Design Parameterization

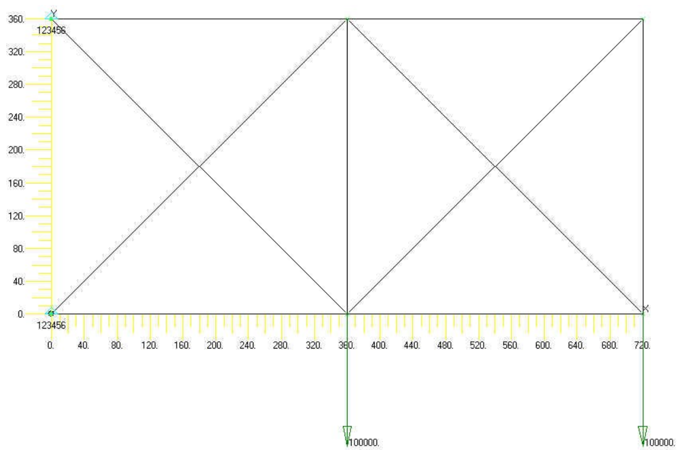
- Sizing design parameters
  - Material properties
    - Density, modulus, viscosity
  - Boundary conditions
    - Applied loads or displacements
  - Element properties
    - Bar cross section, plate thickness, node point locations
- Shape design parameters
  - Coordinates of a single node point location ☹
  - Relationship between a shape parameter and each of the node locations
    - Geometry-based mesh parameterization
    - Reduced basis approach

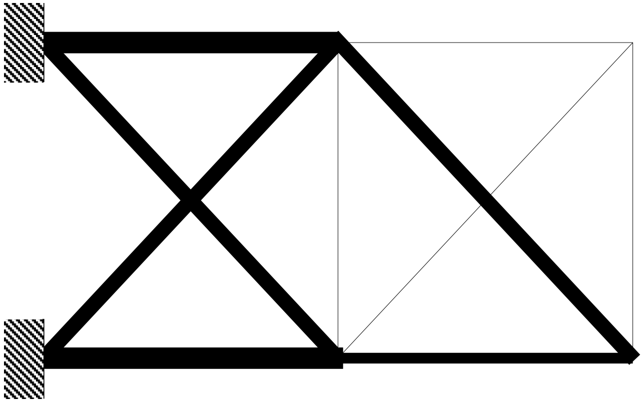
$$\min_{\text{design}} \int_{\Omega} \rho \, d\Omega$$

$$\begin{aligned} &\text{subject to} \\ &a(\boldsymbol{u},\boldsymbol{v}) = f(\boldsymbol{v}) \; \forall \boldsymbol{v} \\ &\bar{\sigma} \le \sigma_{\max} \\ &|\boldsymbol{u}| \le u_{\max} \end{aligned}$$

Size Optimization

- 2nd ASCE Conference on Electronic Computation, Pittsburgh, PA, September 8-9, 1960
  - R. W. Clough, The Finite Element Method in Plane Stress Analysis
  - L. A. Schmit, Structural Design by Systematic Synthesis
- Weight minimization with stress and displacement constraints
  - Truss, beam, other space frame type structures
  - Aerospace and civil engineering



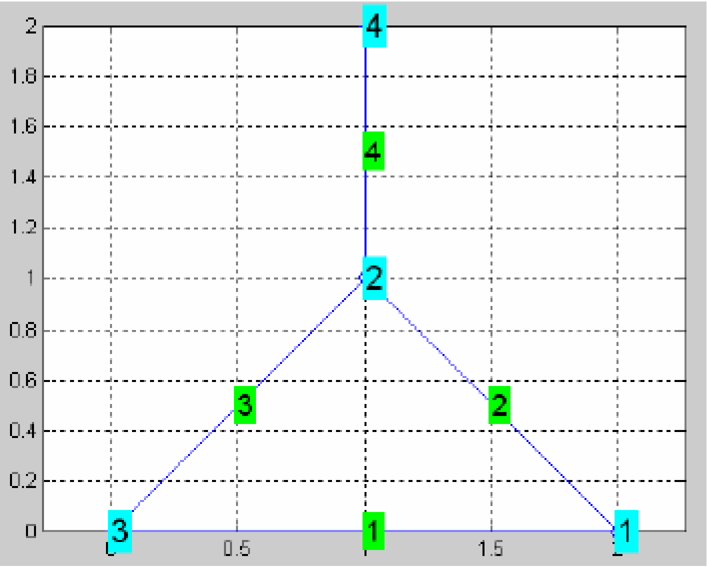


10 bar truss

Truss System

- Tensile bar of length *l*
  - Uniform material (Young's modulus E), cross section (area A)
- Axially loaded bars joined together at pinned joints
  - Transfer forces, but not moments

$$\sigma = \frac{P}{A}, \varepsilon = \frac{\sigma}{E} = \frac{P}{EA}, \Delta l = l\varepsilon = \frac{Pl}{EA}$$




```
Truss.NodeXY = [2 1 0 1; 0 1 0 2];
Truss.Connectivity = [1 3; 1 2; 2 3; 2 4]';
Truss.E = 2e11*[1 1 1 1];
Truss.Area = 0.0001*[1 1 1 1];
Truss.FreeNodes = [1 2];
Truss.Force = [0 -100;0 0]';
```

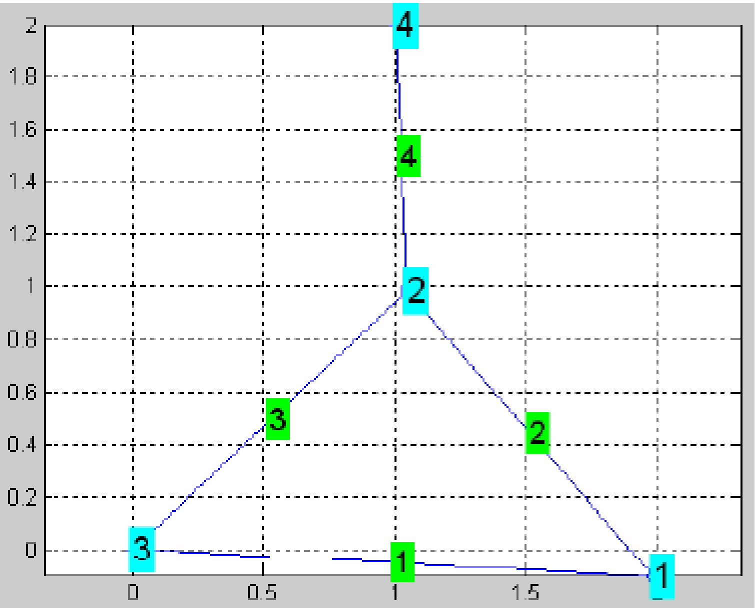
Node 1: downward vertical force P
Node 3, 4: pinned

Truss Analysis

- Find the displacements $u \equiv \{u_1, v_{1,} u_2, v_{2,} \ldots\}^T$
- Linear approximation of spring systems

$$\boldsymbol{Ku} = \boldsymbol{f} \quad \text{where} \quad \boldsymbol{K} = \sum_{i=1}^{n} k_i \quad (\text{stiffness matrix})$$

$$k_i = \frac{EA_i}{l_l}\begin{bmatrix} & \end{bmatrix} \quad (\text{element stiffness matrix})$$



$$\{u_1, v_{1,} u_2, v_2\} = 10^4 [-0.1 \quad -0.583 \quad 0.24 \quad -0.1]$$

$$\sigma = E\frac{\Delta l}{l}$$

$$\{\sigma_1, \sigma_{2,} \sigma_3, \sigma_4\} = 10^6 [-1 \quad 1.41 \quad 1.41 \quad 2]$$

$$\sigma_1 = -\frac{P}{A_1}, \sigma_2 = \frac{\sqrt{2}P}{A_2}, \sigma_3 = \frac{\sqrt{2}P}{A_3}, \sigma_4 = \frac{2P}{A_4}$$

Truss Optimization

– Find the optimal areas such that the total volume of the truss system is minimized, with the following constraints: (1) the tip displacement of node-1 does not exceed 0.01, and (2) the stresses within the members lies between 160MPa and -80 MPa.

$$
\left.\begin{array}{l}
\underset{A_i}{Minimize} \sum_{i=1}^{4} A_i l_i \\
v_1 - 0.01 \le 0 \\
\sigma_i - 160\times10^6 \le 0, \quad i = 1,\dots,4 \\
-\sigma_i + 80\times10^6 \le 0, \quad i = 1,\dots,4
\end{array}\right\}
\xrightarrow[positive]{A_i}
\left\{\begin{array}{l}
\underset{A_i}{Minimize} \sum_{i=1}^{4} A_i l_i \\
v_1 - 0.01 \le 0 \\
\sigma_i - 160\times10^6 \le 0, \quad i = 1,\dots,4 \\
-\sigma_i + 80\times10^6 \le 0, \quad i = 1,\dots,4 \\
-A_i + 10^{-8} \le 0, \quad i = 1,\dots,4
\end{array}\right\}
\xrightarrow{scaling}
\left\{\begin{array}{l}
\underset{A_i}{Minimize} \left(\sum_{i=1}^{4} A_i l_i \right) \times 10^6 \\
\dfrac{v_1}{0.01} - 1 \le 0 \\
\dfrac{\sigma_i}{160\times10^6} - 1 \le 0, \quad i = 1,\dots,4 \\
-\dfrac{\sigma_i}{80\times10^6} + 1 \le 0, \quad i = 1,\dots,4 \\
-\dfrac{A_i}{10^{-8}} + 1 \le 0, \quad i = 1,\dots,4
\end{array}\right.
$$

Truss Optimization: Matlab

```
Truss = PreprocessTruss(Truss);
Truss.SigmaTensionMax = 160e6;
Truss.SigmaCompressionMax = 80e6;
Truss.DeflectionMax = 0.01;
Truss.AMin = 1e-8;
Truss.VolumeScale = 1e6;
options = optimset('LargeScale','off','Diagnostics','Off');
N =length(Truss.Area);
xBar0 = Truss.Area;
LB = Truss.AMin*ones(1,N);
[FinalArea,Vol,Flag,output,lambda] = fmincon(@(x)fObj1(x,Truss),xBar0,...
    [],[],[],[],LB,[],@(x)fCon1(x,Truss),options);
```

Truss Sensitivity

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \rightarrow \frac{\partial(\boldsymbol{K}\boldsymbol{u})}{\partial A_j} = \frac{\partial(\boldsymbol{f})}{\partial A_j} \rightarrow \frac{\partial \boldsymbol{K}}{\partial A_j}\boldsymbol{u} + \boldsymbol{K}\frac{\partial \boldsymbol{u}}{\partial A_j} = 0 \rightarrow \boldsymbol{K}\frac{\partial \boldsymbol{u}}{\partial A_j} = -\frac{\partial \boldsymbol{K}}{\partial A_j}\boldsymbol{u}$$

$$\frac{\partial \boldsymbol{K}}{\partial A_j} = \sum_{i=1}^{n} \frac{\partial k_i}{\partial A_j} \quad \text{where} \quad \frac{\partial k_i}{\partial A_j} = \begin{cases} \frac{E}{l_l}\left[\quad \right], & i = j \\ 0, & i \neq j \end{cases}$$

```
Truss = PreprocessTruss(Truss);
Truss.SigmaTensionMax = 160e6;
Truss.SigmaCompressionMax = 80e6;
Truss.DeflectionMax = 0.01;
Truss.AMin = 1e-8;
Truss.VolumeScale = 1e6;
options = optimset('LargeScale','off','Diagnostics','Off',...
                   'GradObj','On','GradConstr','On');
N =length(Truss.Area);
xBar0 = Truss.Area;
LB = Truss.AMin*ones(1,N);
AInEq = -eye(N,N)/Truss.AMin; % Can use these instead of LB
BInEq = -ones(N,1); % Can use these instead of LB
[FinalArea,Vol,Flag,output,lambda] = ...
fmincon(@(x)fObjWithGradients1(x,Truss),xBar0,...
    [],[],[],[],LB,[],@(x)fConWithGradients1(x,Truss),options);
```

Continuum Tensile Bars

- Non-uniform cross section: A(x)
- Non-uniform material: E(x)
- Body force: b(x)
- Find the displacement of the bar u(x)

- Force balance: $$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b(x)A(x)=0+2B.C.s$$

- Minimum potential energy:

$$\underset{u(x)}{Minimize}\left[\frac{1}{2}\int_0^l EA\left(\frac{du}{dx}\right)^2 dx-\int_0^l buAdx\right]$$

$$\xrightarrow{\text{Euler-Lagrange equation}}\frac{d}{dx}\left(EA\frac{du}{dx}\right)+bA=0$$

- Force balance

$$(\sigma+\Delta\sigma)\left(A(x)+\Delta A\right)-\sigma A(x)+b(x)A(x)\Delta x=0$$

$$\sigma\frac{dA}{dx}+\frac{d\sigma}{dx}A(x)+b(x)A(x)=0$$

$$\frac{d(A\sigma)}{dx}+b(x)A(x)=0$$

$$\xrightarrow{\sigma=E\varepsilon}\frac{d(EA\varepsilon)}{dx}+b(x)A(x)=0$$

$$\xrightarrow{\varepsilon=\frac{du}{dx}}\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b(x)A(x)=0$$

$$B.C.\begin{Bmatrix}u(0)=0\\ \sigma|_{x=l}=0\end{Bmatrix}\rightarrow\begin{Bmatrix}u(0)=0\\ \left.\dfrac{du}{dx}\right|_{x=l}=0\end{Bmatrix}$$

- Minimum potential energy

$$U=\frac{1}{2}\int_V\varepsilon\sigma dV=\frac{1}{2}\int_0^l\varepsilon\sigma Adx$$

$$=\frac{1}{2}\int_0^l\frac{du}{dx}E\frac{du}{dx}Adx=\frac{1}{2}\int_0^l EA\left(\frac{du}{dx}\right)^2dx$$

$$W=\int_0^l buAdx$$

$$\Pi=\frac{1}{2}\int_0^l EA\left(\frac{du}{dx}\right)^2dx-\int_0^l buAdx$$

$$\underset{u(x)}{Minimize}\left[\frac{1}{2}\int_0^l EA\left(\frac{du}{dx}\right)^2dx-\int_0^l buAdx\right]$$

$$u(0)=0$$

$$\underset{u(x)}{Minimize}\quad F\left(x,u,\frac{du}{dx}\right)\xrightarrow[\text{equation}]{\text{Euler-Lagrange}}\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0$$

$$F=\frac{1}{2}EA\left(\frac{du}{dx}\right)^2-buA\rightarrow\left\{\begin{aligned}&\frac{\partial F}{\partial u}=-bA\\&\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=\frac{d}{dx}\left(EA\frac{du}{dx}\right)\end{aligned}\right\}\rightarrow -bA-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=0$$

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+bA=0$$

Examples

$$(\mathrm{i})\ A(x) = A_0,\ E(x) = E_0,\ b(x) = b_0$$

$$\rightarrow u(x) = \frac{b_0 l}{E_0} x - \frac{b_0}{2E_0} x^2$$

$$(\mathrm{ii})\ A(x) = A_0 e^{-x},\ E(x) = E_0,\ b(x) = \frac{b_0}{1+x^2},\ l = 1$$

$$\rightarrow u(x) = 0.8613 \frac{b_0}{E_0} x - 0.3744 \frac{b_0}{E_0} x^2$$

Necessary Condition for an Extremum

- Real-valued function f: slope of the tangent

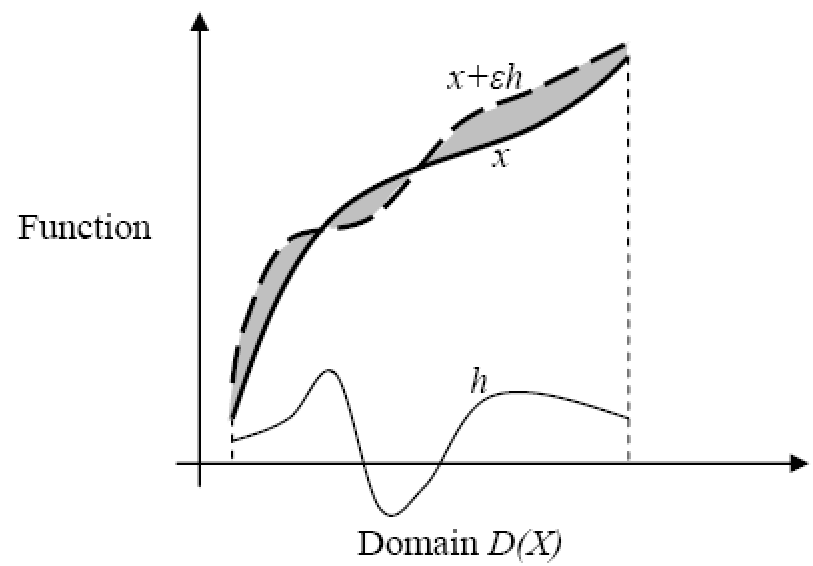
$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

- Functional J: (Gâteaux) variation

$$\delta J(x;h) = \lim_{\varepsilon \to 0} \frac{J(x+\varepsilon h) - J(x)}{\varepsilon}$$

$$= \frac{d}{d\varepsilon} J(x+\varepsilon h)\Big|_{\varepsilon=0}$$

where h is any vector in a vector space, X



Calculus of Variations (1)

$$I = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx, \qquad \underbrace{y\left(x_1\right) = y_1 \ and \ y\left(x_2\right) = y_2}_{\text{end condition}}$$

$$Min\left(Max\right) \rightarrow y = y\left(x\right)?$$

$$y + \delta y = y + \varepsilon h\left(x\right)$$

$$\begin{cases} \varepsilon : \text{ amplitude parameter} \\ h\left(x\right) : \text{ shape function, } h\left(x_1\right) = h\left(x_2\right) = 0 \end{cases}$$

$$I\left(\varepsilon\right) = \int_{x_1}^{x_2} F\left(x, y + \varepsilon h, y' + \varepsilon h'\right) dx \rightarrow \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \\ \dfrac{1}{y'}\left[\dfrac{d}{dx}\left(F - \dfrac{\partial F}{\partial y'}\dfrac{dy}{dx}\right) - \dfrac{\partial F}{\partial x}\right] = 0 \end{array}\right\} \rightarrow \text{Euler-Lagrange equation}$$

Calculus of Variations (2)

$$\left.\begin{aligned}&\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\frac{J\left(x^{*}+\varepsilon h\right)-J\left(x^{*}\right)}{\varepsilon}\ge 0\\ &\lim_{\substack{\varepsilon\to 0\\ \varepsilon<0}}\frac{J\left(x^{*}+\varepsilon h\right)-J\left(x^{*}\right)}{\varepsilon}\le 0\end{aligned}\right\}\underbrace{\lim_{\varepsilon\to 0}\frac{J\left(x^{*}+\varepsilon h\right)-J\left(x^{*}\right)}{\varepsilon}}_{\substack{\text{existence of Gateaux variation}\\ \text{ensures the existence of this limit}}}=\delta J\left(x;h\right)=0$$

$$I\left(\varepsilon\right)=\int_{x_1}^{x_2}F\left(x,y+\varepsilon h,y'+\varepsilon h'\right)dx\rightarrow\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}=0$$

$$\frac{dI}{d\varepsilon}=\int_{x_1}^{x_2}\frac{dF\left(x,y+\varepsilon h,y'+\varepsilon h'\right)}{d\varepsilon}dx$$

$$=\int_{x_1}^{x_2}\left(\frac{\partial F\left(x,y+\varepsilon h,y'+\varepsilon h'\right)}{\partial x}\frac{\partial x}{\partial\varepsilon}+\frac{\partial F\left(x,y+\varepsilon h,y'+\varepsilon h'\right)}{\partial\left(y+\varepsilon h\right)}\frac{\partial\left(y+\varepsilon h\right)}{\partial\varepsilon}+\frac{\partial F\left(x,y+\varepsilon h,y'+\varepsilon h'\right)}{\partial\left(y'+\varepsilon h'\right)}\frac{\partial\left(y'+\varepsilon h'\right)}{\partial\varepsilon}\right)dx$$

extermum: $\dfrac{dI}{d\varepsilon}=0$ at $\varepsilon=0$

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}h+\frac{\partial F}{\partial y'}h'\right)dx=0$$

$$\rightarrow\int_{x_1}^{x_2}\frac{\partial F}{\partial y}hdx+\left[\frac{\partial F}{\partial y'}h\right]_{x_1}^{x_2}-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)hdx=\underbrace{\left[\frac{\partial F}{\partial y'}h\right]_{x_1}^{x_2}}_{\text{end condition }(=0)}+\int_{x_1}^{x_2}\underbrace{\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}_{=0}\underbrace{h}_{\substack{\text{arbitray}\\ \text{functinon}}}dx=0$$

Calculus of Variations (3)

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx}+\frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{dy'}{dx}-\frac{\partial F}{\partial y}=0$$

$$\left(\frac{\partial^2 F}{\partial y'^2}\right)\frac{d^2 y}{dx^2}+\frac{\partial^2 F}{\partial y \partial y'}\frac{dy}{dx}+\left(\frac{\partial^2 F}{\partial x \partial y'}-\frac{\partial F}{\partial y}\right)=0 \Leftrightarrow ay''+by'+c=0 \text{ with two end points}$$

$$\to \frac{1}{y'}\left[\frac{d}{dx}\left(F-\frac{\partial F}{\partial y'}\frac{dy}{dx}\right)-\frac{\partial F}{\partial x}\right]=0$$

$$\begin{cases} F=F\left(y,y'\right)\to \dfrac{\partial F}{\partial x}\to F-\dfrac{\partial F}{\partial y'}\dfrac{dy}{dx}=const \\ F=F\left(x,y'\right)\to \dfrac{\partial F}{\partial y}\to \dfrac{\partial F}{\partial y'}=const \end{cases}$$

Euler-Lagrange Equation

$$\int_{x_1}^{x_2} F(x, y, y')dx \quad \text{min or max} \ \underset{?}{\overset{\text{O}}{\rightleftarrows}} \ \text{Euler-Lagrange equation}$$

- Boundary conditions
  - Essential: *y*(*x*) is specified at end points, $y(x_1) = y_1$ and/or $y(x_2) = y_2$
  - Natural: *y*(*x*) is NOT specified at end points, $\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0$ and/or $\left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0$

$F = F(x, y, y')$	$F = F(y, y')$	$F = F(x, y')$
$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$	$F - \left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} = const$	$\frac{\partial F}{\partial y'} = const$

Applications

- Geodesic: path minimization
  - In a plane $$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$
  - On a general surface ? $$I = \int_{x_1}^{x_2} ds \quad \text{subject to} \quad g(x, y, z) = 0$$
- Brachistochrone: shortest time (John Bernoulli in 1696)

$$I = \int_A^B dt = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{1 + y'^2}}{v} dx$$

$$\frac{1}{2} m v^2 - \frac{1}{2} m v_1^{\,2} = mg\left(y_1 - y\right) \qquad \begin{cases} v_1 = 0 \rightarrow v = \sqrt{2g\left(y_1 - y\right)} \\ v_1 \neq 0 \rightarrow v = \sqrt{2g\left(y_0 - y\right)} \qquad y_0 = y_1 + \left(\dfrac{v_1^{\,2}}{2g}\right) \end{cases}$$

$$I = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{y_0 - y}} dx$$

Geodesic Problem

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$$

$$F = \sqrt{1+y'^2} \rightarrow \left\{ \begin{array}{l} \dfrac{\partial F}{\partial y} = 0 \\ \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = \dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+y'^2}}\right) \end{array} \right\} \rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\rightarrow \frac{y'}{\sqrt{1+y'^2}} = C \rightarrow y' = \sqrt{\frac{C}{1-C}} = A \rightarrow y = Ax + B$$

Brachistochrone Problem

$$I=\int_A^B dt=\int_A^B \frac{ds}{v}=\int_A^B \frac{\sqrt{1+y'^2}}{v}dx \rightarrow I=\frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1+y'^2}{y_0-y}}dx$$

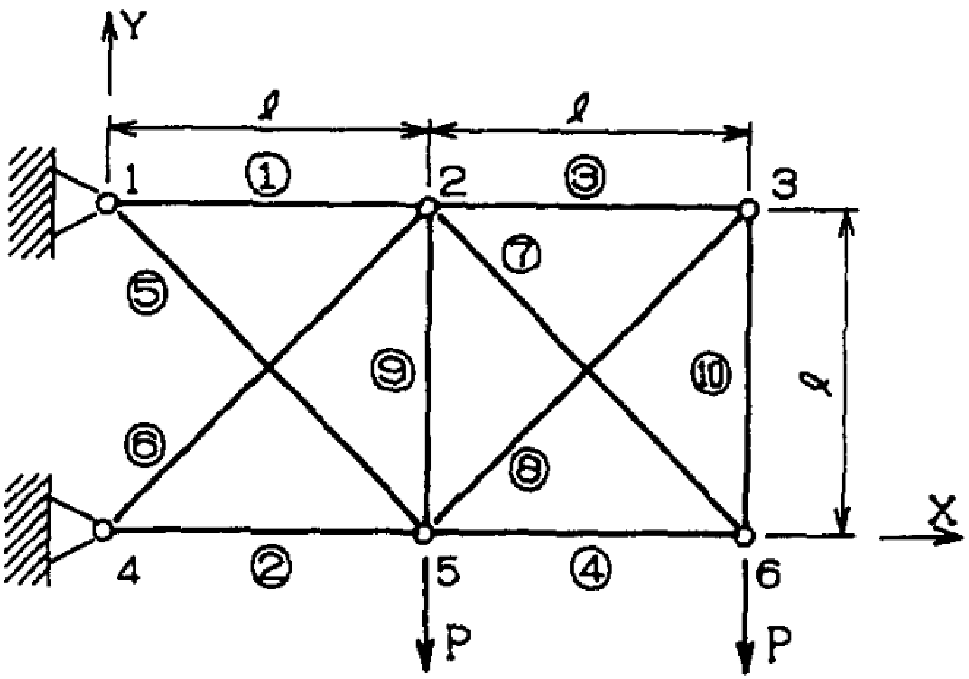
$$\frac{1}{2}mv^2-\frac{1}{2}mv_1^{\ 2}=mg\left(y_1-y\right) \qquad \begin{cases} v_1=0\rightarrow v=\sqrt{2g\left(y_1-y\right)} \\ v_1\neq 0\rightarrow v=\sqrt{2g\left(y_0-y\right)} \qquad y_0=y_1+\left(\dfrac{v_1^{\ 2}}{2g}\right) \end{cases}$$

$$F=\sqrt{\frac{1+y'^2}{y_0-y}}\xrightarrow{\frac{1}{y'}\left[\frac{d}{dx}\left(F-\frac{\partial F}{\partial y'}\frac{dy}{dx}\right)-\frac{\partial F}{\partial x}\right]=0}\sqrt{\frac{1+y'^2}{y_0-y}}-y'\frac{\dfrac{2y'}{y_0-y}}{2\sqrt{\dfrac{1+y'^2}{y_0-y}}}=c\rightarrow\sqrt{\frac{1+y'^2}{y_0-y}}-\frac{y'^2}{\sqrt{\left(y_0-y\right)\left(1+y'^2\right)}}=c$$

$$1=c^2\left(y_0-y\right)\left(1+y'^2\right)\rightarrow\frac{1}{c^2\left(y_0-y\right)}=1+y'^2\rightarrow y'^2=\frac{1-c^2\left(y_0-y\right)}{c^2\left(y_0-y\right)}$$

$$\rightarrow y'=\sqrt{\frac{1-c^2\left(y_0-y\right)}{c^2\left(y_0-y\right)}}\xrightarrow{a=\frac{1}{2c^2}}x=\int\frac{\sqrt{y_0-y}}{\sqrt{2a-\left(y_0-y\right)}}dy\xrightarrow[cycloid]{y_0-y=2a\sin^2\frac{\theta}{2}}\begin{cases} x=2a\displaystyle\int\sin^2\frac{\theta}{2}d\theta=a\left(\theta-\sin\theta\right)+x_0 \\ y=y_0-2a\sin^2\dfrac{\theta}{2}=-a\left(1-\cos\theta\right)+y_0 \end{cases}$$

Optimal Weight Design Problem: 10 Bar Truss



$$\min \quad W(A) = \rho \sum_{i=1}^{10} l_i A_i$$

$$\text{s. t.} \quad G_i = \sigma_i \le b_i, (i = 1, 2, .., 10)$$

$$G_k = v_k \le b_k, (k = 2, 3, 5, 6)$$

$$A_i^L \le A_i \le A_i^U, \quad (i = 1, 2, .., 10),$$

$$\sigma_i^L \le \sigma_i \le \sigma_i^U, \quad (i = 1, 2, .., 10),$$

$$v_k^L \le v_k \le v_k^U, \quad (k = 2, 3, 5, 6),$$

where

$$\sigma_i = \epsilon_i E, (i = 1, 2, .., 10)$$

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \{F\} [K]^{-1}, (k = 2, 3, 5, 6)$$

$11.5 \le A_1 \le 12.5$	$8.0 \le A_2 \le 9.0$
$0.1 \le A_3 \le 1.0$	$5.5 \le A_4 \le 6.5$
$5.5 \le A_5 \le 6.0$	$8.0 \le A_6 \le 9.0$
$8.0 \le A_7 \le 9.0$	$0.1 \le A_8 \le 1.0$
$0.1 \le A_9 \le 1.0$	$0.1 \le A_{10} \le 1.0$
$E = 10^7$	$\rho = 0.1$
$\lvert\sigma\rvert \le 25000$	$\lvert v_6\rvert \le 5.0$
$l_{1-4,9,10} = 360$	$P = 10^5$
$l_{5-8} = 360\sqrt{2}$	

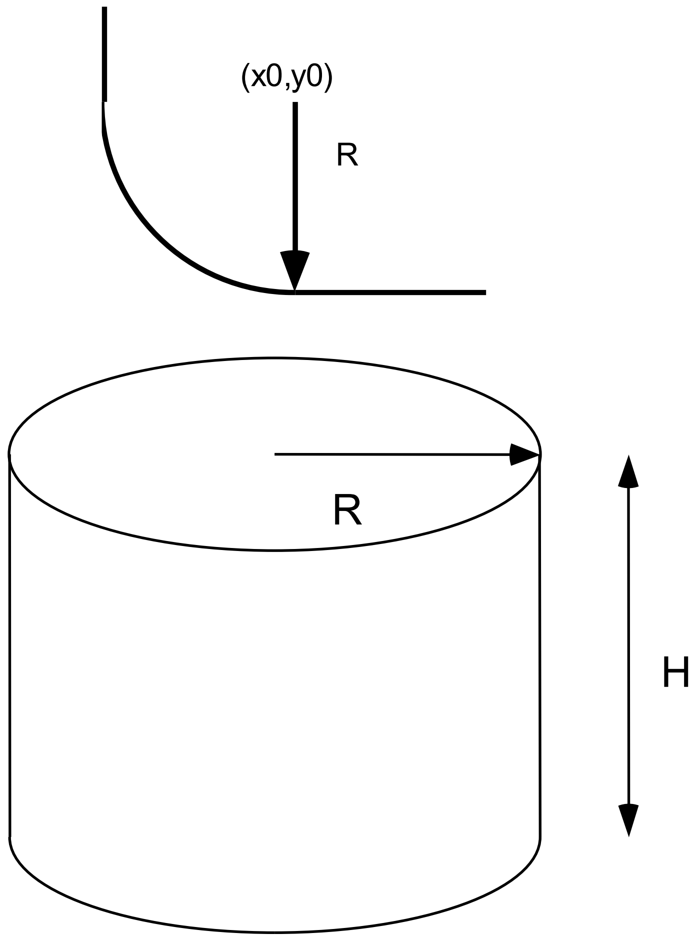
Results: 10 Bar Truss

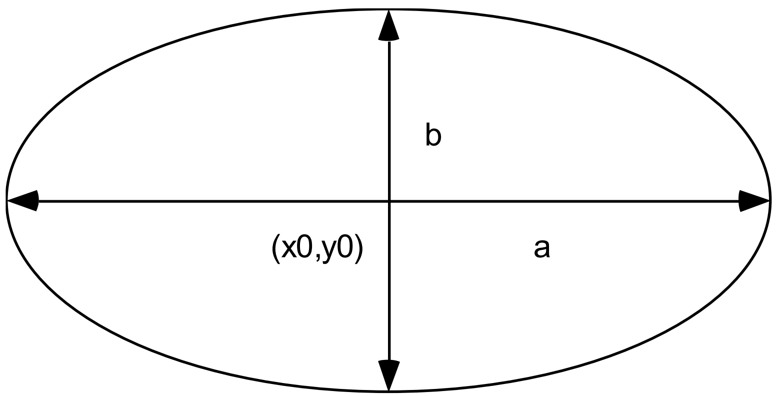
	improved GA	DCOC	Dual	DOC-FSD
A_1	12.131896	12.161173957	12.161173956	12.126576172
A_2	8.794619	8.707029023	8.707029026	8.827450732
A_3	0.100000	0.100000000	0.100000000	0.100000000
A_4	6.065801	6.040579884	6.040579884	6.046585281
A_5	5.100000	5.560164853	5.560164853	5.564322434
A_6	8.539911	8.573640198	8.573640196	8.497882192
A_7	8.575261	8.542669996	8.542669996	8.551162911
A_8	0.100000	0.100000000	0.100000000	0.100000000
A_9	0.100000	0.100000000	0.100000000	0.100000000
A_{10}	0.100000	0.100000000	0.100000000	0.100000000
W(lb)	2118.626	2139.105	2139.105	2139.198

	σ_i	f_i
l_1	166.2779	20215.11096
l_2	-2249.6584	-19784.88904
l_3	475.6522	47.56522
l_4	-1640.7454	-9952.43478
l_5	2713.3182	13837.92279
l_6	-1691.6275	-1446.34846
l_7	1641.3341	14074.86824
l_8	-672.6738	-67.26738
l_9	2626.7618	262.67618
l_{10}	475.6522	47.56522

node	u_k	v_k
1	0	0
2	0.606673	-1.817000
3	0.768973	-4.83595
4	0	0
5	-0.827898	-2.78003
6	-1.422710	-4.99826

Size Design in MCAE

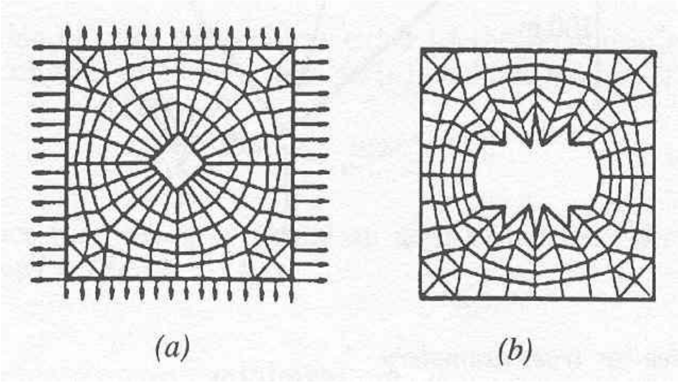




Sizing in Mechanical Design is always related to the shape of a structure !

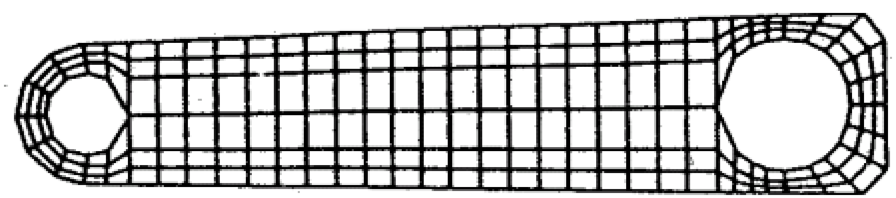
Shape Optimization (1)

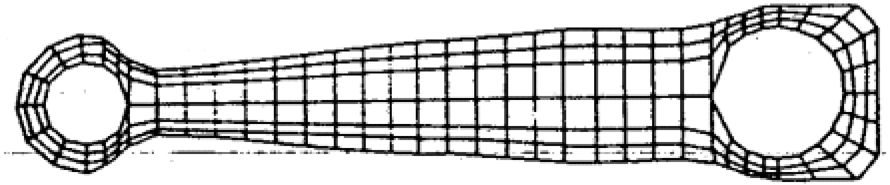
- FEM + Design Sensitivity + SLP
  - O. C. Zienkiewicz and J. S. Campbell, Shape Optimization and Sequential Linear Programming, International Symposium on Optimization of Structural Design, University of Wales, Swansea, January 1973
- Adaptation of Nodal Points on the Boundary
- Without using parametric representation, they adapted the nodes of the finite element model $\rightarrow$ a lot of problem !
  - possibility of non-smoothed optimum shape due to non-smooth stresses on the design boundary
  - possibility of excessive element distortion
  - unclear adaptation schemes



Shape Optimization (2)

- Reducing stresses at a boundary by changing that boundary
- Difficulties in shape optimization
  - Accuracy of the FE analysis? continuously changing FE model
  - Good sensitivity derivatives w.r.t. shape design variables? expensive

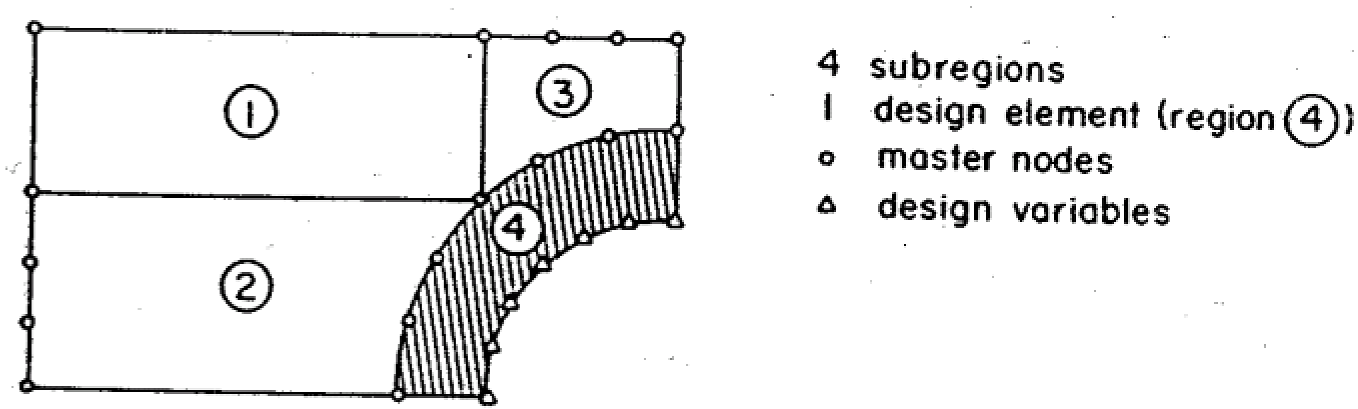




R.T. Haftka and R.V. Grandhi, Structural Shape Optimization–A Survey, *Computer Methods in Applied Mechanics and Engineering*, 57, pp.91-10, 1986

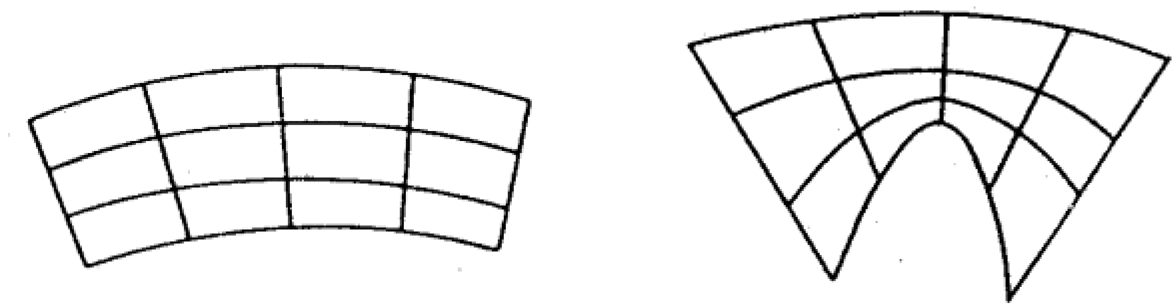
Design Variables

- Coordinates of the boundary nodes of FE model
  - Accuracy problem
- Polynomial coefficients: polynomial describing boundaries
  - High-order? Oscillation
- (cubic) spline, Bezier and B-spline blending functions
- Design element concept



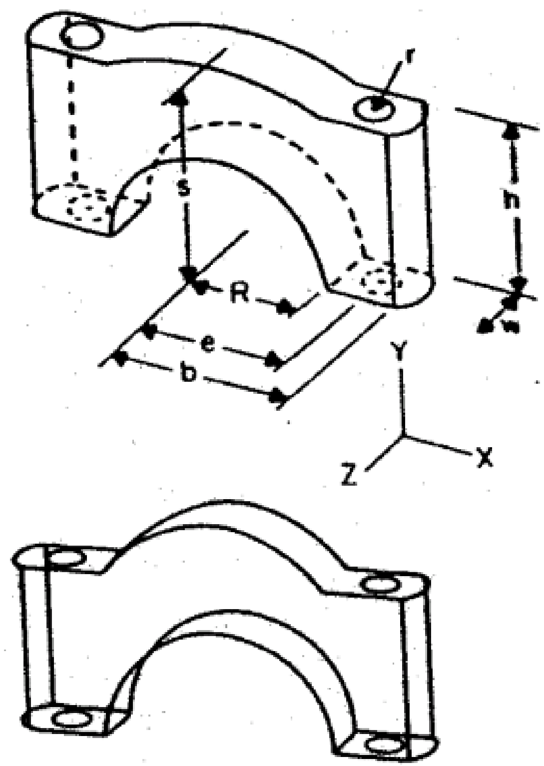
FE mesh generation (1)

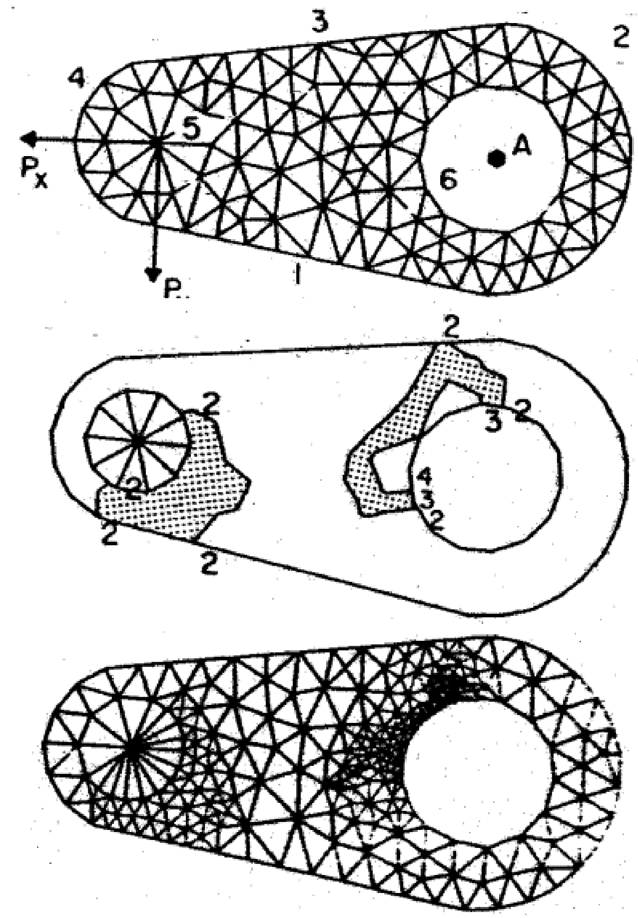
- Manual mesh refinement: simple modification rules for deforming the initial mesh



- Automated mesh generation: adaptive mesh refinement
  - Adding additional elements in the area to be refined
  - Increasing the order of the FE
  - FE mesh points are relocated
  - FE analysis + automatic mesh generation + structural optimization

FE mesh generation (2)





- Optimum mesh refinement
  - Optimum mesh + optimum shape

Sensitivity derivatives calculation

- Differentiation of the discretized (FE) system

$$\boldsymbol{KU} = \boldsymbol{F} \rightarrow \boldsymbol{K}\frac{d\boldsymbol{U}}{dx} = \frac{d\boldsymbol{F}}{dx} - \boldsymbol{U}\frac{d\boldsymbol{K}}{dx}$$

  – Calculation of dK/dx is quite costly even small changes of the boundary
  – Spurious component: changes in shape $\rightarrow$ distortion of the FE
- Differentiation of continuum equations
  – Concept of material derivatives
  – Differentiation $\rightarrow$ discretization: avoid spurious errors
  – Numerical difficulties in boundary integrals

Solution techniques

- Calculus of variations and optimality criteria methods
  - Optimality criteria: either rigorous or intuitive (uniform tangential stress or the strain energy density along the boundary)
- Pattern transformation methods
  - Transforming the shape of the boundary based on the stress ratio in the boundary finite elements
- The photoelasticity technique
  - Optimize the shape of the holes experimentally
  - Modifying hole boundaries in a two-dimensional photoelastic model until the tensile and compressive boundary stresses were approximately constant
- Boundary element method
  - Overcome two major drawbacks: FE mesh regeneration and difficulties in sensitivity derivatives calculation
  - Not as reliable as the FEM

Shape: Formulation (1)

Typical Setting of Optimization

$$\min_{\text{design}} \int_\Omega \rho \, d\Omega$$

subject to

$$a(\boldsymbol{u},\boldsymbol{v}) = f(\boldsymbol{v}) \ \forall \boldsymbol{v}$$
$$\bar{\sigma} \le \sigma_{\max}$$
$$|\boldsymbol{u}| \le u_{\max}$$

Ω: variable unknown domain

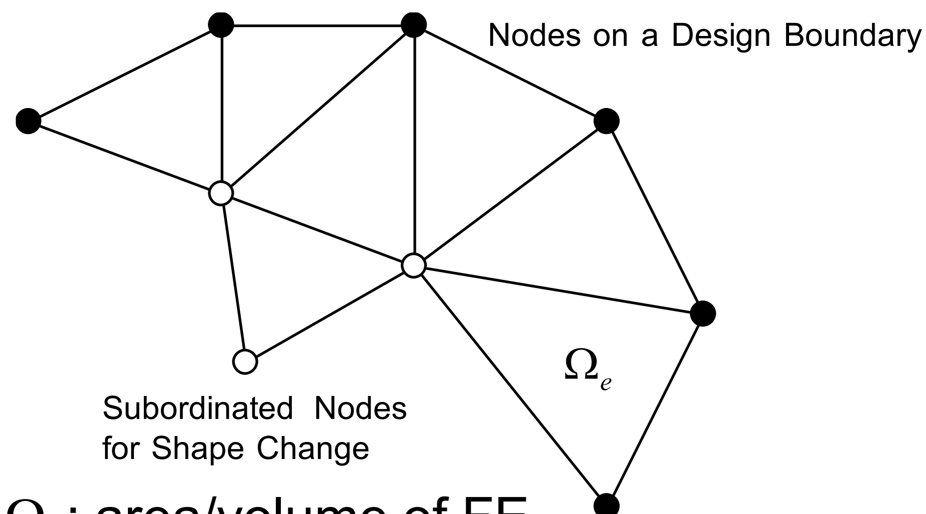
Finite Element Representation

$$\min_{\text{design}} \int_\Omega \rho \, d\Omega = \sum_{e=1}^{nel} \rho_e \Omega_e$$

subject to

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$$
$$\bar{\sigma}_e \le \sigma_{\max}, \ e=1,\ldots,nel$$
$$|\boldsymbol{u}_i| \le u_{\max}, \ i=1,\ldots,node$$

Varying in Shape Design



Ω_e: area/volume of FE

Shape: Formulation (2)

- Virtual Work Principle

$$a(\boldsymbol{u},\boldsymbol{v}) = f(\boldsymbol{v}) \quad \forall \boldsymbol{v}$$

$$\Leftrightarrow \underbrace{\int_\Omega \varepsilon(\boldsymbol{v})^T \boldsymbol{E}\varepsilon(\boldsymbol{u})d\Omega}_{\text{internal virtual work}} = \underbrace{\int_\Omega \varepsilon(\boldsymbol{v})^T \boldsymbol{E\alpha}d\Omega}_{\text{thermal load}} + \underbrace{\int_\Omega \boldsymbol{v}^T \rho\boldsymbol{b}d\Omega}_{\text{body force}} + \underbrace{\int_{\Gamma_t} \boldsymbol{v}^T \boldsymbol{t}d\Gamma}_{\text{applied traction}} \quad \forall \boldsymbol{v}$$

- Finite Element Approximation: $\boxed{\boldsymbol{Ku} = \boldsymbol{f}}$

- Design Sensitivity: $\text{Design Variable} = \text{Control Points } \boldsymbol{x}_{cp}$
  - A lot of mathematical evaluation is necessary to compute required design sensitivity for shape design

Design Sensitivity

- Step 1 : Relation between FE nodes and Design control points
- Step 2 : Design Sensitivity w.r.t. control points $\boxed{\boldsymbol{x} = \boldsymbol{T}\boldsymbol{x}_{cp}}$

$$\frac{D}{D\boldsymbol{x}_{cp}}(\boldsymbol{K}\boldsymbol{u}) = \frac{D\boldsymbol{K}}{D\boldsymbol{x}_{cp}}\boldsymbol{u} + \frac{D\boldsymbol{u}}{D\boldsymbol{x}_{cp}}$$

$$\frac{D\boldsymbol{K}}{D\boldsymbol{x}_{cp}} = \frac{D}{D\boldsymbol{x}_{cp}}\sum_{e=1}^{nel}\boldsymbol{K}_e = \frac{D}{D\boldsymbol{x}_{cp}}\sum_{e=1}^{nel}\int_{\Omega_e}\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}d\Omega = \frac{D}{D\boldsymbol{x}_{cp}}\sum_{e=1}^{nel}\int_{\Omega_R}\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}Jd\Omega_R$$

$$= \sum_{e=1}^{nel}\int_{\Omega_R}\left(\frac{D\boldsymbol{B}}{D\boldsymbol{x}_{cp}}\right)^T\boldsymbol{E}\boldsymbol{B}Jd\Omega_R + \sum_{e=1}^{nel}\int_{\Omega_R}\boldsymbol{B}^T\boldsymbol{E}\left(\frac{D\boldsymbol{B}}{D\boldsymbol{x}_{cp}}\right)Jd\Omega_R + \sum_{e=1}^{nel}\int_{\Omega_R}\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}\left(\frac{DJ}{D\boldsymbol{x}_{cp}}\right)d\Omega_R$$

- Design Sensitivity Analysis must be in FEA codes
- Design Sensitivity Analysis must be linked with spline representation of design segments/surfaces

$$\frac{D\boldsymbol{B}}{D\boldsymbol{x}_{cp}} = \frac{D\boldsymbol{B}}{D\boldsymbol{x}}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{x}_{cp}} = \frac{D\boldsymbol{B}}{D\boldsymbol{x}}\boldsymbol{T}$$

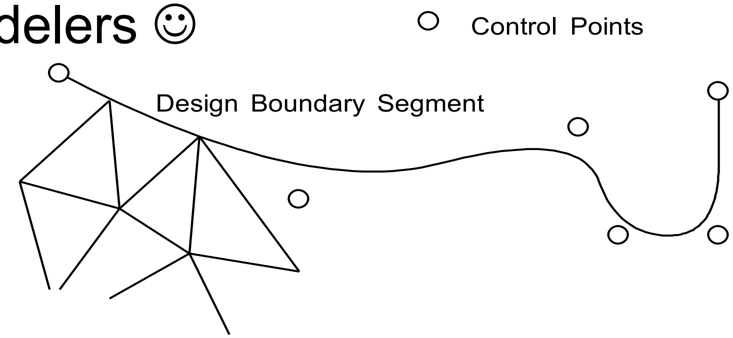
Practical Approach

- Difficulties
  - Every FEA code does have their own special finite elements, and then design sensitivity must be performed in such a FEA code
  - Geometric representation of the control points and the FE nodes must be related, and then this requires full link with CAD representation and mesh generation scheme
  - Full integration of
    - CAD like representation of Design Segments
    - Control Point Adaptation
    - Adaptive Finite Element Method
    - Full Automatic Mesh Generation Method

    is not realistic in practice.
- What is a possible alternate ?
  - GENESIS Approach / Bio-mechanical Growth Approach

Shape Design Parameters (1)

- Geometry-based mesh parameterization
  - Higher-level geometry data: surface control points, fillet radii
  - Mapped / free meshes
  - Integration with parametric solid modelers ☺
  - Mesh generator must be included ☹



- Reduced basis approach
  - Base configuration with a distinct mesh topology that remains fixed during the optimization
  - How to generate the design velocities (design base shapes for complex FE meshes)?

$$\{X(b)\} = \underbrace{\{X_0\}}_{\text{original nodal coordinates}} + \sum_{k=1}^{N_s} b_k \{V_k\}$$

$\{V_k\} = \{X_k\} - \{X_0\}$: k-th design velocity vector

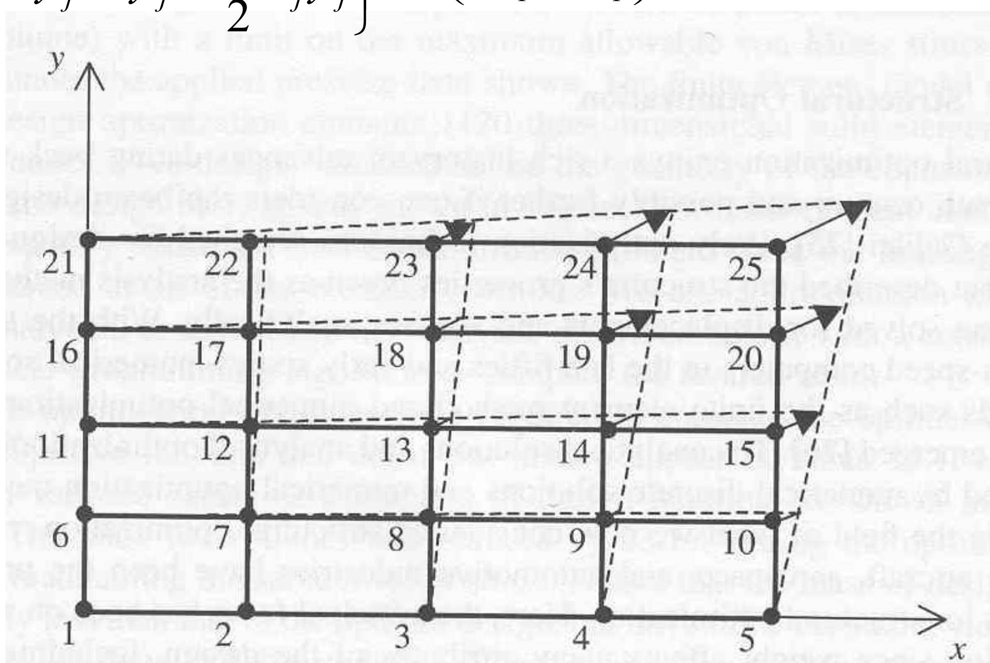
b_k : shape parameters

Shape Design Parameters (2)

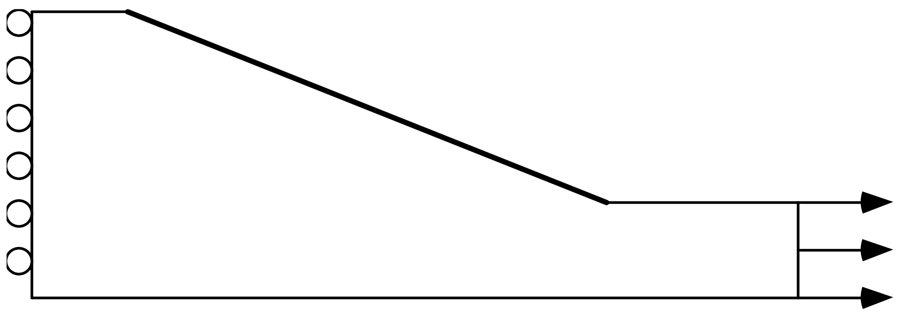
– Design velocities giving shape changes as a function of shape design parameters

$$\left(\hat{x}_j, \hat{y}_j\right) \xrightarrow{\Delta x_j = b_1 \hat{x}_j \hat{y}_j,\, \Delta y_j = \frac{b_1}{2} \hat{x}_j \hat{y}_j} \left(x_j, y_j\right)$$

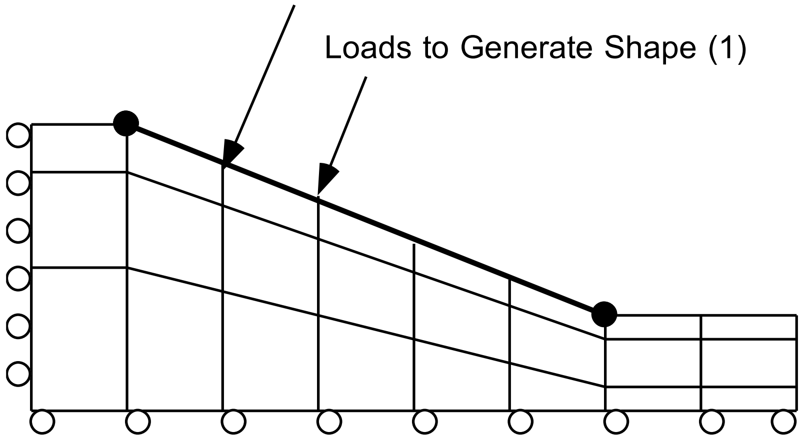
$$\left.\begin{aligned} x_j &= \hat{x}_j + b_1 \hat{x}_j \hat{y}_j \\ y_j &= \hat{y}_j + \frac{b_1}{2} \hat{x}_j \hat{y}_j \end{aligned}\right\} \rightarrow \left(\frac{\partial x_j}{\partial b_1}, \frac{\partial y_j}{\partial b_1}\right) = \left(\hat{x}_j \hat{y}_j, \frac{1}{2} \hat{x}_j \hat{y}_j\right)$$

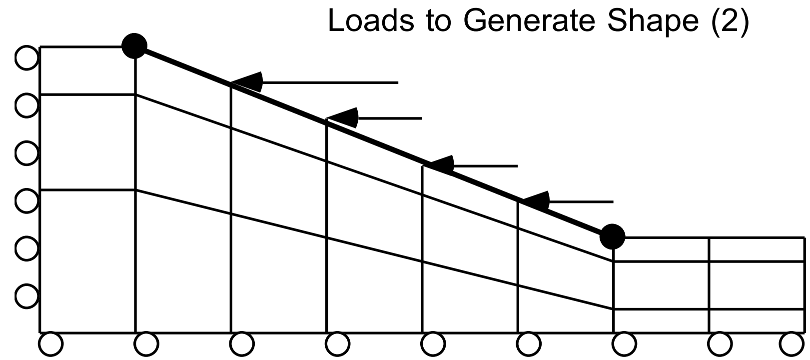


GENESIS Approach (1)



Linear Combination of Base Shapes generated by FE deformation by artificial loads





GENESIS Approach (2)

- Advantage
  - Design boundary change is smooth and can be controlled, since elastic deformation due to fictitious load is regarded as a base design change
  - Finite element distortion is minimized
  - Remeshing methods need not be integrated, since the initial finite element connectivity is maintained during optimization
- Success
  - By creating interactive preprocessor to define the base shapes for the design change, but it is independent of CAD S/W
  - Three-dimensional curved design segments are treated by the same way
  - FORD extensively uses this after Topology Design results to make detailed design

Bio-mechanical Growth (1)

- Similarity with thermal deformation
  - increasing temperature results expansion
  - cooling results shrinkage of a structure
- (temperature change) = (difference between the current stress and the targeted one) in the optimality criteria method
- Characteristics
  - Bio-mechanical growth approach is quite powerful for the fully stressed design approach and also for the optimality criteria method for design optimization
  - It is similar to GENESIS approach in the sense that fictitious loadings are considered to adapt the design shape and no need to make remeshing schemes

Bio-mechanical Growth (2)

For the Fully Stressed Design

thermal loading

$$\boldsymbol{f}^{e}_{thermal} = \int_{\Omega_e} \boldsymbol{B}^T \boldsymbol{D} \Delta T \alpha d\Omega$$

fictitious loading

$$\boldsymbol{f}^{e}_{thermal} = \int_{\Omega_e} \boldsymbol{B}^T \boldsymbol{D} \left(\frac{\sigma - \sigma_{\text{target}}}{\sigma_{\text{target}}} \right)^{\eta} \alpha d\Omega$$

Shrink if stress is too low, enlarge if stress is too high

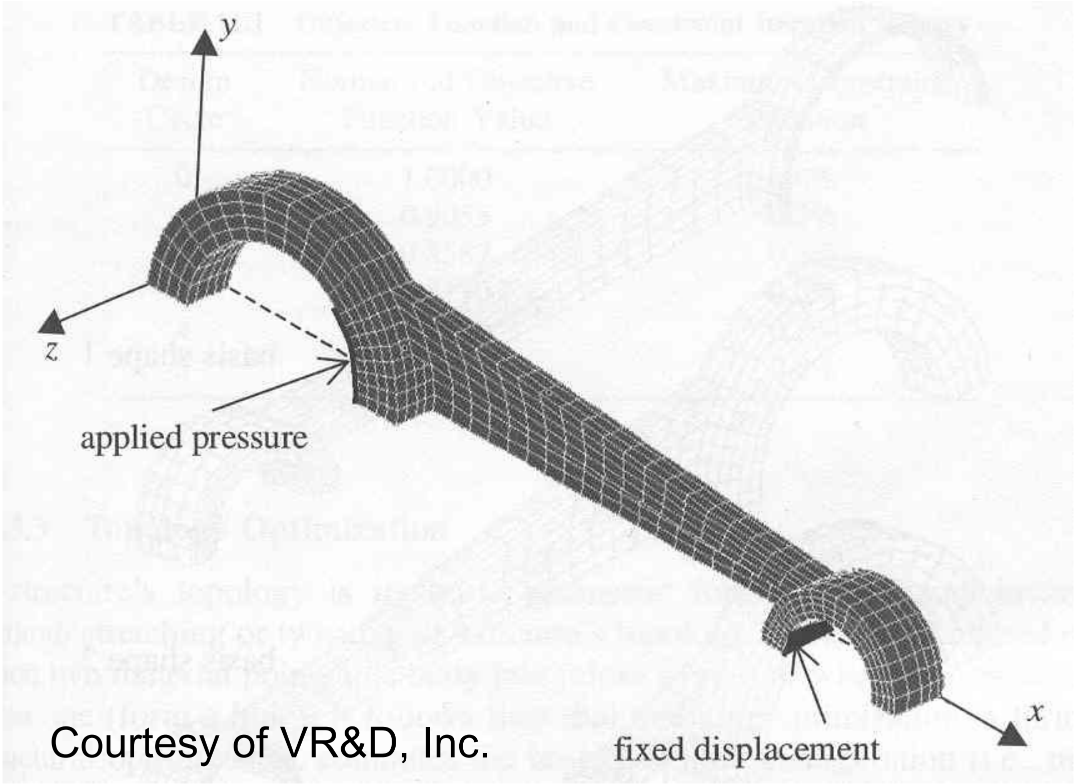
Nature of Shape Change

- If dramatic shape change is not required, CAD linked remeshing scheme with full automatic mesh generation methods is not quite essential.
- Thus, if shape design is considered after topology optimization, then both GENESIS and Bio-mechanical growth approaches are sufficiently powerful.

- Shape design optimization
  - Since Topology Design Optimization does not include many design constraints, Shape Design stage should involve all kind of design restriction not only for
    - stiffness, strength, local buckling
  - but also
    - manufacturability
    - geometric constrains

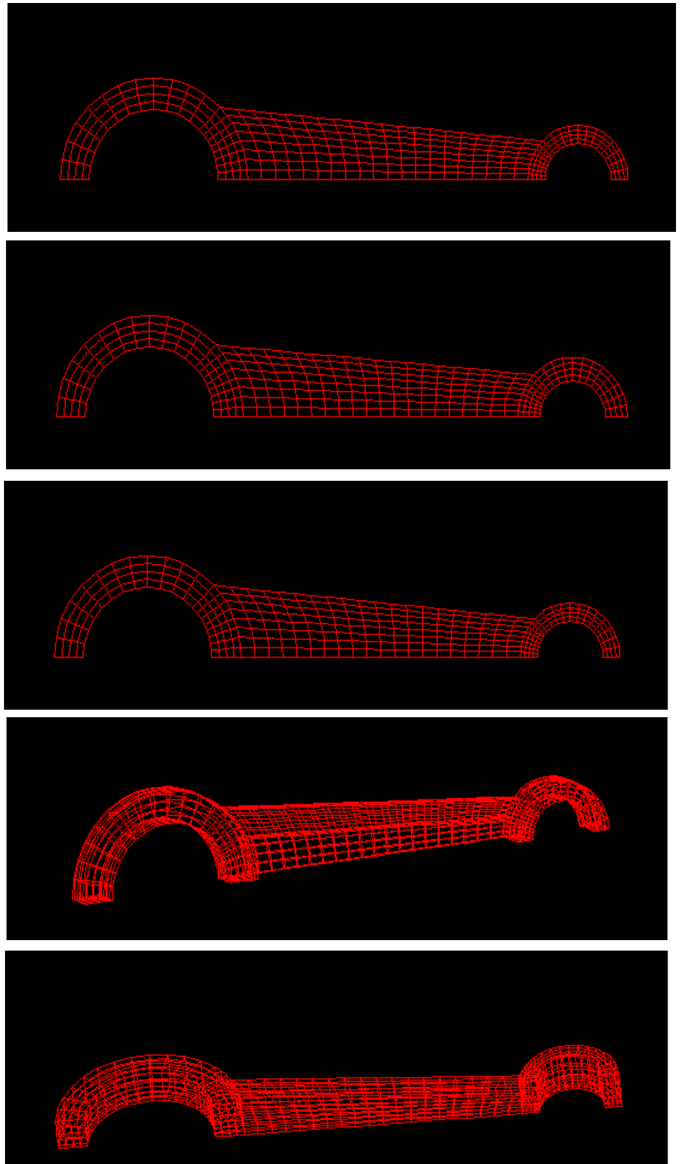
Engine Connecting Rod: Problem Description

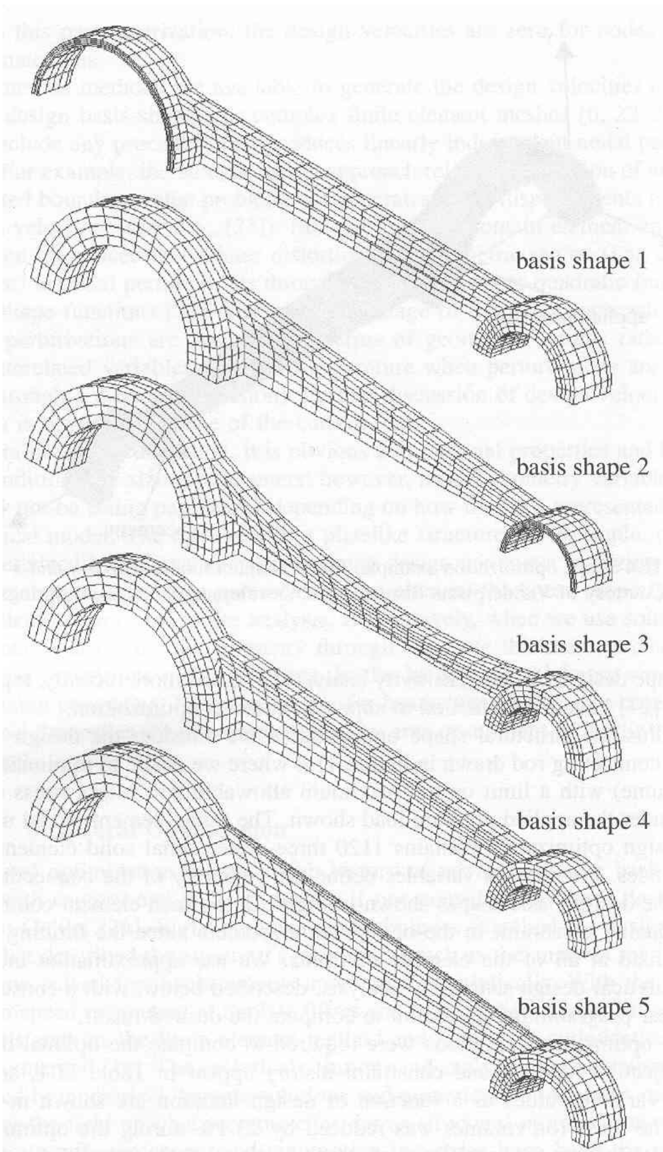
– Minimize mass with a limit on the maximum allowable von Mises stress developed under the applied pressure load
– 1120 3D solid elements
– Design variables
    • Outer radius at crank
    • Outer radius at piston
    • Rod body curvature
    • Flange thickness
    • Flange width



Courtesy of VR&D, Inc.

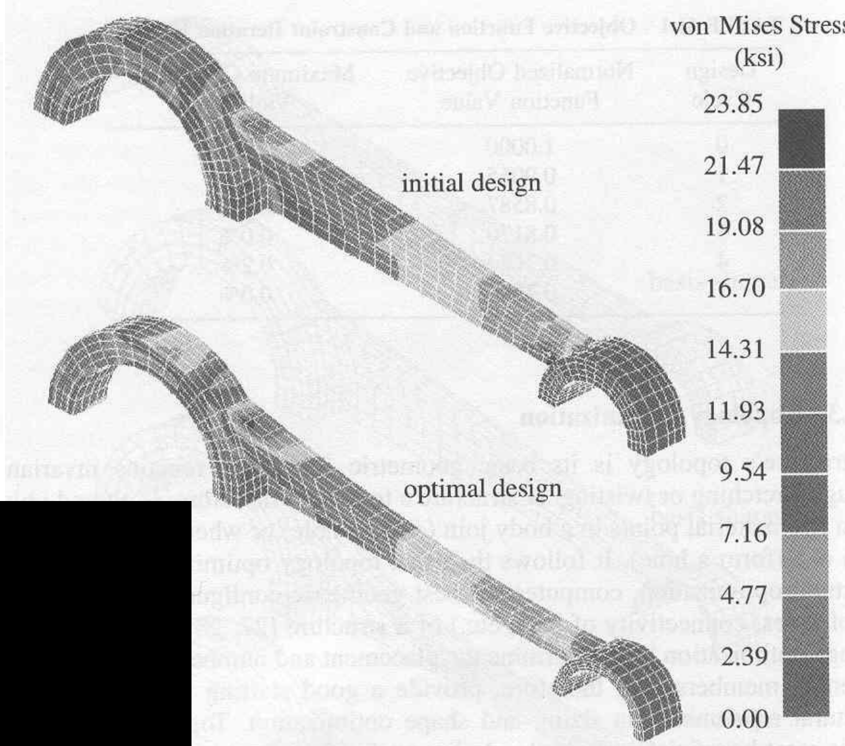
Engine Connecting Rod: Basis Vectors

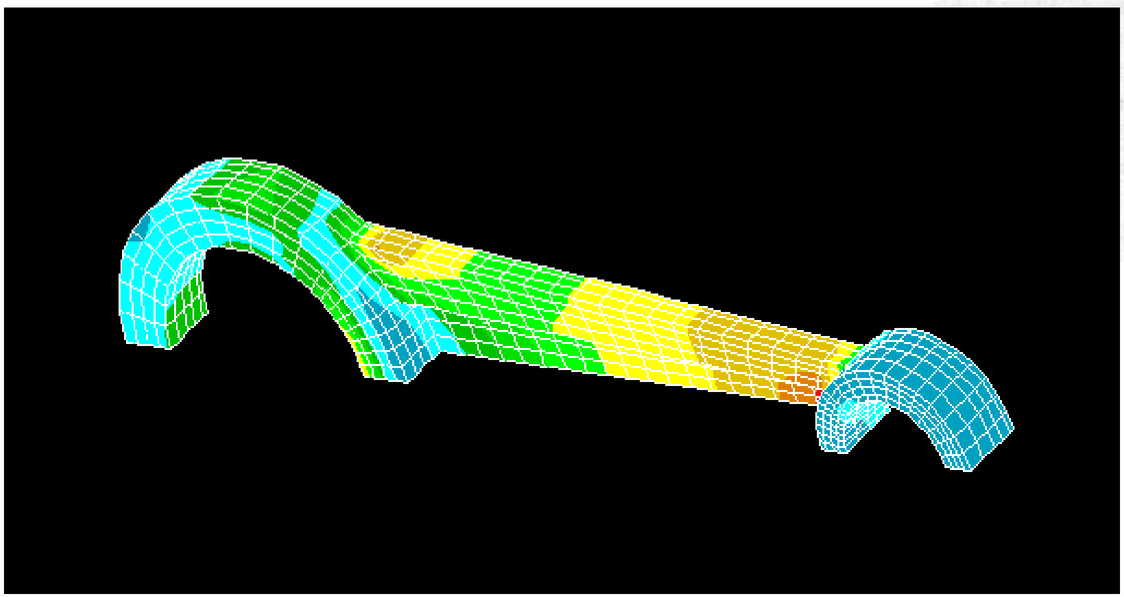




Engine Connecting Rod: Results

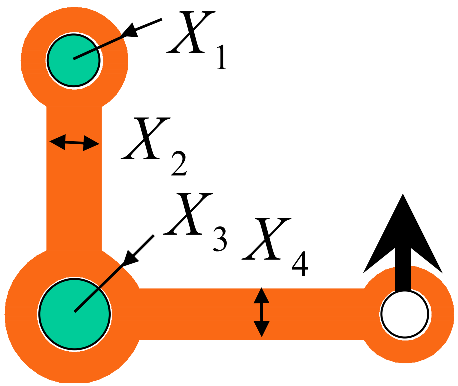
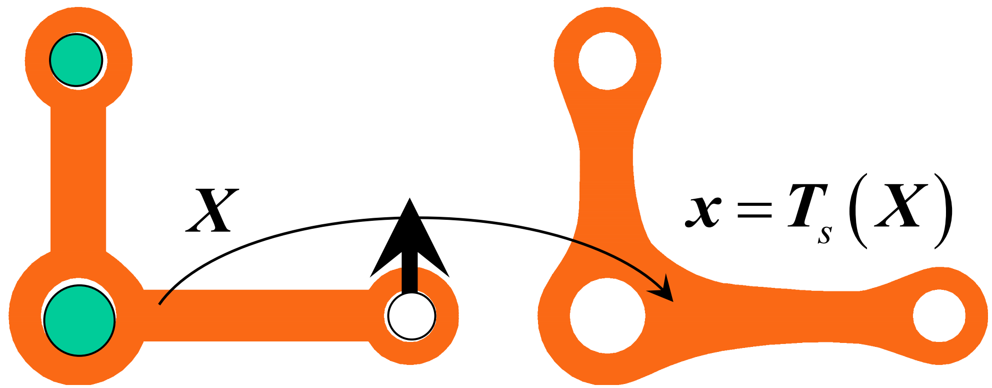
Design Cycle	b_1	b_2	b_3	b_4	b_5
0	0.00000	0.00000	0.00000	0.00000	0.00000
1	0.10000	0.10000	0.10000	0.00397	−0.01313
2	0.15121	0.20000	0.20000	−0.01992	−0.11313
3	0.18213	0.33333	0.33333	−0.09861	−0.21313
4	0.21461	0.56275	0.37941	−0.10697	−0.35258
5	0.21317	0.56275	0.37876	−0.10733	−0.35314



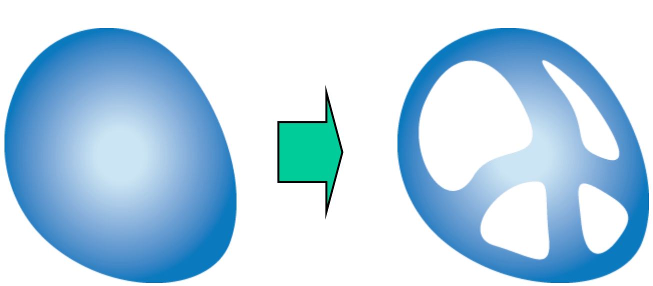


weight reduction (23.1%)

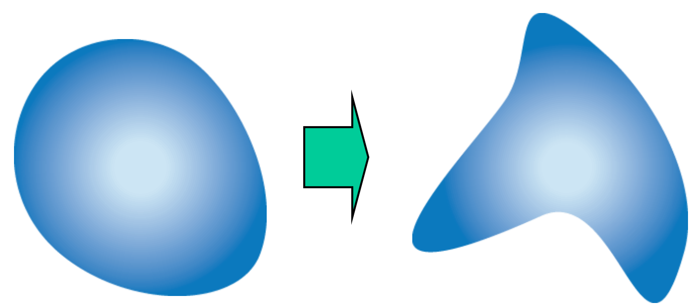
Classification of Shape Optimization

Parametric (value)	Nonparametric (function)
	

- Nonparametric optimization



Topology change



Boundary change

Nonparametric Optimization

Deign Variable	value	function
Objective	$J_0 = \boldsymbol{P} \cdot \boldsymbol{u}$	$J_0 = \int_\Gamma \boldsymbol{P} \cdot \boldsymbol{u} \, \mathrm{d}\Gamma$
Constraint	$J_1 = m(\boldsymbol{X}) - m_0 \leq 0$	$J_1 = \int_{\boldsymbol{T}_s(\Omega)} dx - m_0 \leq 0$
Equilibrium	$K(\boldsymbol{X})\boldsymbol{u} = \boldsymbol{P}$	$\int_{\boldsymbol{T}_s(\Omega)} \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx = \int_\Gamma \boldsymbol{P} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma$

Design Variable: Value

– Lagrange multiplier method

$$L\left(\boldsymbol{u},\boldsymbol{v},\Lambda,\boldsymbol{X}\right)=\boldsymbol{P}\cdot\boldsymbol{u}-\boldsymbol{v}\cdot\left(K\boldsymbol{u}-\boldsymbol{P}\right)+\Lambda\left(m-m_0\right)$$

$$\frac{\partial L}{\partial \boldsymbol{u}}=\boldsymbol{P}-K\boldsymbol{v}=0$$

Adjoint equation: ***v***

$$\frac{\partial L}{\partial \boldsymbol{v}}=\boldsymbol{P}-K\boldsymbol{u}=0$$

State equation: ***u***

$$\frac{\partial L}{\partial \Lambda}=m-m_0=0,\ \Lambda\geq 0,\ \Lambda\left(m-m_0\right)=0$$

Lagrange multiplier: ***Λ***

$$\Delta L\Big|_{\boldsymbol{u},\boldsymbol{v},\Lambda}=\Delta J_0\Big|_{\boldsymbol{u},\boldsymbol{v},\Lambda}=\frac{\partial L}{\partial \boldsymbol{X}}\cdot\Delta\boldsymbol{X}=\left(-\boldsymbol{v}\cdot\frac{\partial K}{\partial \boldsymbol{X}}\boldsymbol{u}+\Lambda\frac{\partial m}{\partial \boldsymbol{X}}\right)\cdot\Delta\boldsymbol{X}=\boldsymbol{G}_X\cdot\Delta\boldsymbol{X}$$

Gradient

Design Variable: Function

- Lagrange multiplier method
- Adjoint variable method
- Material derivatives

$$\dot{J}_0(\boldsymbol{u}) = \int_\Gamma \boxed{G\nu} \cdot \boldsymbol{V} \; \mathrm{d}\Gamma \qquad \boldsymbol{V} \equiv \frac{\partial \boldsymbol{T}_s}{\partial s}\left(\boldsymbol{T}_s^{-1}(\boldsymbol{x})\right)$$

Shape Gradient

Velocity of domain change

Lagrange multiplier

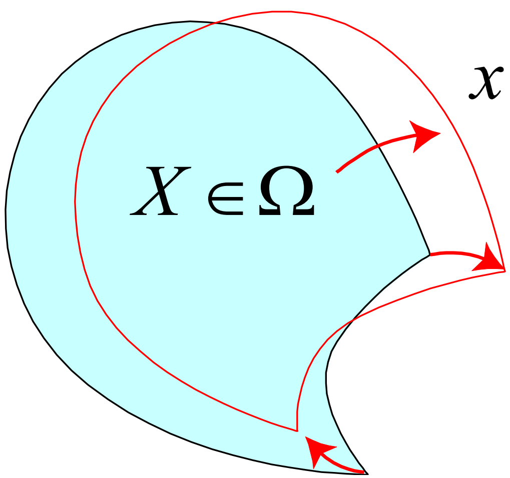
Shape gradient density

$$G = G^{[0]}\left(\boldsymbol{u}, \boldsymbol{v}^{[0]}\right) + \sum_{m=1}^{q} \Lambda^{[m]} G^{[m]}\left(\boldsymbol{u}, \boldsymbol{v}^{[m]}\right)$$

Adjoint variable

Representation of Shape Change

- Domain mapping (Zolesio 1981)
- Velocity V: derivative of T_s w.r.t. s



$$x = T_s(X) \in \Omega_s = T_s(\Omega)$$

$$X \in \Omega \xrightarrow{T_s} x \in \Omega_s$$

$$x = T_s(X)$$

$$V(x) = \frac{\partial x}{\partial s} = \frac{\partial T_s(X)}{\partial s} = \frac{\partial T_s}{\partial s}\left(T_s^{-1}(x)\right)$$

Dirichlet Problem

- Elliptic boundary value problem
  - Strong form

$$\nabla \equiv \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{Bmatrix}$$

$$\begin{cases} -\nabla \cdot \nabla u(x) = f(x) \quad x \in \Omega \\ u(x) = 0 \quad x \in \Gamma \end{cases}$$

Elliptic P.D.E.

B.C.

  - Weak form

$$a(u,v) = l(v) \qquad \forall v \in U = H_0^1(\Omega) = \left\{ u \in H^1(\Omega) \middle| \; u|_\Gamma = 0 \right\}$$

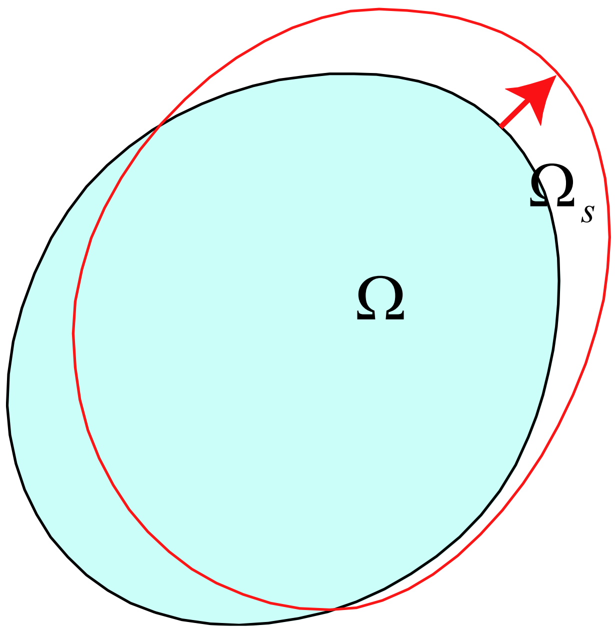
$$\text{where} \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

$$l(v) = \int_\Omega fv \, dx$$

Shape Optimization Problem Formulation

$$J(u) = \int_{\Omega} F(u)\, dX$$

$$\boxed{\begin{aligned} &\min_{\Omega \subset R^n} J(u) \text{ such that} \\ &a(u,v) = l(v) \quad \forall v \in U \\ &\int_{\Omega} dx \le M_0 \end{aligned}}$$



Adjoint variable function

$$L(u,v) = J(u) - a(u,v) + l(v) + \Lambda\left(\int_{\Omega} dx - M_0\right)$$

Lagrange multiplier

Material Derivatives

$$\phi(X) \mapsto \phi^s(X) \equiv \phi^s\left(T_s^{-1}(x)\right) = \phi_s(x) = \phi_s\left(T_s(X)\right) \quad \left(X \in \Omega \mapsto x \in \Omega_s\right)$$

- Distribution function

$$\dot{\phi}(X) \equiv \underbrace{\phi'(X)}_{\text{shape derivative}} + \left[\nabla \phi(X)\right] \cdot V(X)$$

- Domain Functional

$$I_\Omega(\phi) = \int_\Omega \phi dx \rightarrow \dot{I}_\Omega(\phi) = \int_\Omega \phi'(X) dX + \int_\Gamma \phi(X) \nu \cdot V(X) d\Gamma$$

- Boundary Functional

$$I_\Gamma(\phi) = \int_\Gamma \phi d\Gamma \rightarrow \dot{I}_\Gamma(\phi) = \int_\Gamma \left\{\phi'(X) + \left[\left(\nabla \phi(X)\right) \cdot \nu + \phi(X) \kappa\right] \nu \cdot V(X)\right\} d\Gamma$$

Shape Gradient

$$L(u,v) = J(u) - a(u,v) + l(v) + \Lambda\left(\int_\Omega dx - M_0\right)$$

$$\dot{L} = \int_\Omega \frac{\partial F}{\partial u} u' \mathrm{d}X - a(u',v) + l(v') - a(u,v') + \dot{\Lambda}\left(\int_\Omega dx - M_0\right)$$
$$+ \int_\Gamma \left(F(u) - (\nabla u)\cdot \nabla v + fv + \Lambda\right) \nu \cdot V \ \mathrm{d}\Gamma$$

$$a(u,v') = l(v') \quad \forall v' \in U$$

$$a(u',v) = \int_\Omega \frac{\partial F}{\partial u} u' dx \quad \forall u' \in U$$

$$\Lambda\left(\int_\Omega dx - M\right) = 0$$

$$\Lambda \geq 0$$

$$\int_\Omega dx \leq M_0$$

$$\dot{L}\Big|_{u,v,\Lambda} = \dot{J}\Big|_{u,v,\Lambda} = l_G(V) = \int_\Gamma \boxed{G\nu} \cdot V \ \mathrm{d}\Gamma$$

Shape Gradient

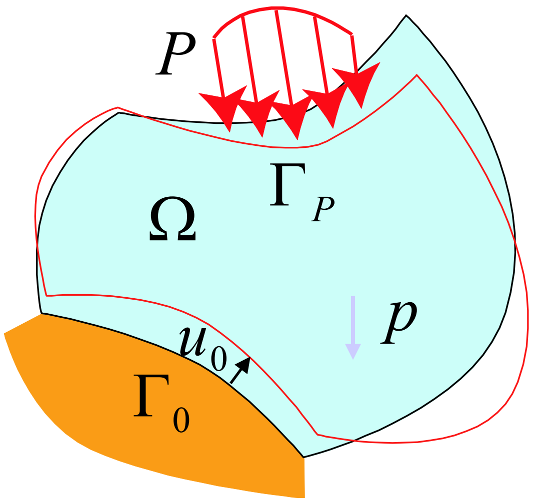
$$G = F(u) - \nabla u \cdot \nabla v + \boxed{fv} + \Lambda$$

Shape Gradient Density

$v|_\Gamma = u|_\Gamma = 0$ in Dirichlet Problem

Stiffness Maximization for Linear Elastic Problem

$$\boxed{\begin{array}{l} a(u,v) = l(v) \quad \forall v \in U \\ u_0 \in U_0,\; u - u_0 \in U \end{array}}$$



$$a(u,v) = \int_\Omega C_{ijkl}\, u_{k,l}\, v_{i,j}\, dx$$

$$l(v) = \int_\Omega p_i v_i \,\mathrm{d}x + \int_{\Gamma_P} P_i\, v_i \,\mathrm{d}\Gamma$$

$$U_0 = \left\{ u_0 \in \left(H^1(\Omega)\right)^n \,\middle|\, u|_{\Gamma \setminus \Gamma_0} = 0 \right\}$$

$$U = \left\{ u \in \left(H^1(\Omega)\right)^n \,\middle|\, u|_{\Gamma_0} = 0 \right\}$$

$$\Pi(u) = \frac{1}{2} a(u,u) - l(u)$$

$$\boxed{\begin{array}{l} \max\limits_{\Omega \subset \mathbb{R}^n} \min\limits_{u - u_0 \in U} \Pi(u) \\ \text{such that } \int_\Omega dx \le M_0 \end{array}}$$

Shape Gradient: Linear Elastic

$$L(u,\Lambda) = -\frac{1}{2}a(u,u) + l(u) + \Lambda\left(\int_\Omega dx - M_0\right)$$

$$\dot{L} = -a(u,u') + l(u') + \dot{\Lambda}\left(\int_\Omega dx - M_0\right)$$

$$+\int_\Gamma \left\{-\frac{1}{2}C_{ijkl}u_{k,l}v_{i,j} + p_i u_i + \Lambda\right\} \nu_m V_m \, d\Gamma$$

$$= -a(u,u') + l(u') + \dot{\Lambda}\left(M(\Omega) - M_0\right) + l_G(V)$$

$$a(u,u') = l(u') \quad \forall u' \in U$$

$$\Lambda\left(\int_\Omega dx - M_0\right) = 0$$

$$\Lambda \ge 0$$

$$\int_\Omega dx \le M_0$$

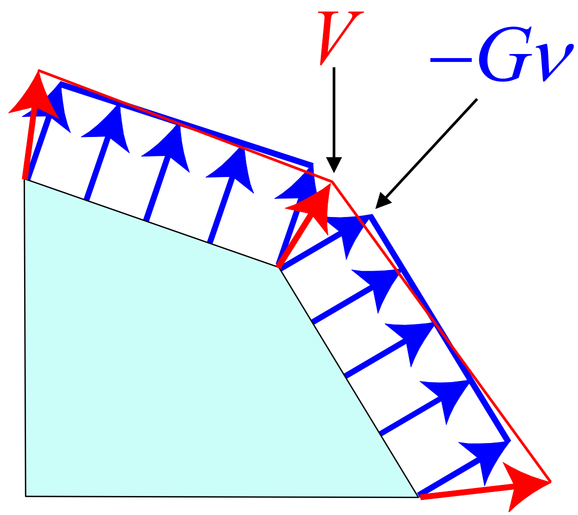
$$\dot{L}\Big|_{u,\Lambda} = \dot{\Pi}\Big|_{u,\Lambda} = l_G(V) = \int_\Gamma \boxed{G\nu} \cdot V \, \mathrm{d}\Gamma$$

$$G = -\boxed{\frac{1}{2}C_{ijkl}u_{k,l}u_{i,j}} + p_i u_i + \Lambda$$

Direct Gradient Method

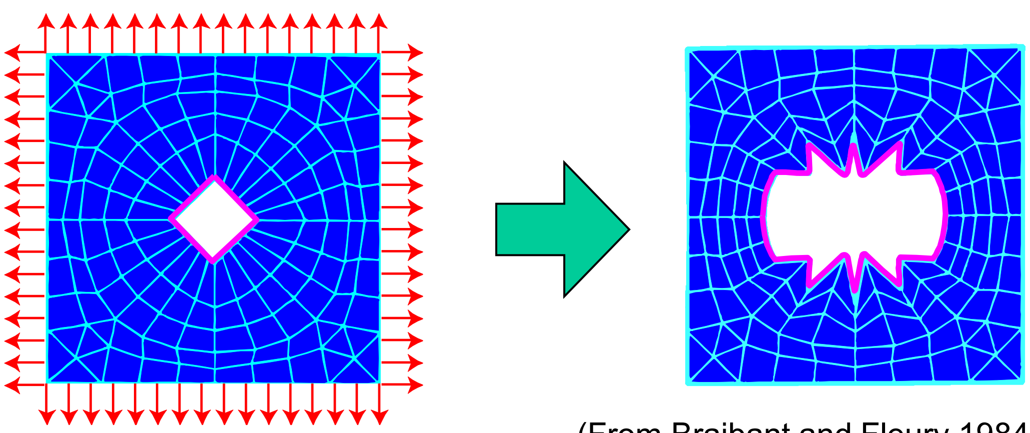
$$
\boxed{V \in D:\ b(V,y)_{(H^0(\Gamma))^n} = -(Gv, y)_{(H^0(\Gamma))^n} \quad \forall y \in Y}
$$

V needs to be continuous.



$$
D = \left\{ V \in \left(C^1\left(\bar{\Omega}\right)\right)^n \,\middle|\, V\big|_{\Gamma_0 \cup \Gamma_P} = 0 \right\}
$$

$$
Y = \left\{ y \in \left(H^0(\Gamma)\right)^n \,\middle|\, y\big|_{\Gamma_0 \cup \Gamma_P} = 0 \right\}
$$



Problems when nodes are relocated proportional to −Gv

(From Braibant and Fleury 1984,1985)

Traction Method

– Apply tensional boundary force proportional to −Gv
– Shape gradient: Neumann B.C. → Solve boundary value problem

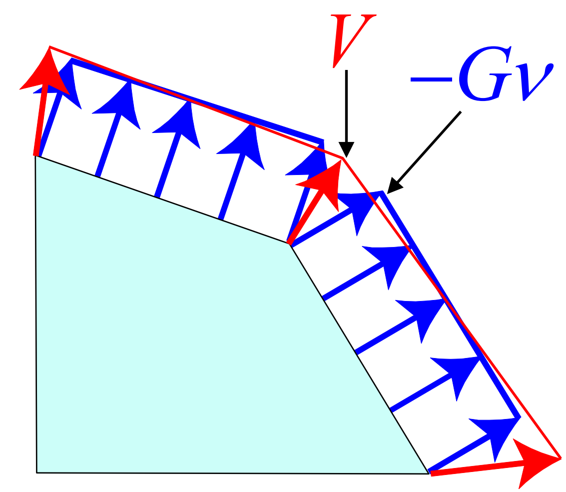
$$\boxed{V \in D:\ a(V, y) = -l_G(y) \quad \forall y \in Y}$$

$$a(u, v) = \int_\Omega C_{ijkl} u_{k,l} v_{i,j}\ \mathrm{d}x$$

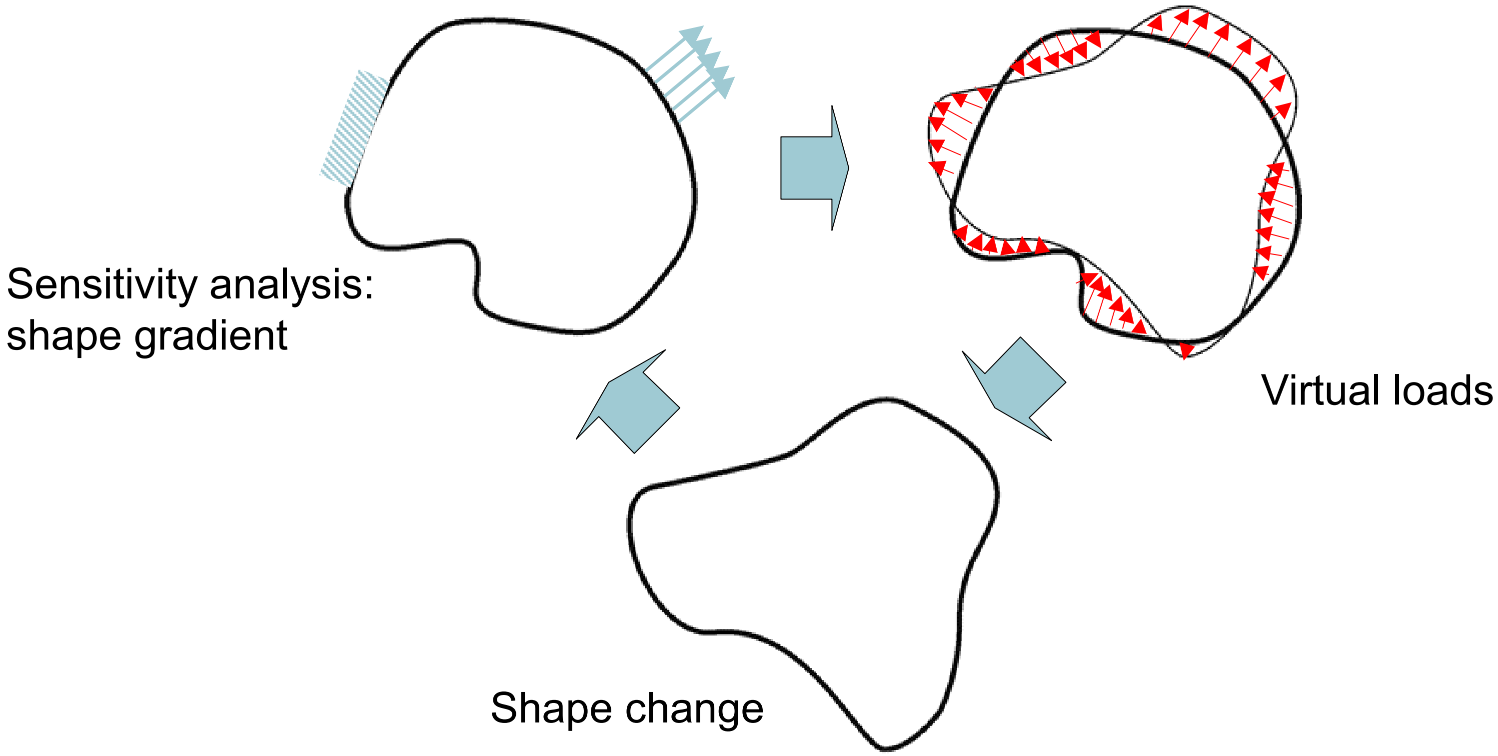
$$l_G(v) = \int_\Gamma G \nu_i v_i\ \mathrm{d}\Gamma$$

$$D = \left\{ V \in \left(C^1\left(\bar{\Omega}\right)\right)^n \middle|\ V\big|_{\Gamma_0 \cup \Gamma_P} = 0 \right\}$$

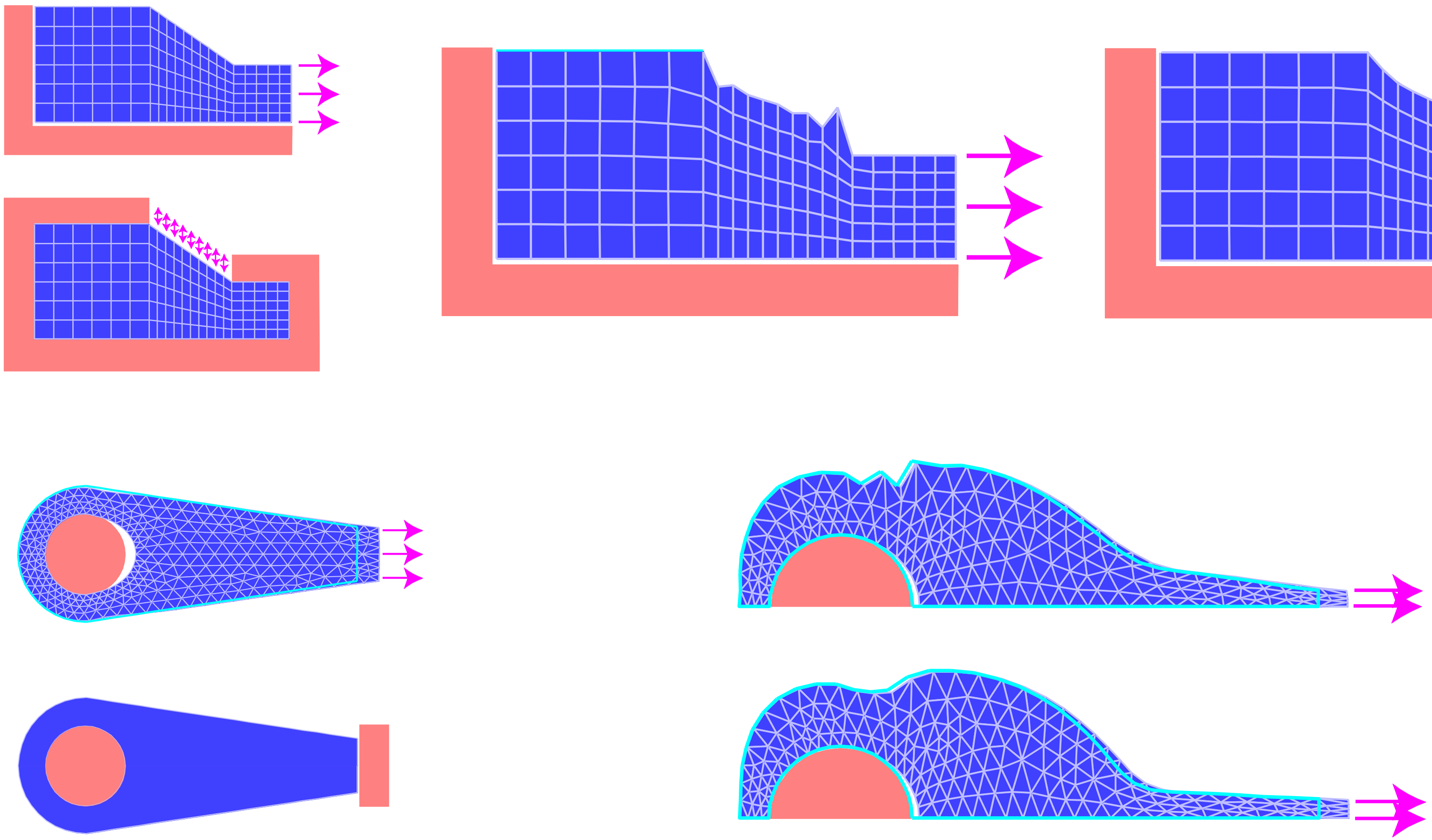
$$Y = \left\{ y \in \left(H^1\left(\Omega\right)\right)^n \middle|\ y\big|_{\Gamma_0 \cup \Gamma_P} = 0 \right\}$$



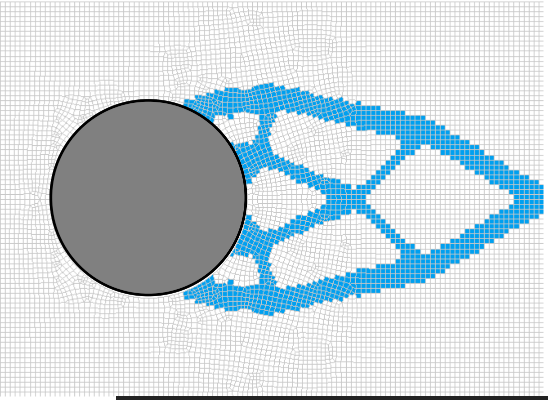
Traction Method: Procedure

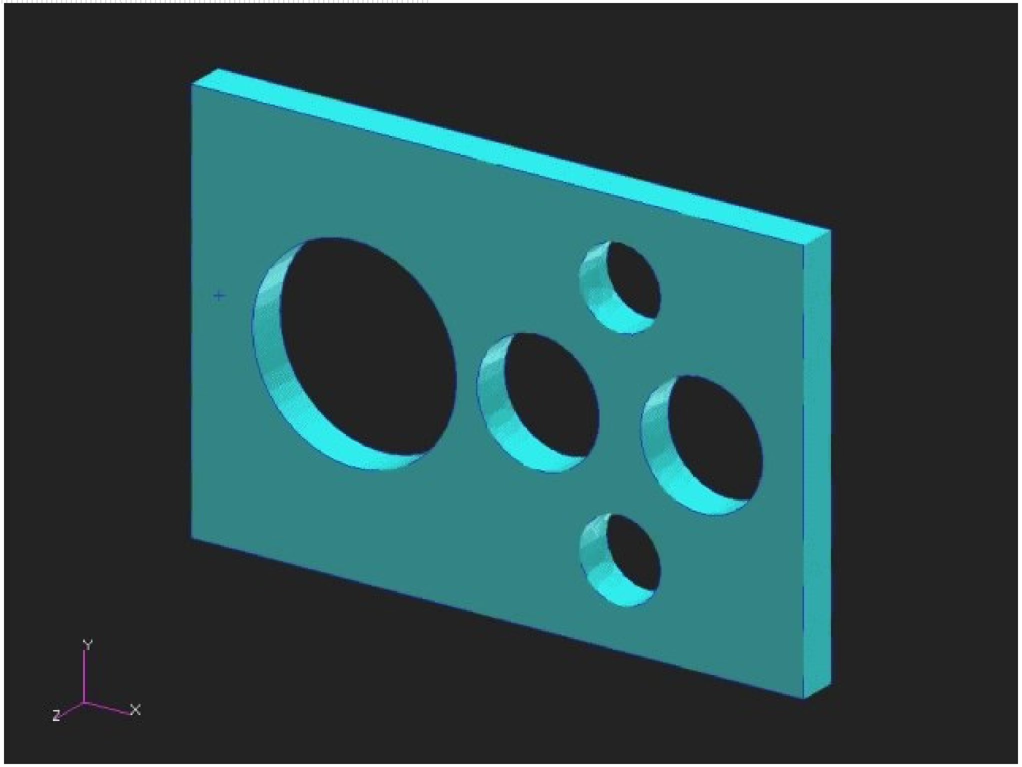


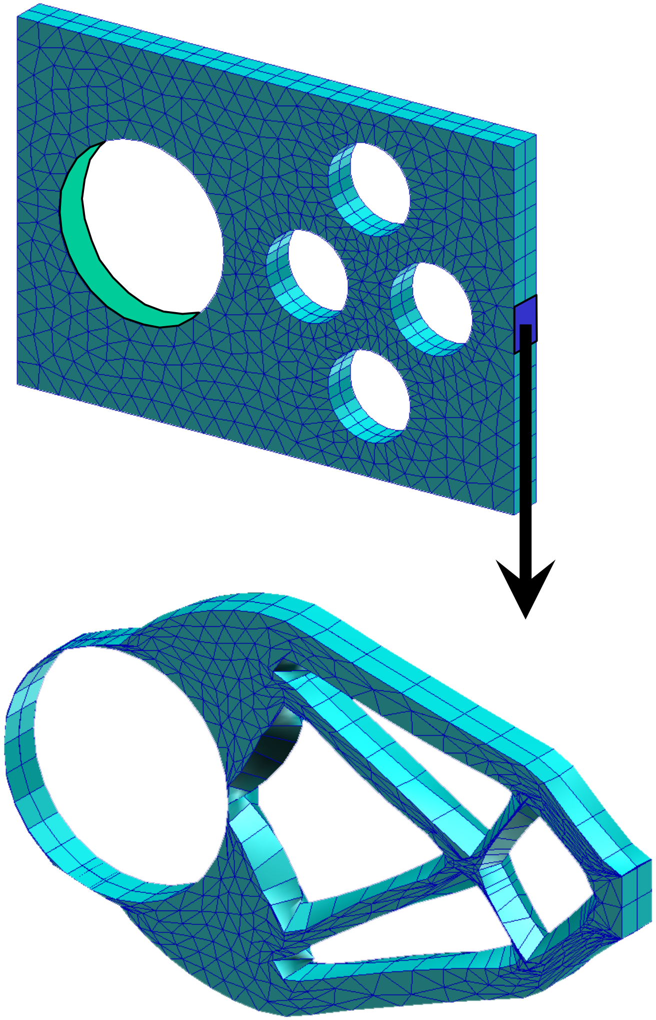
Traction Method: Example



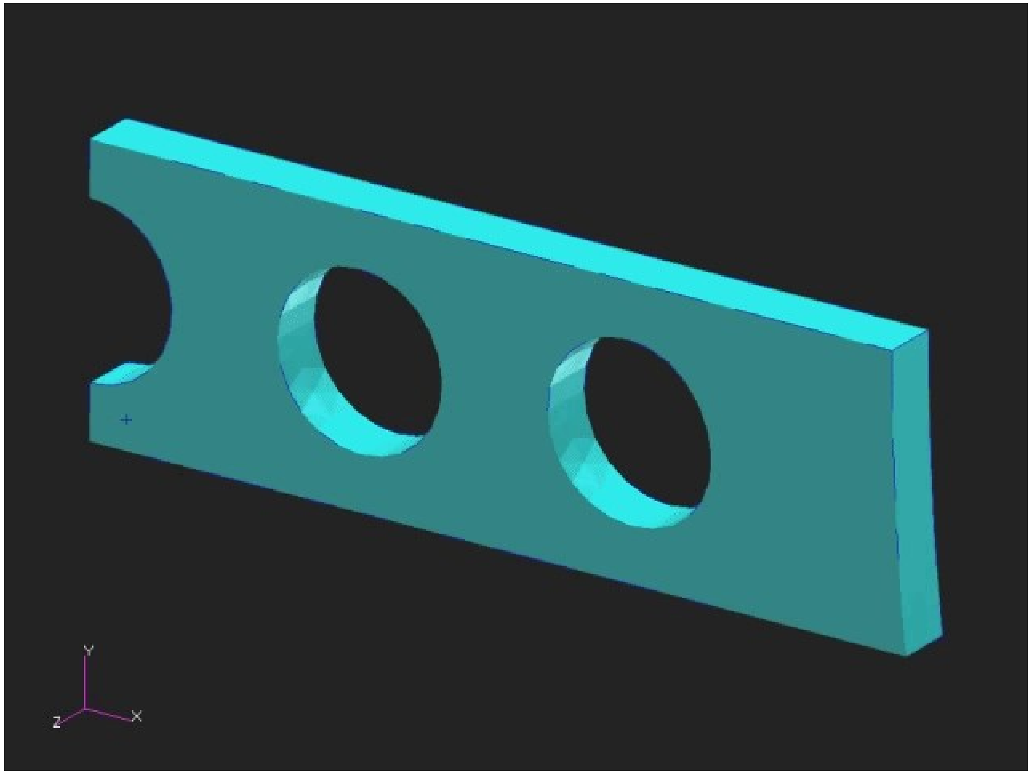
Michell Truss

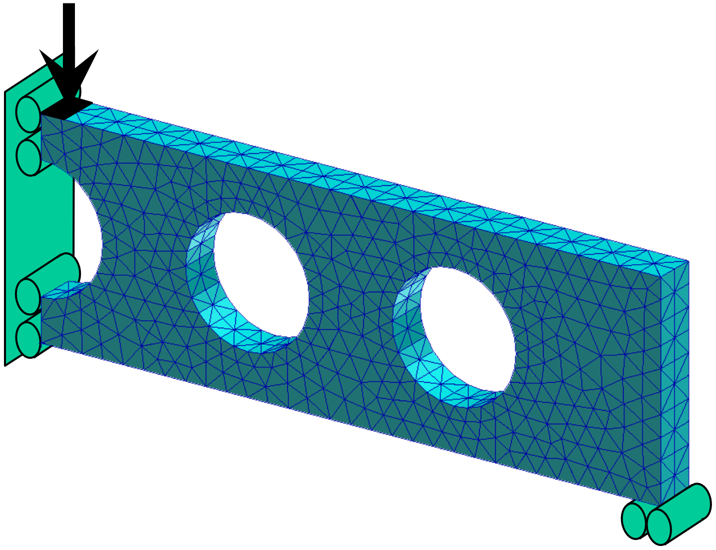


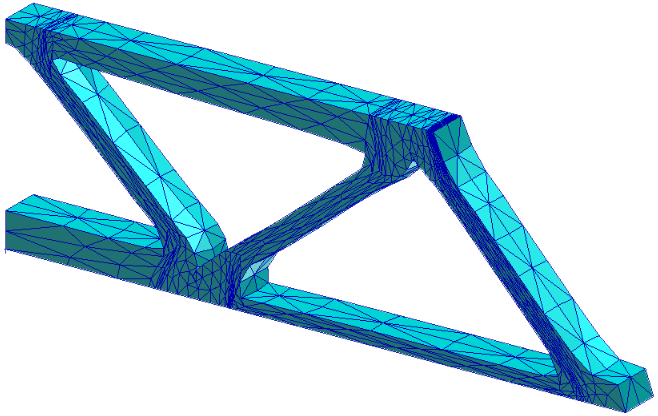




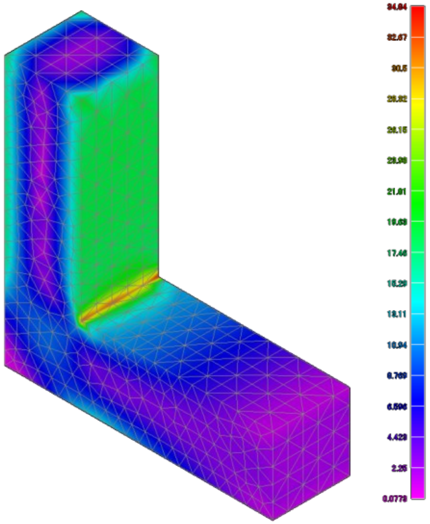
MBB beam



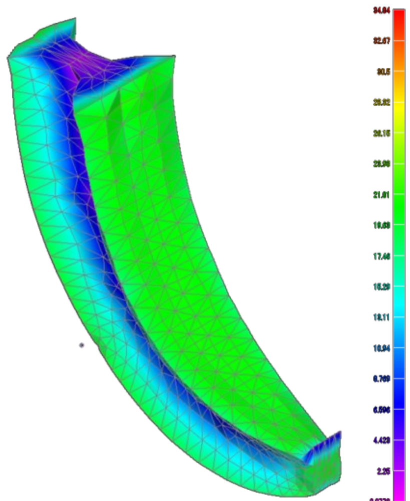


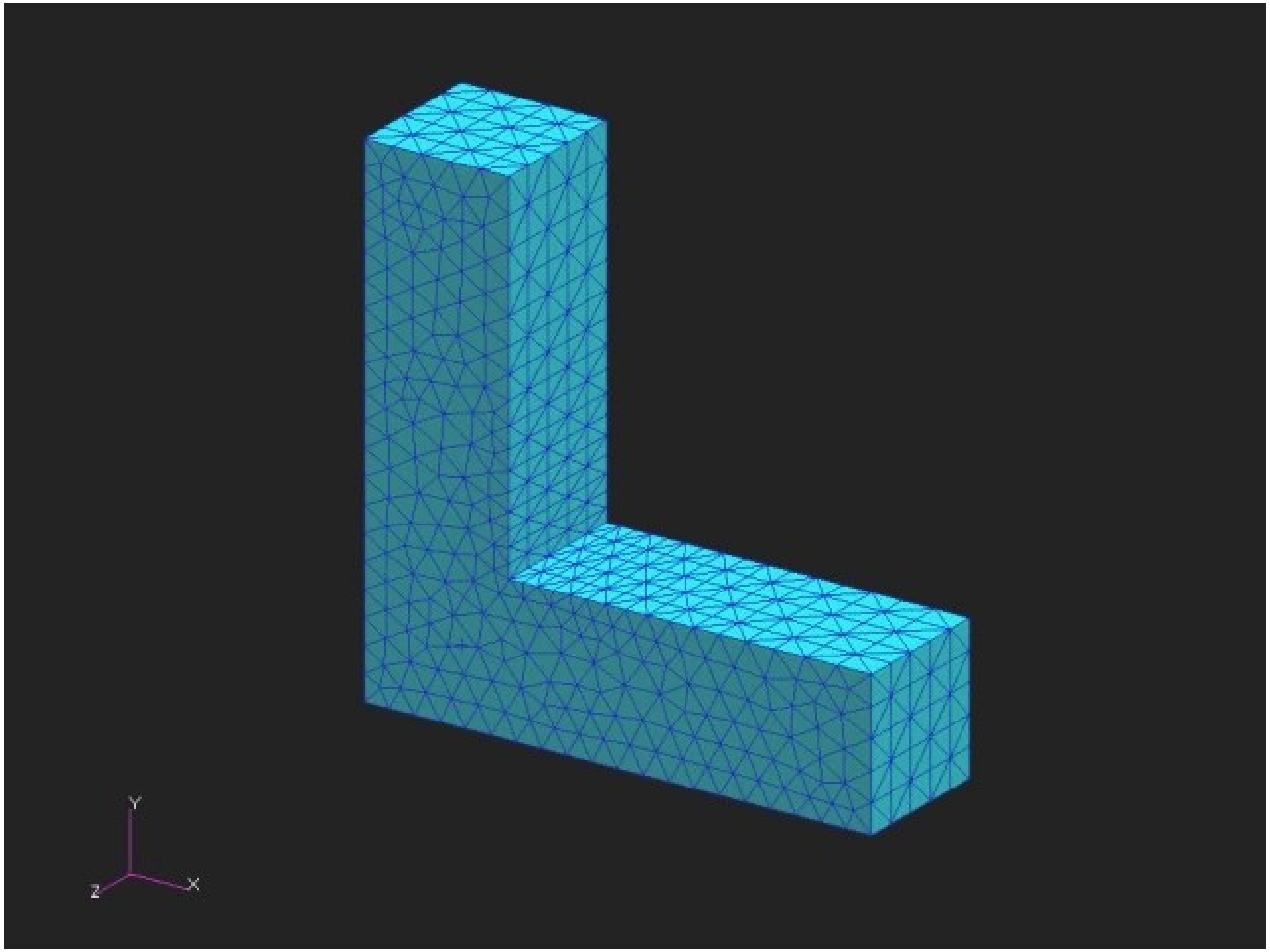


L-shape

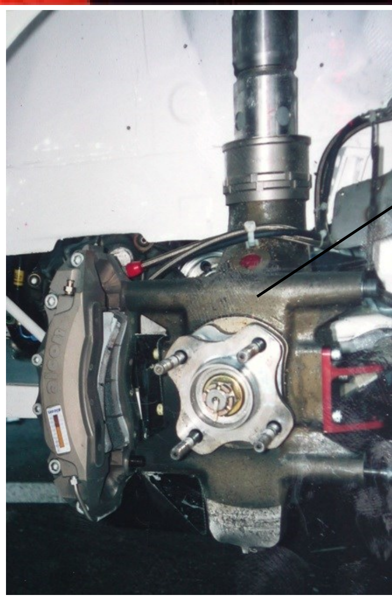


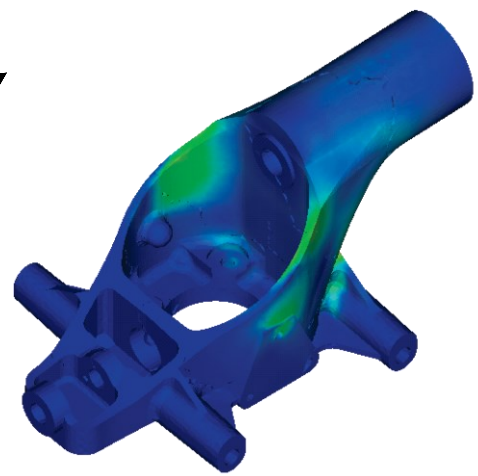
Max. Stress : 50%
Volume : 35.1%



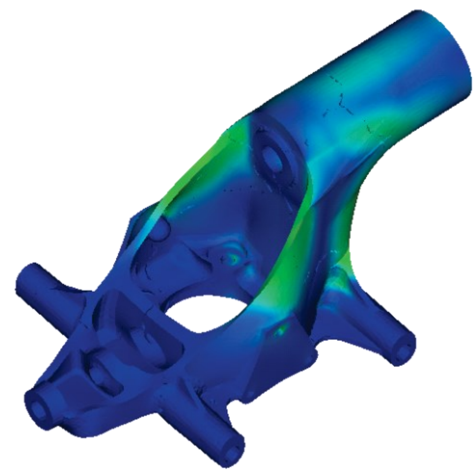


Knuckle Joint





Initial Shape



Optimal Shape
Volume 73.5%