mathbf{v}\in V^{\Omega_d}}{\text{minimize}}\ F(\mathbf{v}) = \frac{1}{2} a(\mathbf{u},\mathbf{u})$

$$\underset{\text{design variables}}{\text{maximize}}\ l_{m.c.} = \underset{\text{design variables}}{\text{maximize}}\ a(\mathbf{u},\mathbf{u}) = \underset{\text{design variables}}{\text{maximize}}\ 2F(\mathbf{u})$$

$$\rightarrow \underset{\text{design variables}}{\text{maximize}}\ F(\mathbf{u}) = \underset{\text{design variables}}{\text{maximize}}\ \underset{\mathbf{v}\in V^{\Omega_d}}{\text{minimize}}\ F(\mathbf{v})$$

Stress-Strain Relation

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}\\
\varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}\\
\varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}\\
\gamma_{yz} &= \frac{\tau_{yz}}{G}\\
\gamma_{zx} &= \frac{\tau_{zx}}{G}\\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}\right\} \rightarrow \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{Bmatrix} = \frac{\mathbf{1}}{\boldsymbol{E}}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{Bmatrix} \rightarrow \boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma} \xrightarrow{G=\frac{E}{2(1+\nu)}} \boldsymbol{\sigma} = \mathbf{C}^{-1}\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0\\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0\\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}$$

Optimization Problem Formulation

$$\underset{design}{\text{minimize}} \int_D \rho d\Omega$$
$$g' = l_{m.c.} - \bar{l}_{m.c.} \le 0$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \mathbf{u} \in V^D, \forall \boldsymbol{v} \in V^D$$

→

$$\underset{design}{\text{minimize}}\ l_{m.c.} = l(\boldsymbol{u})$$
$$g = \int_D \rho d\Omega - \Omega_0 \le 0$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \mathbf{u} \in V^D, \forall \boldsymbol{v} \in V^D$$

→

$$\underset{\rho}{\text{maximize}}\ F(\boldsymbol{u})$$
$$g = \int_D \rho d\Omega - \Omega_0 \le 0$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \mathbf{u} \in V^D, \forall \boldsymbol{v} \in V^D$$
$$0 \le \rho \le 1$$

$$L = -F(\boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{w}) - l(\boldsymbol{w}) + \Lambda\left(\int_D \rho d\Omega - \Omega_0\right) + \int_D [\lambda_0(-\rho) + \lambda_1(\rho - 1)] d\Omega$$

$$\begin{aligned}
\delta L &= -\frac{\partial F(\boldsymbol{u})}{\partial \boldsymbol{u}} \delta \boldsymbol{u} - \frac{\partial F(\boldsymbol{u})}{\partial \boldsymbol{\rho}} \delta \rho + \frac{\partial a(\boldsymbol{u}, \boldsymbol{w})}{\partial \boldsymbol{u}} \delta \boldsymbol{u} + \Lambda \int_D \delta \rho d\Omega + \int_D [\lambda_0(-\delta\rho) + \lambda_1(\delta\rho)] d\Omega \\
&= -\int_D \boldsymbol{\varepsilon}(\boldsymbol{u}) : \chi_\Omega \boldsymbol{E} : \boldsymbol{\varepsilon}\left(\delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial \rho} \delta \rho\right) d\Omega - \frac{1}{2} \int_D \boldsymbol{\varepsilon}(\boldsymbol{u}) : \frac{\partial \chi_\Omega \boldsymbol{E}}{\partial \rho} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \delta \rho d\Omega + \int_D \boldsymbol{\varepsilon}(\boldsymbol{w}) : \chi_\Omega \boldsymbol{E} : \boldsymbol{\varepsilon}\left(\delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial \rho} \delta \rho\right) d\Omega \\
&\quad + \Lambda \int_D \delta \rho d\Omega + \int_D [\lambda_0(-\delta\rho) + \lambda_1(\delta\rho)] d\Omega
\end{aligned}$$

$$\begin{aligned}
\delta L &= -\frac{1}{2} \int_D \boldsymbol{\varepsilon}(\boldsymbol{u}) : \frac{\partial \chi_\Omega \boldsymbol{E}}{\partial \rho} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \delta \rho d\Omega + \Lambda \int_D \delta \rho d\Omega + \int_D [\lambda_0(-\delta\rho) + \lambda_1(\delta\rho)] d\Omega \\
&= \int_D \left\{ -\frac{1}{2} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \frac{\partial \chi_\Omega \boldsymbol{E}}{\partial \rho} : \boldsymbol{\varepsilon}(\boldsymbol{u}) + \Lambda - \lambda_0 + \lambda_1 \right\} \delta \rho d\Omega = 0
\end{aligned}$$

KKT Conditions

$$
\begin{cases}
-\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}):\frac{\partial \chi_\Omega \boldsymbol{E}}{\partial \rho}:\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathit{\Lambda} - \lambda_0 + \lambda_1 = 0 \\
a(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}) \quad \mathbf{u} \in V^D, \forall \boldsymbol{v} \in V^D \\
\mathit{\Lambda} \geq 0,\ \int_D \rho d\mathit{\Omega} - \mathit{\Omega}_0 \leq 0,\ \Lambda\left(\int_D \rho d\mathit{\Omega} - \mathit{\Omega}_0\right) = 0 \\
\lambda_0 \geq 0,\ -\rho \leq 0,\ \lambda_0(-\rho) = 0 \\
\lambda_1 \geq 0,\ \rho - 1 \leq 0,\ \lambda_1(\rho - 1) = 0
\end{cases}
$$

Discrete Formulation

$$\boxed{\min_{\substack{\boldsymbol{u}^T\boldsymbol{f}\le l_{\max}\\ \boldsymbol{Ku}=\boldsymbol{f}}} \quad \text{Total Weight}}$$

- Shifted stiffness matrix
  $$\boldsymbol{K} = \underbrace{\int_\Omega \boldsymbol{B}^T \boldsymbol{E}\boldsymbol{B}\, d\Omega}_{\text{stiffness matrix}} - \omega^2 \underbrace{\int_\Omega \boldsymbol{N}^T \rho \boldsymbol{N}\, d\Omega}_{\text{mass matrix}}$$

- Generalized load vector
  $$\boldsymbol{f} = \int_\Omega \boldsymbol{B}^T \boldsymbol{\sigma}_0\, d\Omega + \int_\Omega \boldsymbol{N}^T \rho \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}^T \boldsymbol{t}\, d\Gamma$$

- Dual problems
  - Minimize the volume with the mean compliance constraint
  - Minimize the mean compliance with the volume constraint

$$\min_{\substack{\boldsymbol{u}^T\boldsymbol{f}\le l_{\max}\\ \boldsymbol{Ku}=\boldsymbol{f}}} \int_\Omega \rho\, d\Omega \xleftrightarrow{\text{dual problem}} \min_{\substack{\boldsymbol{Ku}=\boldsymbol{f}\\ \int_\Omega \rho d\Omega \le \Omega_s}} \boldsymbol{u}^T \boldsymbol{f}$$

Compliance and Energy

– Mean compliance = twice of the total strain energy

$$\boldsymbol{u}^T \boldsymbol{f} = work\ done \qquad \boldsymbol{Ku} = \boldsymbol{f} \quad \Rightarrow \quad \boldsymbol{u}^T \boldsymbol{Ku} = \boldsymbol{u}^T \boldsymbol{f}$$

$$\min_{\boldsymbol{v}} \ I(\boldsymbol{v}) = \underbrace{I(\boldsymbol{u})}_{\substack{\text{minimum potential energy} \\ \text{at equilibrium}}} = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{Ku} - \boldsymbol{u}^T \boldsymbol{f} = -\frac{1}{2}\boldsymbol{u}^T \boldsymbol{f}$$

– Using the relation $\min\limits_{design} \ \boldsymbol{u}^T \boldsymbol{f} = \min\limits_{design} \left\lfloor -2 \min\limits_{\boldsymbol{v}} \ I(\boldsymbol{v}) \right\rfloor = 2 \max\limits_{design} \min\limits_{\boldsymbol{v}} \ I(\boldsymbol{v})$

– We can define the optimum design problem by the total potential energy $\max\limits_{design} \min\limits_{\boldsymbol{v}} \ I(\boldsymbol{v})$

– The most fundamental structural design problem can be stated as the maximization of the minimum potential energy of a structural system with respect to designs and admissible displacements.

Optimization Algorithm

- Optimization problem

$$\begin{aligned}
&\underset{(a,b,\theta)}{\text{maximize}}\ \underbrace{\underset{\boldsymbol{v}}{\text{minimize}}\ \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v})-f(\boldsymbol{v})}_{I(\boldsymbol{u})}\\
&\text{subject to}\quad \int_\Omega (1-ab)\,d\Omega \le \Omega_s\\
&\qquad\qquad\quad 0 \le a, b < 1
\end{aligned}$$

- Lagrangian

$$\begin{aligned}
L &= I(\boldsymbol{u}) - \Lambda\left[\int_\Omega (1-ab)\,d\Omega - \Omega_s\right]\\
&\quad - \int_\Omega \{\lambda_{a0}(-a)+\lambda_{a1}(a-1)\}\,d\Omega - \int_\Omega \{\lambda_{b0}(-b)+\lambda_{b1}(b-1)\}\,d\Omega
\end{aligned}$$

- Principle of virtual displacement: equilibrium

$$\delta\boldsymbol{u} : a(\boldsymbol{u},\boldsymbol{v}) = f(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V$$

Optimality Conditions

- for design variables

$$
\begin{cases}
\delta a : \varepsilon(\boldsymbol{u})^T \dfrac{\partial \chi_\Omega \boldsymbol{E}}{\partial a} \varepsilon(\boldsymbol{u}) - \boldsymbol{u}^T \dfrac{\partial \boldsymbol{f}}{\partial a} + \Lambda b + \lambda_{a0} - \lambda_{a1} = 0 \\
\delta b = 0 : \varepsilon(\boldsymbol{u})^T \dfrac{\partial \chi_\Omega \boldsymbol{E}}{\partial b} \varepsilon(\boldsymbol{u}) - \boldsymbol{u}^T \dfrac{\partial \boldsymbol{f}}{\partial b} + \Lambda a + \lambda_{b0} - \lambda_{b1} = 0 \\
\delta \theta = 0 : \varepsilon(\boldsymbol{u})^T \dfrac{\partial \chi_\Omega E}{\partial \theta} \varepsilon(\boldsymbol{u}) = 0
\end{cases}
$$

- Switching conditions

$$
\begin{cases}
\Lambda\left[\int_\Omega (1 - ab)\, d\Omega - \Omega_s\right] = 0 : \quad \Lambda \le 0, \quad \int_\Omega (1 - ab)\, d\Omega - \Omega_s \le 0 \\
\lambda_{a0}(-a) = 0 : \lambda_{a0} \le 0, -a \le 0 \\
\lambda_{a1}(a - 1) = 0 : \lambda_{a1} \le 0, a - 1 \le 0 \\
\lambda_{b0}(-b) = 0 : \lambda_{b0} \le 0, -b \le 0 \\
\lambda_{b1}(b - 1) = 0 : \lambda_{b1} \le 0, b - 1 \le 0
\end{cases}
$$

Resizing Scheme

$$a_{k+1} = \begin{cases} \max\{(1-\zeta)a_k, 0\} & \text{if } \; a_k\left(E_k^a\right)^\eta \le \max\{(1-\zeta)a_k, 0\} \\ a_k\left(E_k^a\right)^\eta & \text{if } \; \min\{(1+\zeta)a_k, 1\} \le a_k\left(E_k^a\right)^\eta \le \max\{(1-\zeta)a_k, 0\} \\ \min\{(1+\zeta)a_k, 1\} & \text{if } \; \min\{(1+\zeta)a_k, 1\} \le a_k\left(E_k^a\right)^\eta \end{cases}$$

$$b_{k+1} = \begin{cases} \max\{(1-\zeta)b_k, 0\} & \text{if } \; b_k\left(E_k^b\right)^\eta \le \max\{(1-\zeta)b_k, 0\} \\ b_k\left(E_k^b\right)^\eta & \text{if } \; \min\{(1+\zeta)b_k, 1\} \le b_k\left(E_k^b\right)^\eta \le \max\{(1-\zeta)b_k, 0\} \\ \min\{(1+\zeta)b_k, 1\} & \text{if } \; \min\{(1+\zeta)b_k, 1\} \le b_k\left(E_k^b\right)^\eta \end{cases}$$

where $E_k^a = \dfrac{1}{-\Lambda_k b_k}\varepsilon(\boldsymbol{u}_k)^T \left.\dfrac{\partial \chi_\Omega \boldsymbol{E}}{\partial a}\right|_k \varepsilon(\boldsymbol{u}_k)$ and $E_k^b = \dfrac{1}{-\Lambda_k a_k}\varepsilon(\boldsymbol{u}_k)^T \left.\dfrac{\partial \chi_\Omega \boldsymbol{E}}{\partial b}\right|_k \varepsilon(\boldsymbol{u}_k)$

$$\Lambda_k \xleftarrow{\text{bisection}} \int_\Omega (1-ab)\,d\Omega - \Omega_s = 0$$

η : weighting factor (0.7 ~ 0.85)

ζ : move limit

Optimum Angle (1)

- Optimum angle is the one for the principal stress direction
  - P. Pedersen, On Optimal Orientation of Orthotropic Materials, Struct Optim 1, pp.101-106, 1989
- Orthotropic material

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} (\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 - \varepsilon_2)\cos 2\phi \\ (\varepsilon_1 + \varepsilon_2) - (\varepsilon_1 - \varepsilon_2)\cos 2\phi \\ -2(\varepsilon_1 - \varepsilon_2)\sin 2\phi \end{Bmatrix}$$

- Minimize compliance = Maximize strain energy

$$\Phi = \frac{1}{2} \begin{Bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{Bmatrix} \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\rightarrow \begin{cases} \dfrac{\partial \Phi}{\partial \phi} = -\dfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\phi\left[\alpha(\varepsilon_1 + \varepsilon_2) + \beta(\varepsilon_1 - \varepsilon_2)\cos 2\phi\right] = 0 \\ \text{where } \alpha = E_{11} - E_{22} \text{ and } \beta = E_{11} + E_{22} - 2E_{12} - 4E_{66} \end{cases}$$

Optimum Angle (2)

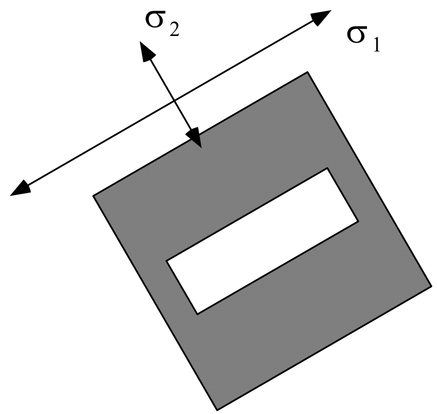
$$\frac{\partial \Phi}{\partial \theta} = 0 \rightarrow \sin 2\theta = 0 \text{ or } \cos 2\theta = -\frac{\alpha(\varepsilon_1 + \varepsilon_2)}{\beta(\varepsilon_1 - \varepsilon_2)}$$

$$\frac{\partial^2 \Phi}{\partial \theta^2} = -(\varepsilon_1 - \varepsilon_2)[\alpha(\varepsilon_1 + \varepsilon_2)\cos 2\theta + \beta(\varepsilon_1 - \varepsilon_2)\cos 4\theta]$$

β : measure of shear stiffness

(for low shear stiffness : $\beta \geq 0$) $\rightarrow$ global minimum compliance$(\theta = 0)$

- Rectangular hole should be aligned in the principal stress direction
  - Large hole can be assumed in the small principal stress direction
  - Small hole must be placed in the large principal stress direction



Multiple Loading

$$\min_{\{a,b,\theta\}\in K_d} \sum_{i=1}^{m} w_i \left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\}^p$$

$$\text{where}\ \begin{cases} K_d = \left\{(a,b,\theta)\,\middle|\, 0 \le a,b < 1,\ \int_\Omega (1-ab)\,d\Omega \le \Omega_s\right\} \\ 0 \le w_i \le 1,\quad \sum_{i=1}^{m} w_i = 1 \\ I^i(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - f^i(\boldsymbol{v}) \end{cases}$$

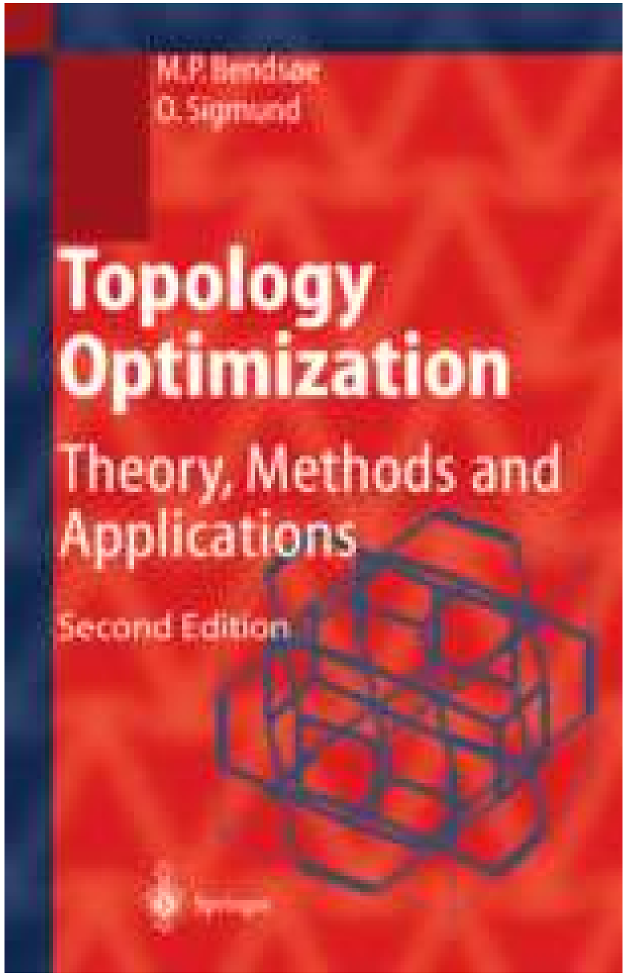
$$w_i = 1:\ \lim_{p\to\infty} \sum_{i=1}^{m} \left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\}^p = \max_{i=1,\ldots,m} \left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\} = \max_{i=1,\ldots,m} l^i(\boldsymbol{u})$$

– p = 1: linear combination of minimum value of the total potential energy

– p = ∞: minimization of the maximum value of the mean compliances

$$\frac{\partial}{\partial a}\sum_{i=1}^{m} w_i \left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\}^p = \sum_{i=1}^{m} \underbrace{w_i \left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\}^{p-1}}_{\overline{w}_i} \frac{\partial}{\partial a}\left\{-2\min_{\boldsymbol{v}} I^i(\boldsymbol{v})\right\}$$

Density or SIMP Approach



1.1 Problem Formulation

- To find the optimal lay-out of a structure within a specified region
- Known quantities
  - Applied loads
  - Possible support conditions
  - Volume of the structure to be constructed
  - Some additional design restrictions: location and size of prescribed holes or solid areas
- Unknown
  - Physical size and shape and connectivity of the structure
- Representation
  - Not by standard parametric function, but by a set of distributed functions defined on a fixed design domain

Minimum Compliance Design

- Problem set-up
  - Well known formulations for sizing problems for discrete and continuum structures
  - Large scale both in state and in the design variables
- Simplest type
  - Minimum compliance (maximum global stiffness) under simple resource constraints
  - Problem of finding the optimal choice of stiffness tensor $E_{ijkl}(x)$ which is a variable over the domain

$$\left.\begin{aligned} &\min_{u\in U,E} l(u) \\ &a_E(u,v) = l(v), \quad \text{for all } v\in U,\ E\in \mathrm{E}_{\mathrm{ad}} \end{aligned}\right\} \xrightarrow[\textit{form}]{\textit{discrete}} \left\{\begin{aligned} &\min_{\boldsymbol{u},E_e} \boldsymbol{f}^T\boldsymbol{u} \\ &\boldsymbol{K}(E_e)\boldsymbol{u} = \boldsymbol{f},\ E_e\in \mathrm{E}_{\mathrm{ad}},\ \boldsymbol{K} = \sum_{e=1}^{N} \boldsymbol{K}_e(E_e) \end{aligned}\right.$$

$$a(u,v) = \int_\Omega E_{ijkl}(x)\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,d\Omega : \text{energy bilinear form} \xleftarrow[\text{strain}]{\text{linearized}} \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$l(u) = \int_\Omega fud\Omega + \int_{\Gamma_T} tuds : \ \text{load linear form}$$

Design Parametrization (1)

- Determination of the optimal placement of a given isotropic material in space
  - Optimal subset of material points
- Geometric representation of a structure = black-white rendering of an image
- E_{ad}: distributed, discrete valued design problem (0-1 problem)
  - Replace the integer variables with continuous variables and then introduce some form of penalty

$$E_{ijkl} = 1_{\Omega^{mat}} E^0_{ijkl}, 1_{\Omega^{mat}} = \begin{cases} 1 \text{ if } x \in \Omega^{mat} \\ 0 \text{ if } x \in \Omega \setminus \Omega^{mat} \end{cases}$$

E^0_{ijkl}: stiffness tensor for the given isotropic material

$$E_{ijkl} \in L^\infty(\Omega)$$

$$\int_\Omega 1_{\Omega^{mat}} \, d\Omega = \text{Vol}\left(\Omega^{mat}\right) \le V$$

Design Parametrization (2)

- Sizing problem by modifying the stiffness matrix so that it depends continuously on a function (design variable) interpreted as a density of material
  - SIMP-model: penalized, proportional stiffness model

$$E_{ijkl}\left(x\right)=\rho\left(x\right)^{p}\,E_{ijkl}^{0},\; p>1\rightarrow\begin{cases}E_{ijkl}\left(\rho=0\right)=0\\ E_{ijkl}\left(\rho=1\right)=E_{ijkl}^{0}\end{cases}$$

$$\int_{\Omega}\rho\left(x\right)d\Omega\le V,\; 0\le\rho\left(x\right)\le 1,\; x\in\Omega$$

  - Physical relevance: Hashin-Shtrikman bounds for two-phase materials: limits of possible isotropic material properties one can achieve by constructing composites (materials with microstructure) from two given, linearly elastic, isotropic materials.

$$\left.\begin{aligned}&p\ge\max\left\{\frac{2}{1-\nu^{0}},\frac{4}{1+\nu^{0}}\right\}\left(2D\right)\\ &p\ge\max\left\{15\frac{1-\nu^{0}}{7-5\nu^{0}},\frac{3}{2}\frac{1-\nu^{0}}{1-2\nu^{0}}\right\}\left(3D\right)\end{aligned}\right]\xrightarrow{\nu^{0}=\frac{1}{3}}p\ge 3$$

ν^{0} : Poisson ratio of the given base material with stiffness tensor E_{ijkl}^{0}

Design Parametrization (3)

- Role of composites
  - Homogenization method: viewed as an interpolation model
  - Inspired by theoretical studies on generalized shape design in conduction and torsion problems

$$\left.\begin{array}{l} \min\limits_{u\in U,E} l(u) \\ a_E(u,v) = l(v), \quad \text{for all } v\in U,\ E\in \mathrm{E}_{\text{ad}} \end{array}\right\}$$

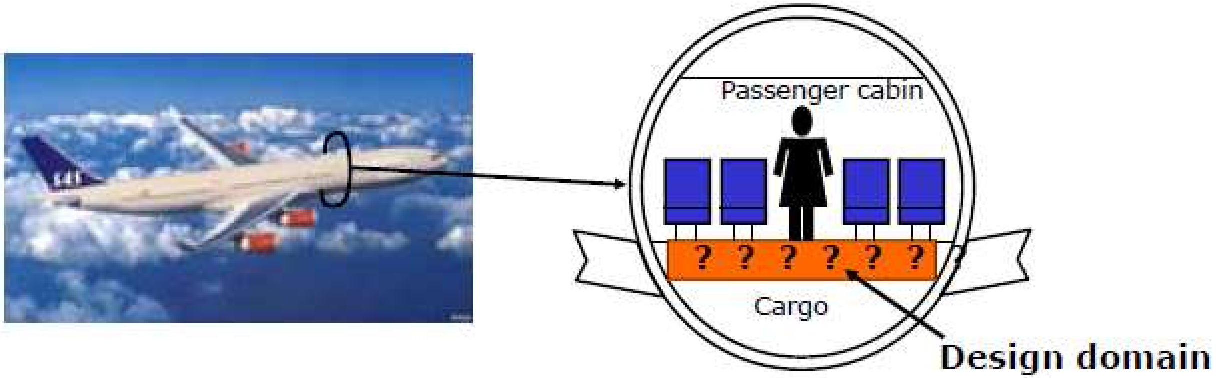
$$\xrightarrow[\text{minimizer of the functional } F(v)=\frac{1}{2}a_E(v,v)-l(v) \text{ on } U]{\text{equilibrium condition}\rightarrow\text{principle of minimim potential energy}} \max_{E\in \mathrm{E}_{\text{ad}}} \min_{u\in U} \left\{ \frac{1}{2} a_E(u,u) - l(u) \right\}$$

$$\xrightarrow{\text{principle of minimim complementart energy}} \min_{E\in \mathrm{E}_{\text{ad}}} \min_{\sigma\in S} \left\{ \frac{1}{2} \int_\Omega C_{ijkl}\sigma_{ij}\sigma_{kl} d\Omega \right\}$$

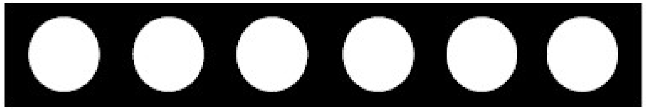
$$C_{ijkl} = \left(E^{-1}\right)_{ijkl}$$

$$S = \left\{ \sigma \middle| div\sigma + f = 0 \text{ in } \Omega, \sigma\cdot n = t \text{ on } \Gamma_T \right\}$$

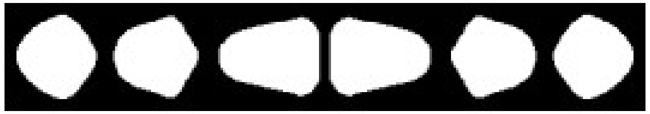
Design Space



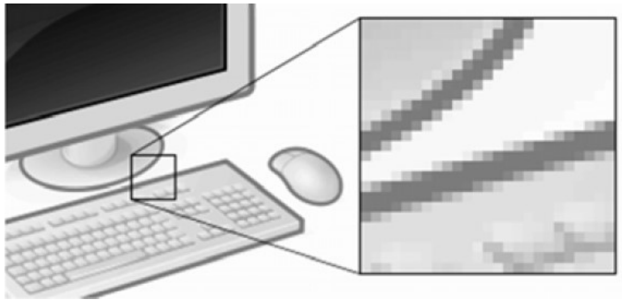
- Design parametrization 1



  - 6 circular holes: one center and one radius per hole (18 variables)
- Design parametrization 2

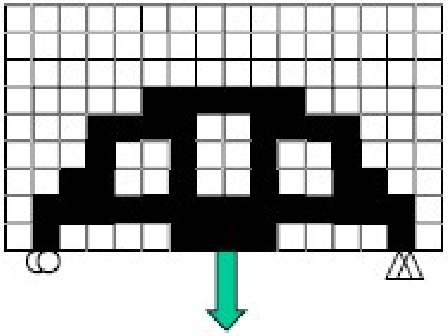


  - 6 slightly more complicated holes: splines (hundred variables)
- Design parametrization 3
  - Raster representation: use pixels



Integer Form vs. Continuous Form

Black and white [정수(0/1) 문제]
- 모든 조합? Ill-conditioned



$$\min_{\boldsymbol{\rho}} \ \Phi(\boldsymbol{\rho}, \boldsymbol{U}(\boldsymbol{\rho})) = \boldsymbol{F}^T\boldsymbol{U}$$

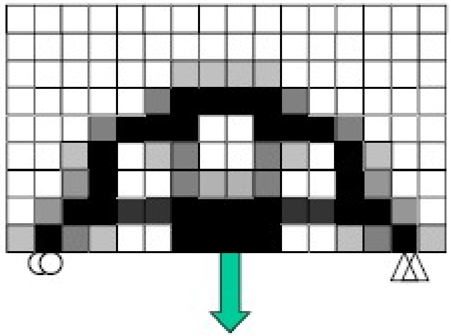
$$\text{subject to} \ \sum_{e=1}^{N} v_e\rho_e = \boldsymbol{v}^T\boldsymbol{\rho} \le V^*$$

$$g_i(\boldsymbol{\rho}) \le g_i^* \quad i = 1,\ldots,M$$

$$\rho_e = \begin{cases} 0 \ \ \text{(void)} \\ 1 \ \ \text{(material)} \end{cases} \quad e = 1,\ldots,N$$

$$\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}$$

Gray sale [실수(0~1) 문제]
- 완화, 중간밀도?



$$\min_{\boldsymbol{\rho}} \ \Phi(\boldsymbol{\rho}, \boldsymbol{U}(\boldsymbol{\rho})) = \boldsymbol{F}^T\boldsymbol{U}$$

$$\text{subject to} \ \sum_{e=1}^{N} v_e\rho_e = \boldsymbol{v}^T\boldsymbol{\rho} \le V^*$$

$$g_i(\boldsymbol{\rho}) \le g_i^* \quad i = 1,\ldots,M$$

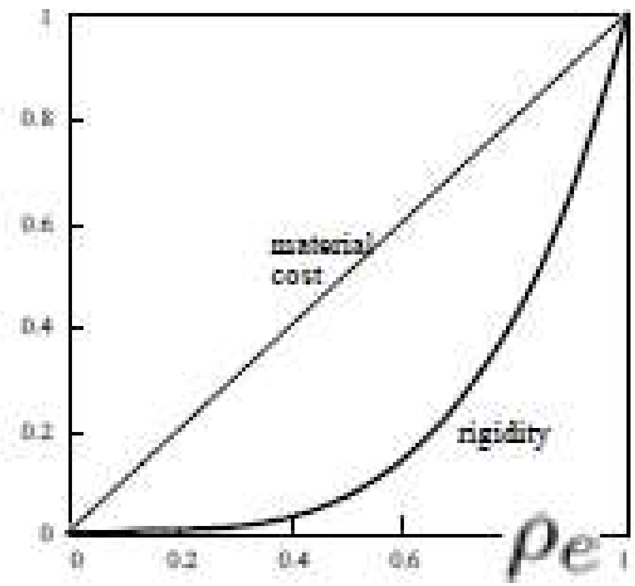
$$0 < \rho_{\min} \le \boldsymbol{\rho} \le 1$$

$$\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}$$

Material Density Distribution

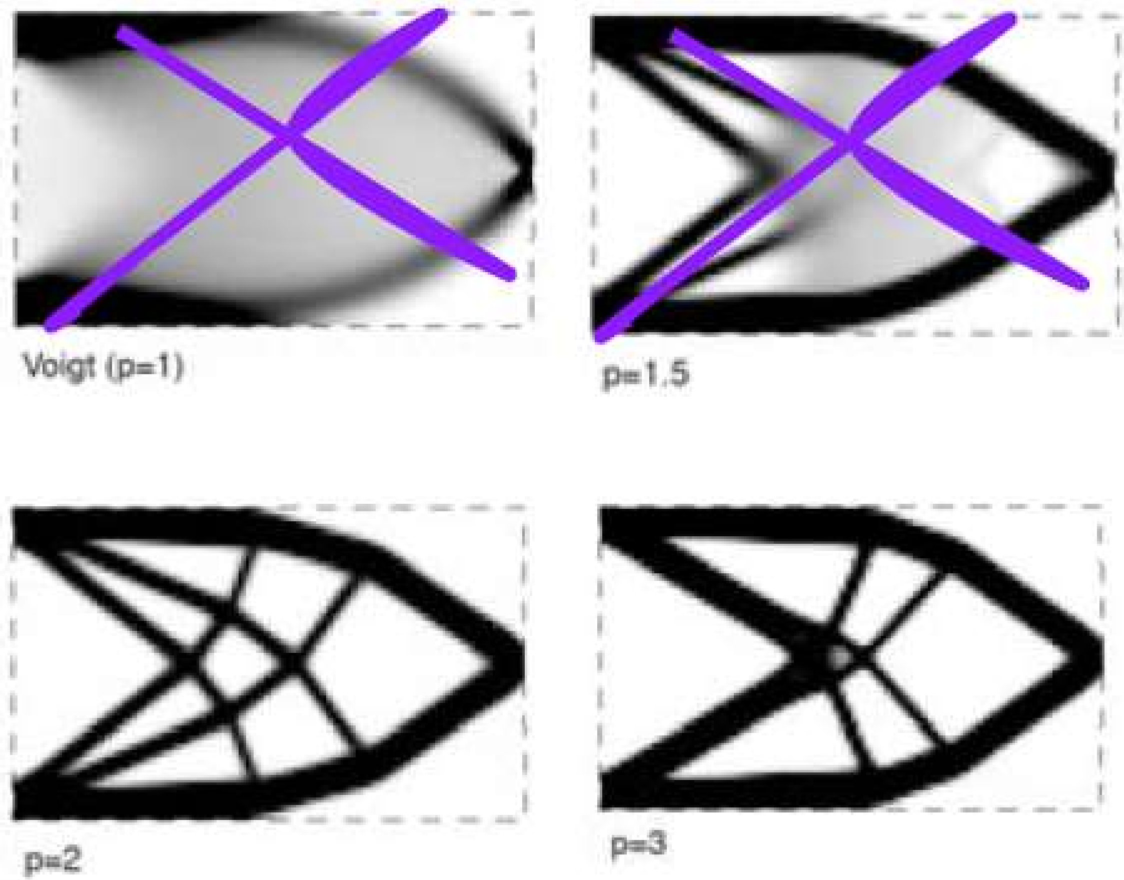
SIMP

Simplified Isotropic Material with Penalization



$$\boxed{E(\rho_e) = \rho_e^{\,p} E_0}$$

$$p \geq 1$$



Material Interpolation Model

$p \geq 1$ SIMP $\longrightarrow$ RAMP $q \geq 1$

$$E(\rho_e) = \begin{cases} {\rho_e}^p E_0 \\ E_{min} + {\rho_e}^p (E_0 - E_{min}) \end{cases}$$

$$E(\rho_e) = \frac{\rho_e}{1 + q(1 - \rho_e)} E_0$$

Convexity in compliance !

$$\begin{aligned} &\min_{\boldsymbol{\rho}} \ \Phi(\boldsymbol{\rho}, \boldsymbol{U}(\boldsymbol{\rho})) = \boldsymbol{F}^T \boldsymbol{U} \\ &\text{subject to } \sum_{e=1}^{N} v_e \rho_e = \boldsymbol{v}^T \boldsymbol{\rho} \leq V^* \\ &\qquad g_i(\boldsymbol{\rho}) \leq g_i^* \quad i = 1, \ldots, M \\ &\qquad 0 < \rho_{\min} \leq \boldsymbol{\rho} \leq 1 \\ &\qquad \boldsymbol{K}(\boldsymbol{\rho}) \boldsymbol{U} = \boldsymbol{F} \end{aligned}$$

$+$ $\boxed{E(\rho_e)}$

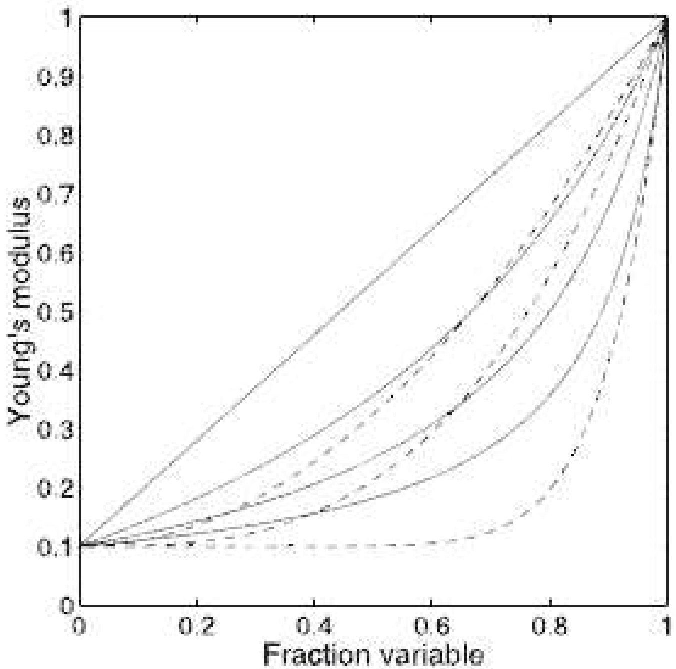


Fig. 1 A comparison of the SIMP model and the alternative model when $E_1 = 1$ and $E_0 = 0.1$. $E_p(x_j)$ for $p = 2, 3, 10$ are shown as dashed lines; $E_q(x_j)$ for $q = 0, 1.5, 4, 9$ are shown as solid lines

1.2 Solution Methods

- Optimal topology problem → sizing problem on a fixed domain
  - Typically very large number of design variables
  - Efficiency of the optimization procedure is crucial
- Conditions of optimality

$$\left.\begin{aligned}
&\min_{u\in U,\rho} l(u)\\
&a_E(u,v)=l(v),\quad \text{for all } v\in U\\
&E_{ijkl}(x)=\rho(x)^p\,E^0_{ijkl}\\
&\int_\Omega \rho(x)\,d\Omega\le V\\
&0<\rho_{\min}\le\rho(x)\le 1
\end{aligned}\right\}
\xrightarrow[\bar{u}:\ \text{Lagrange multiplier for the equilibrium constraint}]{L=l(u)+\left[a_E(u,\bar{u})-l(\bar{u})\right]+\Lambda\left[\int_\Omega\rho(x)d\Omega-V\right]+\int_\Omega\lambda^+(x)\left[\rho(x)-1\right]d\Omega+\int_\Omega\lambda^-(x)\left[\rho_{\min}-\rho(x)\right]d\Omega}
\begin{cases}
\dfrac{\partial L}{\partial\rho}=-\dfrac{\partial E_{ijkl}}{\partial\rho}\varepsilon_{ij}(u)\varepsilon_{kl}(u)+\Lambda+\lambda^+-\lambda^-=0\\
\lambda^+\left[\rho(x)-1\right]=0\\
\lambda^-\left[\rho_{\min}-\rho(x)\right]=0\\
\lambda^+\ge 0,\ \lambda^-\ge 0
\end{cases}$$

$$\frac{\partial E_{ijkl}}{\partial\rho}\varepsilon_{ij}(u)\varepsilon_{kl}(u)=\Lambda+\lambda^+-\lambda^-$$

$$p\rho(x)^{p-1}\,E^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(u)=\Lambda\quad\text{for intermediate densities }\left(\rho_{\min}<\rho(x)<1\right)$$

Sensitivity Analysis

- Design problem as an optimization problem in the design variables only
  - Displacement fields given implicitly in terms of these design variables through the equilibrium equation
  - Derivatives of the displacements with respect to the design variables
  - Additional material in any element decreases compliance

$$\left.\begin{aligned}&\min_{\boldsymbol{u},\rho_e} \boldsymbol{f}^T\boldsymbol{u}\\&\left(\sum_{e=1}^{N}\rho_e{}^p\boldsymbol{K}_e\right)\boldsymbol{u}=\boldsymbol{f}\\&\sum_{e=1}^{N}v_e\rho_e\le V\\&0<\rho_{\min}\le\rho_e\le 1\end{aligned}\right\}\xrightarrow[\textit{only}]{\text{design variables}}\left\{\begin{aligned}&\min_{\rho_e} c(\rho_e)=\boldsymbol{f}^T\boldsymbol{u}\ \text{ where } \boldsymbol{u} \text{ solves: } \left(\sum_{e=1}^{N}\rho_e{}^p\boldsymbol{K}_e\right)\boldsymbol{u}=\boldsymbol{f}\\&\sum_{e=1}^{N}v_e\rho_e\le V\\&0<\rho_{\min}\le\rho_e\le 1\end{aligned}\right.$$

$$c(\rho)=\boldsymbol{f}^T\boldsymbol{u}-\tilde{\boldsymbol{u}}^T\left(\boldsymbol{K}\boldsymbol{u}-\boldsymbol{f}\right)$$

$$\frac{\partial c}{\partial\rho_e}=\left(\boldsymbol{f}^T-\tilde{\boldsymbol{u}}^T\boldsymbol{K}\right)\frac{\partial\boldsymbol{u}}{\partial\rho_e}-\tilde{\boldsymbol{u}}^T\frac{\partial\boldsymbol{K}}{\partial\rho_e}\boldsymbol{u}+\frac{\partial f}{\partial\rho_e}\left(\boldsymbol{u}+\tilde{\boldsymbol{u}}\right)\xrightarrow[\text{adjoint equation}]{\boldsymbol{f}^T-\tilde{\boldsymbol{u}}^T\boldsymbol{K}=0,\ \frac{\partial f}{\partial\rho_e}=0}\frac{\partial c}{\partial\rho_e}=-\tilde{\boldsymbol{u}}^T\frac{\partial\boldsymbol{K}}{\partial\rho_e}\boldsymbol{u}=-p\rho_e{}^{p-1}\boldsymbol{u}^T\boldsymbol{K}_e\boldsymbol{u}$$

Sensitivity Analysis

$$\boxed{\Phi = \Phi\left(\boldsymbol{U}\left(\boldsymbol{\rho}\right)\right),\ \boldsymbol{K}\left(\boldsymbol{\rho}\right)\boldsymbol{U} = \boldsymbol{F}}$$

- Direct method
  - One FE load case per design variable

$$\Phi' = \frac{\partial \Phi}{\partial \boldsymbol{U}}\boldsymbol{U}'$$
$$\boldsymbol{K}'\boldsymbol{U} + \boldsymbol{K}\boldsymbol{U}' = 0$$
$$\rightarrow \boldsymbol{K}\boldsymbol{U}' = -\boldsymbol{K}'\boldsymbol{U}$$

- Adjoint Method
  - One FE load case per constraint

$$\hat{\Phi} = \Phi + \boldsymbol{\lambda}^T\left(\boldsymbol{K}\boldsymbol{U} - \boldsymbol{F}\right)$$
$$\hat{\Phi}' = \frac{\partial \Phi}{\partial \boldsymbol{U}}\boldsymbol{U}' + \boldsymbol{\lambda}^T\left(\boldsymbol{K}'\boldsymbol{U} + \boldsymbol{K}\boldsymbol{U}'\right)$$
$$= \underbrace{\left(\boldsymbol{\lambda}^T\boldsymbol{K} + \frac{\partial \Phi}{\partial \boldsymbol{U}}\right)}_{=0 \rightarrow \boldsymbol{K}^T\boldsymbol{\lambda} = -\frac{\partial \Phi}{\partial \boldsymbol{U}}}\boldsymbol{U}' + \boldsymbol{\lambda}^T\boldsymbol{K}'\boldsymbol{U}$$
$$= \boldsymbol{\lambda}^T\boldsymbol{K}'\boldsymbol{U}$$

Compliance Sensitivity

$$\boxed{\Phi = \boldsymbol{F}^T \boldsymbol{U}(\boldsymbol{\rho}),\ \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F}}$$

$$\Phi = \boldsymbol{F}^T \boldsymbol{U}$$

$$\boldsymbol{K}^T \boldsymbol{\lambda} = -\frac{\partial \Phi}{\partial \boldsymbol{U}} = -\boldsymbol{F} \rightarrow \boldsymbol{\lambda} = -\boldsymbol{U}$$

$$\hat{\Phi}' = \boldsymbol{\lambda}^T \boldsymbol{K}'\boldsymbol{U} \rightarrow \hat{\Phi}' = -\boldsymbol{U}\boldsymbol{K}'\boldsymbol{U} = -p\rho_e{}^{p-1}\boldsymbol{U}_e{}^T \boldsymbol{K}_e^0 \boldsymbol{U}_e$$

$$\boldsymbol{K} = \sum_e \boldsymbol{K}_e$$

$$\boldsymbol{K}_e = \rho_e{}^p \boldsymbol{K}_e^0$$

OC Update

- Optimality criteria method
  - Fully stressed design condition in plastic design
  - Specific strain energy vs. Lagrange multiplier Λ
  - Independent update of design variable

$$\frac{\partial E_{ijkl}}{\partial \rho}\varepsilon_{ij}\left(u\right)\varepsilon_{kl}\left(u\right)=\Lambda \rightarrow B_K = \frac{p\rho\left(x\right)^{p-1}E^0_{ijkl}\varepsilon_{ij}\left(u_K\right)\varepsilon_{kl}\left(u_K\right)}{\Lambda} \rightarrow B_K \begin{cases} >1 \\ =1 \\ <1 \end{cases}$$

$$\rho_{K+1} = \begin{cases} \max\left\{\left(1-\zeta\right)\rho_K, \rho_{\min}\right\} & \text{if } \rho_K {B_K}^{\eta} \le \max\left\{\left(1-\zeta\right)\rho_K, \rho_{\min}\right\} \\ \min\left\{\left(1+\zeta\right)\rho_K, 1\right\} & \text{if } \min\left\{\left(1+\zeta\right)\rho_K, 1\right\} \le \rho_K {B_K}^{\eta} \\ \rho_K {B_K}^{\eta} & \text{otherwise} \end{cases}$$

$$\rho_{\min} = 10^{-3},\ \eta = 0.5\left(\text{tuning parameter}\right),\ \zeta = 0.2\left(\text{move limit}\right)$$

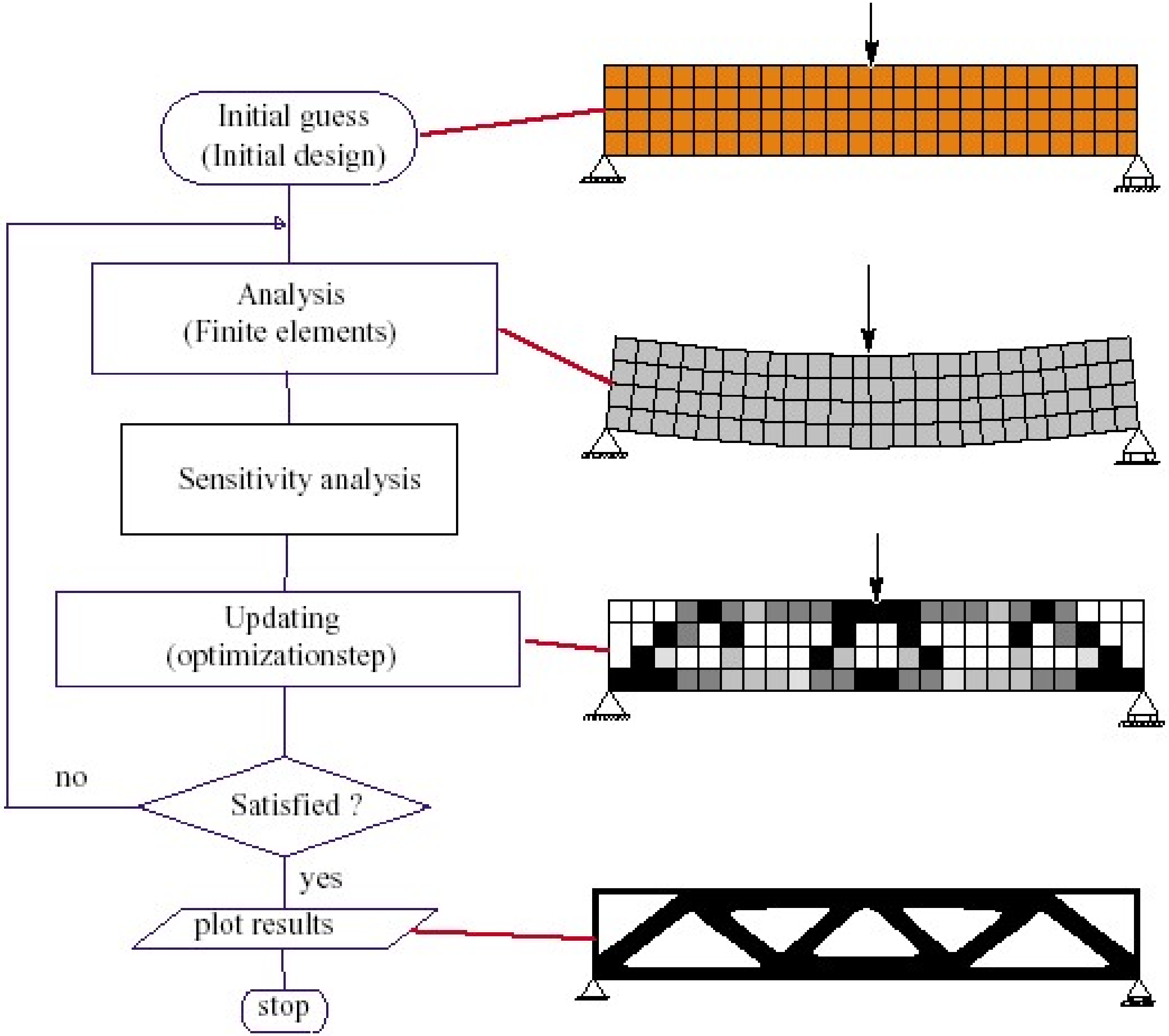
Topology Optimization: Formulation

$$\left.\begin{array}{l} \min\limits_{DV} \int_\Omega \rho d\Omega \to \sum\limits_{e=1}^{N} v_e \rho_e \\ \text{subject to } a(\mathbf{u},\mathbf{v}) = f(\mathbf{v}) \forall \mathbf{v} \to \mathbf{K}(\rho)\mathbf{U} = \mathbf{F} \\ \qquad \left.\begin{array}{l} \bar{\sigma} \le \sigma_{\max} \\ |\mathbf{u}| \le u_{\max} \end{array}\right\} \to \mathbf{u}^T \mathbf{f} \to \mathbf{U}^T \mathbf{F} \end{array}\right\} \xrightarrow[E(\rho_e)=\rho_e{}^p E_0]{\text{dual problem}} \left\{\begin{array}{l} \min\limits_{\rho} \mathbf{U}^T\mathbf{F} = \phi(\mathbf{U}(\rho)) \\ \text{subject to } \begin{cases} \mathbf{K}(\rho)\mathbf{U} = \mathbf{F} \\ \sum\limits_{e=1}^{N} v_e \rho_e \le V^* \\ 0 < \rho_e \le 1 \end{cases} \end{array}\right.$$

$$L(\rho, U, \lambda, \Lambda, \lambda_e^-, \lambda_e^+) = \phi + \lambda^T(\mathbf{KU} - \mathbf{F}) + \Lambda\left(\sum_{e=1}^{N} v_e \rho_e - V^*\right) + \sum_{e=1}^{N} \lambda_e^-(-\rho_e) + \sum_{e=1}^{N} \lambda_e^+(\rho_e - 1)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial \rho} = \underbrace{\dfrac{d\phi}{d\rho} + \dfrac{\partial \phi}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial \rho} + \lambda^T\left(\dfrac{\partial \mathbf{K}}{\partial \rho}\mathbf{U} + \mathbf{K}\dfrac{\partial \mathbf{U}}{\partial \rho} - \dfrac{\partial \mathbf{F}}{\partial \rho}\right)}_{\frac{d\phi}{d\rho} + \left(\frac{\partial \phi}{\partial \mathbf{U}} + \lambda^T \mathbf{K}\right)\frac{\partial \mathbf{U}}{\partial \rho} + \lambda^T \frac{\partial \mathbf{K}}{\partial \rho}\mathbf{U} - \frac{\partial \mathbf{F}}{\partial \rho}} + \Lambda v_e - \lambda_e^- + \lambda_e^+ = 0 \\ \dfrac{\partial L}{\partial \mathbf{U}} = \dfrac{\partial \phi}{\partial \mathbf{U}} + \lambda^T \mathbf{K} = 0 \to \mathbf{F} + \mathbf{K}^T \lambda = 0 \to \lambda = -\mathbf{U} \\ -\mathbf{U}\dfrac{\partial \mathbf{K}}{\partial \rho}\mathbf{U} \to -\mathbf{U}_e^T \dfrac{\partial\left(\sum\limits_e \mathbf{K}_e\right)}{\partial \rho_e}\mathbf{U}_e = -\mathbf{U}_e^T\left(p\rho_e^{p-1}\mathbf{K}_e^0\right)\mathbf{U}_e \end{array}\right\} \to \begin{cases} -\mathbf{U}_e^T\left(p\rho^{p-1}\mathbf{K}_e^0\right)\mathbf{U}_e + \Lambda v_e = 0 \\ 1 = \dfrac{\mathbf{U}_e^T\left(p\rho_e^{p-1}\mathbf{K}_e^0\right)\mathbf{U}_e}{\Lambda v_e} = B_e \\ \rho_e^{new} = \rho_e (B_e)^\eta \end{cases}$$

Implementation



Minimize Compliance vs. Stress

$$l_{m.c.} = a(\boldsymbol{u},\boldsymbol{u}) = \int_D \boldsymbol{\varepsilon}(\boldsymbol{u}):\chi_\Omega \boldsymbol{E}:\boldsymbol{\varepsilon}(\boldsymbol{u})d\Omega = \int_D \boldsymbol{\sigma}:\chi_\Omega \boldsymbol{C}:\boldsymbol{\sigma} d\Omega$$

(double dot products)

$$\boldsymbol{A}:\boldsymbol{B} = A_{ij}B_{ij} = A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}$$

$$\bar{\sigma}^2 = \frac{\mathbf{1}}{\mathbf{2}}[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2] + 3({\sigma_{12}}^2 + {\sigma_{23}}^2 + {\sigma_{31}}^2)$$

$$= \boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \le {\sigma_{max}}^{\mathbf{2}} \text{ where } \boldsymbol{M} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

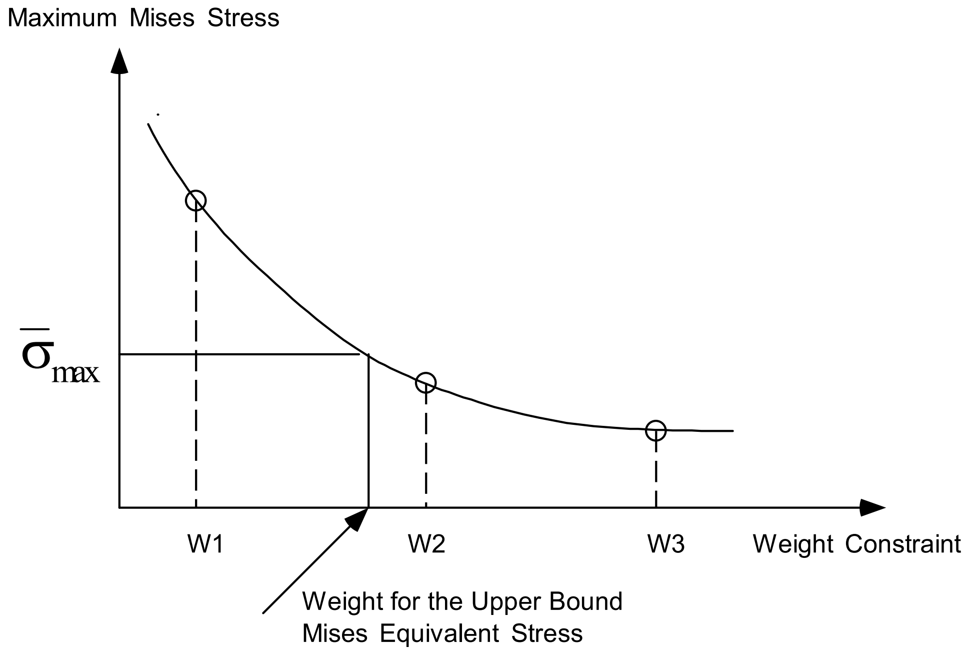
$$(\text{3D}) \begin{cases} \frac{1}{E}\frac{2(1+\nu)}{3}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \le \boldsymbol{\sigma}:\boldsymbol{C}:\boldsymbol{\sigma} \le \frac{1}{E}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \quad \left(\text{if } \nu \ne \frac{1}{2}\right) \to \bar{\sigma} \le \sqrt{\frac{3E}{2(1+\nu)}\boldsymbol{\sigma}:\boldsymbol{C}:\boldsymbol{\sigma}} \\ \frac{1}{E}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} = \boldsymbol{\sigma}:\boldsymbol{C}:\boldsymbol{\sigma}\nu \quad \left(\text{if } \nu = \frac{1}{2}\right) \end{cases}$$

$$\to \begin{cases} \textit{plane stress}: \frac{1}{E}\frac{2(1+\nu)}{3}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \le \boldsymbol{\sigma}:\boldsymbol{C}:\boldsymbol{\sigma} \le \frac{1}{E}2(1-\nu)\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \\ \textit{plane strain}: \frac{1}{E}\frac{2(1+\nu)}{3}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \le \boldsymbol{\sigma}:\boldsymbol{C}:\boldsymbol{\sigma} \le \frac{1}{E}\frac{2(1-\nu+6\nu^2)}{(1+2\nu)^2}\boldsymbol{\sigma}:\boldsymbol{M}:\boldsymbol{\sigma} \end{cases}$$

$$\int_D \chi_\Omega \bar{\sigma}^2 d\Omega \le \frac{3E}{2(1+\nu)}\int_D \boldsymbol{\sigma}:\chi_\Omega \boldsymbol{C}:\boldsymbol{\sigma} d\Omega = \frac{3E}{2(1+\nu)} l_{m.c.}[= {\sigma_{max}}^{\mathbf{2}} mes(\Omega_d)] \to l_{m.c.} = \frac{2(1+\nu)}{3E}{\sigma_{max}}^{\mathbf{2}} mes(\Omega_d)$$

Stress Constraints (1)

- Using the quadratic interpolation of
  - maximum Mises stresses & weights
  - compute the weight for the allowable stress constraints
  - determination of the weight constraint to enforce the stress constraint if exist



Stress Constraints (2)

$$\left.\begin{aligned} &(\text{multiple objectives}) \\ &\min_{\boldsymbol{\rho}} \ \underbrace{\max_{e} (\sigma_e)_{VM}}_{non-differentiable} \\ &\text{subject to } \sum_{e=1}^{N} v_e \rho_e \le V^* \\ &\qquad\qquad 0<\rho_{\min} \le \boldsymbol{\rho} \le 1 \\ &\qquad\qquad \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &(\text{multiple constraints}) \\ &\min_{\boldsymbol{\rho}} \ \sum_{e=1}^{N} v_e \rho_e \\ &\text{subject to } (\sigma_e)_{VM} \le {\rho_e}^p \sigma^* \ \text{ if } \rho_e > 0, \ e = 1, \ldots, N \\ &\qquad\qquad 0<\rho_{\min} \le \boldsymbol{\rho} \le 1 \\ &\qquad\qquad \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{U} = \boldsymbol{F} \end{aligned}\right.$$

$$(\sigma_e)_{VM} \le {\rho_e}^p \sigma^* \rightarrow \left[\frac{(\sigma_e)_{VM}}{{\rho_e}^p \sigma^*} - 1\right] \le 0 \rightarrow \rho_e \left[\frac{(\sigma_e)_{VM}}{{\rho_e}^p \sigma^*} - 1\right] \le 0 \ \ (\text{singularity})$$

$$\rho_e \left[\frac{(\sigma_e)_{VM}}{{\rho_e}^p \sigma^*} - 1\right] \le \varepsilon(1-\rho_e), \ \varepsilon^2 = \rho_{\min} \le \rho_e \ \ (\varepsilon\text{-relaxed stress constraint})$$

Stress Constraints (3)

$$\left.\begin{aligned}&\min_{\boldsymbol{\rho}}\ \underbrace{\max_{e}\left(\sigma_e\right)_{VM}}_{non-differentiable}\\&\text{subject to}\ \sum_{e=1}^{N} v_e\rho_e \le V^*\\&\qquad 0<\rho_{\min} \le \boldsymbol{\rho} \le 1\\&\qquad \boldsymbol{K}\left(\boldsymbol{\rho}\right)\boldsymbol{U} = \boldsymbol{F}\end{aligned}\right\}\xrightarrow{\text{multiple objectives}\rightarrow\text{one objective}}\left\{\begin{aligned}&\min_{\boldsymbol{\rho}}\ \underbrace{\left(\sum_{e=1}^{N}\left(\sigma_e\right)_{VM}{}^{P}\right)^{1/P}}_{p-\text{norm measure}}\\&\text{subject to}\ \sum_{e=1}^{N} v_e\rho_e \le V^*\\&\qquad 0<\rho_{\min} \le \boldsymbol{\rho} \le 1\\&\qquad \boldsymbol{K}\left(\boldsymbol{\rho}\right)\boldsymbol{U} = \boldsymbol{F}\end{aligned}\right.$$

$$\rightarrow\left\{\begin{aligned}&\min_{\boldsymbol{\rho}}\ \sum_{e=1}^{N} v_e\rho_e\\&\text{subject to}\ \left(\sum_{e=1}^{N}\left(\sigma_e\right)_{VM}{}^{P}\right)^{1/P} \le \sigma^*\\&\qquad 0<\rho_{\min} \le \boldsymbol{\rho} \le 1\\&\qquad \boldsymbol{K}\left(\boldsymbol{\rho}\right)\boldsymbol{U} = \boldsymbol{F}\end{aligned}\right.$$

Three significant challenges

- Singularity phenomenon → relaxation
- Local nature of the constraint → single integrated stress constraint, not adequate
- Highly non-linear stress behavior: drastic stress level changes by density

Kreisselmeier-Steinhause (KS) function: $G_{ks} = \frac{1}{p} \ln \sum_{i=1}^{N} e^{p \frac{f_i(\sigma)}{f_{\max}(\sigma)}}$

Park-Kikuchi (KK) function: $G_{kk} = \left\{ \int_{\Omega} \left[\frac{f_i(\sigma)}{f_{\max}(\sigma)} \right]^p d\Omega \right\}^{1/p}$

N : number of finite elements

σ : stress

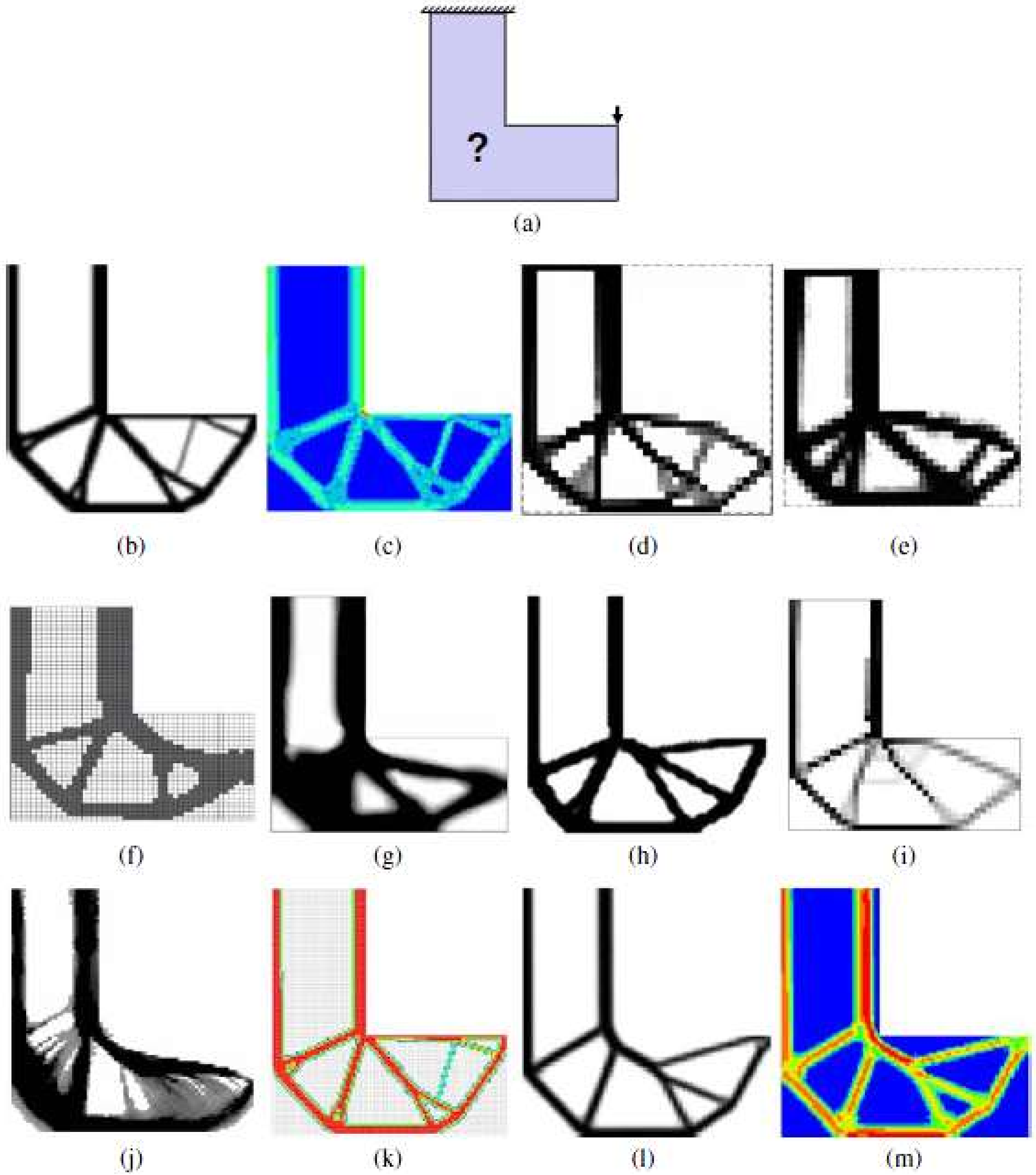
$f_i(\sigma)$: maximum von Mises stress

$f_{\max}(\sigma)$: von Mises stress for each finite element

p : parameter determining the difference the original function and its approximation $(=20)$

$p \uparrow$: peak stress is weighted more heavily $\rightarrow$ oscillation or diverge

Fig. 1 Various L-bracket designs: **b** and **c** are compliance based; **d–m** are stress-based. (Stress plots are based on relaxed von Mises stress, cf. Section 3, and have different color scale ranges). **a** Problem definition, **b** Min. compliance design, **c** Stress distribution in **b**, **d** Duysinx and Bendsøe (1998), **e** Duysinx and Sigmund (1998), **f** Svanberg and Werme (2007), g Pereira et al. (2004), **h** Bruggi and Venini (2008), **i** París et al. (2009), **j** Guilherme and Fonseca (2007), **k** Altair Optistruct (2007) (density distribution), **l** Present work, **m** Stress distribution in **l**



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RESEARCH PAPER

Stress-based topology optimization for continua

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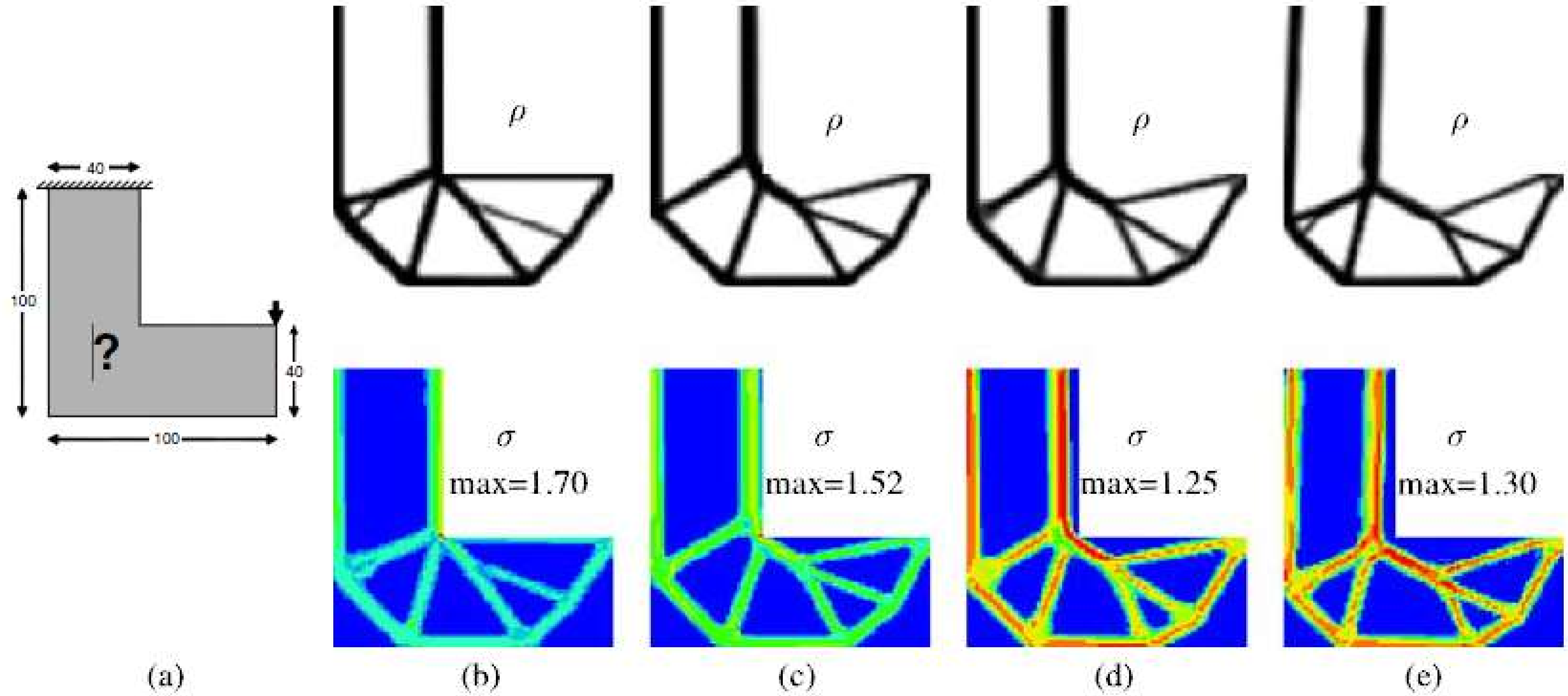

Fig. 3 Effect of stress norm parameter P on the L-bracket. **a** Problem definition, **b** $P = 4$ (63 iter.), **c** $P = 6$ (72 iter.), **d** $P = 8$ (78 iter.), **e** $P = 12$ (235 iter.)