Sensitivity

- Definition
  - Rate of change of a dependent quantity with respect to an independent quantity
- Numerical applications
  - Critical in gradient based optimization
  - Used to construct surrogate models
  - Useful in modeling uncertainties

$$f\left(p, g\left(p\right)\right) = \left\{ f_1 \quad f_2 \quad \dots \quad f_l \right\}^T : \begin{cases} \textit{explicitly} \text{ dependent on a scalar parameter } p \text{ and} \\ \textit{implicitly} \text{ on it via a vector function } g = \left\{ g_1 \quad g_2 \quad \dots \quad g_m \right\}^T \end{cases}$$

$$f' \equiv \frac{df}{dp} \equiv \frac{\partial f}{\partial p} + \left(\nabla_g f\right) \frac{\partial g}{\partial p} \equiv \nabla_p f + \left(\nabla_g f\right) g'$$

$$\text{If } p = \left\{ p_1 \quad p_2 \quad \dots \quad p_m \right\}^T \rightarrow f^{(k)\prime} \equiv \nabla_{p_k} f + \left(\nabla_g f\right) g^{(k)\prime}$$

Sensitivity of Spatial Functions

– Assumption: spatial variables (x, y, z) are independent of the parameter *p*, spatial domain is unaffected by *p*

$$\left(\frac{\partial f}{\partial x}\right)' = \frac{\partial\left(f'\right)}{\partial x} \rightarrow \nabla\left(f'\right) = \left(\nabla f\right)'$$

$$\nabla_p f = \left\{ \frac{\partial f_1}{\partial p} \quad \frac{\partial f_2}{\partial p} \quad \cdots \quad \frac{\partial f_l}{\partial p} \right\}^T \text{: partial derivative}$$

$$\nabla_g f = \begin{bmatrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} & \cdots & \frac{\partial f_1}{\partial g_m} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} & \cdots & \frac{\partial f_2}{\partial g_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_l}{\partial g_1} & \frac{\partial f_l}{\partial g_2} & \cdots & \frac{\partial f_l}{\partial g_m} \end{bmatrix} \text{: functional partial derivative}$$

Sensitivity of Algebraic Systems

- Sensitivity of solution

$$Au = f \xrightarrow{(Au)' = A'u + Au'} Au' = f' - A'u \to Au^{(k)\prime} = f^{(k)\prime} - A^{(k)\prime} u$$

- Sensitivity of performance measures

$$\varphi(p, u(p)) \to \varphi' = \nabla_p \varphi + (\nabla_u \varphi) u' \to \varphi^{(k)\prime} = \nabla_{p_k} \varphi + (\nabla_u \varphi) u^{(k)\prime}$$

- Sensitivity of performance measures via adjoint

increased effort to compute $u^{(k)\prime}$ when k becomes large !!!

$$\varphi^{(k)\prime} = \nabla_{p_k} \varphi + (\nabla_u \varphi) u^{(k)\prime} \xrightarrow{A^T \lambda = (\nabla_u \varphi)^T} \varphi^{(k)\prime} = \nabla_{p_k} \varphi + \left(\lambda^T A\right) u^{(k)\prime} \to \varphi^{(k)\prime} = \nabla_{p_k} \varphi + \lambda^T \left(f^{(k)\prime} - A^{(k)\prime} u\right)$$

- Sensitivity of performance measures via Lagrangian

$$L = \varphi + \lambda^T (f - Au) \xrightarrow{\nabla_p \lambda = \nabla_p u = 0} \nabla_p L = \nabla_p \varphi + \lambda^T (f' - A'u) = \varphi' \to \varphi^{(k)\prime} = \nabla_{p_k} L$$

Size Sensitivity of BVP

– Assumption: Ω does not depend on *p*
– Sensitivity of *u* with respect to a parameter *p*

$$\text{strong form}\ \begin{cases} -\nabla\cdot k\nabla u + cu = f \quad \text{in } \Omega \\ u = \hat{u} \quad \text{in } \Gamma_D \\ -k\nabla u\cdot\hat{n} = q \quad \text{on } \Gamma_N \end{cases}$$

$$\text{weak form}\ \begin{cases} \text{Find } u\in H^1 \text{ such that} \\ \int_\Omega \left[\nabla v\cdot k\nabla u + cuv\right] d\Omega = \int_\Omega fv\,d\Omega - \int_{\Gamma_N} vq\,d\Gamma \quad \forall v\in H_0^1 \end{cases}$$

– For *s* a non-negative integer and $\Omega \subset \mathbf{R}^n$, the Sobolev space $H^s(\Omega)$ contains L^2 functions whose weak derivatives of order up to *s* are also L^2. The inner product in $H^s(\Omega)$ is

$$\langle f, g\rangle = \int_\Omega f(x)\overline{g}(x)\,dx + \int_\Omega Df\cdot D\overline{g}(x)\,dx + \cdots + \int_\Omega D^s f\cdot D^s\overline{g}(x)\,dx$$

$$\int_\Omega \left[-\nabla \cdot k\nabla u + cu - f\right] v d\Omega = 0$$

$$\int_\Omega \left[-v\nabla \cdot k\nabla u + cuv - fv\right] d\Omega = 0 \xrightarrow{\nabla\cdot(vk\nabla u) = \underbrace{v\nabla\cdot(k\nabla u)} + \nabla v\cdot k\nabla u}$$

$$\int_\Omega \left[-\nabla \cdot \left(vk\nabla u\right) + \nabla v \cdot k\nabla u + cuv - fv\right] d\Omega = 0 \xrightarrow{\text{divergence theorem: } \int_\Omega \nabla\cdot(h) d\Omega = \int_{\partial\Omega} h\cdot n d\Gamma}$$

$$\int_\Omega \left[\nabla v \cdot k\nabla u + cuv - fv\right] d\Omega - \int_{\Gamma_N} \left(vk\nabla u\right) \cdot n d\Gamma = 0$$

$$\int_\Omega \left[\nabla v \cdot k\nabla u + cuv\right] d\Omega - \int_\Omega fv d\Omega = \int_{\Gamma_N} \left(vk\nabla u\right) \cdot n d\Gamma$$

$$\int_\Omega \left[\nabla v \cdot k\nabla u + cuv\right] d\Omega = \int_\Omega fv d\Omega - \int_{\Gamma_N} vq d\Gamma$$

Find u where $u = \hat{u}$ on Γ_D and

$$\int_\Omega \left[\nabla v \cdot k\nabla u + cuv\right] d\Omega = \int_\Omega fv d\Omega - \int_{\Gamma_N} vq d\Gamma \quad \forall v, v = 0 \text{ on } \Gamma_D$$

$$H^1 = \left\{ w \middle| w, \nabla w \in C^1, w = \hat{u} \text{ on } \Gamma_D \right\}, H_0^1 = \left\{ w \middle| w, \nabla w \in C^1, w = 0 \text{ on } \Gamma_D \right\}$$

Sensitivity of Solution

$$\left.\begin{aligned}&-\nabla\cdot k\nabla u+cu=f \quad \text{in } \Omega\\&u=\hat{u} \quad \text{in } \Gamma_D\\&-k\nabla u\cdot\hat{n}=q \quad \text{on } \Gamma_N\end{aligned}\right\}\xrightarrow{\text{differentiate}}\begin{cases}-\nabla\cdot k\nabla u'-\nabla\cdot k'\nabla u+cu'+c'u=f' \quad \text{in } \Omega\\\rightarrow -\nabla\cdot k\nabla u'+cu'=\left(f'+\nabla\cdot k'\nabla u-c'u\right) \quad \text{in } \Omega\\u'=0 \quad \text{in } \Gamma_D\\-k\nabla u'\cdot\hat{n}-k'\nabla u\cdot\hat{n}=q' \quad \text{in } \Gamma_N\\\rightarrow -k\nabla u'\cdot\hat{n}=k'\nabla u\cdot\hat{n}+q' \quad \text{on } \Gamma_N\end{cases}$$

$$\left.\begin{aligned}&\text{Find } u\in H^1 \text{ such that}\\&\int_\Omega\left[\nabla v\cdot k\nabla u+cuv\right]d\Omega=\int_\Omega fv\,d\Omega-\int_{\Gamma_N}vq\,d\Gamma \quad \forall v\in H_0^1\end{aligned}\right\}\xrightarrow{\text{differentiate}}$$

$$\begin{cases}\int_\Omega\left[\nabla v\cdot k'\nabla u+\nabla v\cdot k\nabla u'+c'uv+cu'v\right]d\Omega=\int_\Omega f'v\,d\Omega-\int_{\Gamma_N}vq'\,d\Gamma\\\text{Find } u'\in H_0^1 \text{ such that}\\\int_\Omega\left[\nabla v\cdot k\nabla u'+cu'v\right]d\Omega=\int_\Omega f'v\,d\Omega-\int_\Omega\left[\nabla v\cdot k'\nabla u+c'uv\right]d\Omega-\int_{\Gamma_N}vq'\,d\Gamma \quad \forall v\in H_0^1\end{cases}$$

Example: 1-D tension bar

– Consider a 1-D problem that governs the deflection *u*(*x*) of a tensile bar (of possibly varying cross-section and Young's modulus) hanging due to its own weight, and fixed at x=0 and free at *x*=L
– Continuum sensitivity
  • Let the parameter *p* be the cross-sectional area *A*, i.e., a size parameter
  • Let the parameter *p* be the Young's modulus *E*, i.e., a material parameter

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=\rho gA,\quad 0<x<L,\quad u\left(0\right)=0,\quad \left.\frac{du}{dx}\right|_{x=L}=0$$

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=\rho gA,\quad 0<x<L,\quad u(0)=0,\quad \left.\frac{du}{dx}\right|_{x=L}=0$$

$$\rightarrow\int_0^L EA\frac{dv}{dx}\frac{du}{dx}dx=\int_0^L \rho gAv\,dx,\forall v\in H_0^1([0,L])$$

$$\xrightarrow{differentiate}\begin{cases}\text{w.r.t. } A:-\dfrac{d}{dx}\left(EA\dfrac{du'}{dx}\right)=\rho g+\dfrac{d}{dx}\left(E\dfrac{du}{dx}\right),\ \ u'(0)=0,\quad \left.\dfrac{du'}{dx}\right|_{x=L}=0\\ \text{w.r.t. } E:-\dfrac{d}{dx}\left(EA\dfrac{du'}{dx}\right)=\dfrac{d}{dx}\left(A\dfrac{du}{dx}\right),\ \ u'(0)=0,\quad \left.\dfrac{du'}{dx}\right|_{x=L}=0\end{cases}$$

$$E=1,\ A(x)=1,\ \rho g=1\rightarrow-\frac{d^2u}{dx^2}=1\rightarrow\frac{du}{dx}=-x+c_1\rightarrow u(x)=-\frac{x^2}{2}+Lx$$

$$\begin{cases}\text{w.r.t. } A:-\dfrac{d}{dx}\left(\dfrac{du'}{dx}\right)=1+\dfrac{d}{dx}\left(\dfrac{du}{dx}\right)=2,\ \ u'(0)=0,\quad \left.\dfrac{du'}{dx}\right|_{x=L}=0\rightarrow\dfrac{du'}{dx}=-2x+c_1\rightarrow u'=-x^2+2Lx\\ \text{w.r.t. } E:-\dfrac{d}{dx}\left(\dfrac{du'}{dx}\right)=1,\ \ u'(0)=0,\quad \left.\dfrac{du'}{dx}\right|_{x=L}=0\rightarrow\dfrac{du'}{dx}=-x+c_1\rightarrow u'=-\dfrac{x^2}{2}+Lx\end{cases}$$

Sensitivity of Performance Measures

$$\varphi = \int_\Omega g\left(u, \nabla u; p\right) d\Omega \Rightarrow \begin{cases} \varphi = \dfrac{1}{L}\displaystyle\int_0^L u\left(x\right) dx : \text{ average deflection} \\ \varphi = \dfrac{1}{L}\displaystyle\int_0^L E\dfrac{du}{dx} dx : \text{ average stress} \\ \varphi = \dfrac{1}{meas\left(\Omega\right)}\displaystyle\int_\Omega u d\Omega : \text{ average temperature} \\ \varphi = \displaystyle\int_\Omega \left[u^2 + \left(\nabla u\right)^T \nabla u \right] d\Omega \\ \varphi = -\displaystyle\int_\Gamma k \nabla u \cdot \hat{n} d\Gamma : \text{ flux} \end{cases}$$

$$\rightarrow \varphi' = \int_\Omega \left[\nabla_p g + \left(\nabla_u g u' + \nabla_{\nabla u} g \nabla u' \right) \right] d\Omega$$

Example

$$\varphi = \int_{\Omega} \left[pu^2 + (\nabla u)^T \nabla u \right] d\Omega$$

p : independent parameter

$$\left.\begin{aligned} \nabla_p g &= u^2 \\ \nabla_u g &= 2pu \\ \nabla_{\nabla u} g &= 2\nabla u \end{aligned}\right\} \rightarrow \varphi' = \int_{\Omega} \left[u^2 + 2puu' + 2(\nabla u)^T \nabla u' \right] d\Omega$$

Sensitivity of Performance Measures via Adjoint

$$\begin{cases}\text{Find } u \in H^1 \text{ such that} \\ \int_\Omega [\nabla v \cdot k \nabla u + cuv] d\Omega = \int_\Omega f v d\Omega - \int_{\Gamma_N} vq d\Gamma \quad \forall v \in H_0^1\end{cases}$$

$$\begin{cases}\text{Find } u' \in H_0^1 \text{ such that} \\ \int_\Omega [\nabla v \cdot k \nabla u' + cu'v] d\Omega = \int_\Omega f' v d\Omega - \int_\Omega [\nabla v \cdot k' \nabla u + c'uv] d\Omega - \int_{\Gamma_N} vq' d\Gamma \quad \forall v \in H_0^1\end{cases}$$

$$\xrightarrow{\text{Adjoint Weak Form}} \begin{cases}\text{Find } \lambda \in H_0^1 \text{ such that} \\ \int_\Omega [\nabla v \cdot k \nabla \lambda + c\lambda v] d\Omega = \int_\Omega (\nabla_u g v + \nabla_{\nabla u} g \nabla v)\, d\Omega \quad \forall v \in H_0^1\end{cases}$$

$$\xrightarrow{v, u' \in H_0^1} \int_\Omega [\nabla u' \cdot k \nabla \lambda + c\lambda u'] d\Omega = \int_\Omega (\nabla_u g u' + \nabla_{\nabla u} g \nabla u')\, d\Omega$$

$$\xrightarrow{v, \lambda \in H_0^1} \int_\Omega [\nabla \lambda \cdot k \nabla u' + cu'\lambda] d\Omega = \int_\Omega f' \lambda d\Omega - \int_\Omega [\nabla \lambda \cdot k' \nabla u + c'u\lambda] d\Omega - \int_{\Gamma_N} \lambda q' d\Gamma$$

$$\rightarrow \varphi' = \int_\Omega \left[\nabla_p g + (\nabla_u g u' + \nabla_{\nabla u} g \nabla u')\right] d\Omega$$

$$= \int_\Omega \nabla_p g \, d\Omega + \int_\Omega [f' \lambda - (\nabla \lambda \cdot k' \nabla u + c' u \lambda)]\, d\Omega - \int_{\Gamma_N} \lambda q' d\Gamma$$

Sensitivity of Performance Measures via Lagrangian

$$L = \underbrace{(\text{performance measure})}_{\varphi} + \underbrace{(\text{weak form: } v \leftrightarrow \lambda)}_{\int_\Omega [\nabla v \cdot k \nabla u + cuv] d\Omega = \int_\Omega fv d\Omega - \int_{\Gamma_N} vq d\Gamma}$$

$$L = \int_\Omega g d\Omega + \int_\Omega \left[f\lambda - \left(\nabla\lambda \cdot k \nabla u + cu\lambda \right) \right] d\Omega - \int_{\Gamma_N} \lambda q d\Gamma$$

$$\nabla_p u = \nabla_p \lambda = 0, \nabla_p f = f', \nabla_p q = q', \nabla_p k = k', \nabla_p c = c'$$

$$\nabla_p L = \int_\Omega \nabla_p g d\Omega + \int_\Omega \left[f'\lambda - \left(\nabla\lambda \cdot k' \nabla u + c' u\lambda \right) \right] d\Omega - \int_{\Gamma_N} \lambda q' d\Gamma$$

$$\varphi' = \nabla_p L$$

Shape Sensitivity of BVP

- Shape parameter *p* influence the shape, size and location of Ω
- Assumption: *p* does not influence the coefficients k, f, q
- Sensitivity of *u* with respect to a parameter *p*

$$\text{strong form} \begin{cases} -\nabla \cdot k\nabla u + cu = f \quad \text{in } \Omega \\ u = \hat{u} \quad \text{in } \Gamma_D \\ -k\nabla u \cdot \hat{n} = q \quad \text{on } \Gamma_N \end{cases}$$

$$\text{weak form} \begin{cases} \text{Find } u \in H^1 = \left\{ w \middle| w = \hat{u} \quad \text{on } \Gamma_D \right\} \text{ such that} \\ \int_\Omega \left[\nabla v \cdot k\nabla u + cuv \right] d\Omega = \int_\Omega fv d\Omega - \int_{\Gamma_N} vqd\Gamma \quad \forall v \in H_0^1 = \left\{ w \middle| w = 0 \quad \text{on } \Gamma_D \right\} \end{cases}$$

Material to Spatial Domain Transformation

- Reference or material domain Ω_0
  - Independent of *p*
- Spatial domain Ω

$$T(p): \underbrace{\Omega_0(X,Y,Z)}_{u(X,Y,Z;p)} \rightarrow \underbrace{\Omega(x,y,z)}_{u(x,y,z;p)\rightarrow u}$$

Spatial and Material Sensitivity (1)

- Spatial sensitivity
  - derivative of the spatial representation *u* with respect to *p*, where the spatial variables (x,y,z) are considered fixed
- Material sensitivity
  - derivative of the material representation *u*(X,Y,Z; p) with respect to *p*, where the material variables (X,Y,Z) are considered fixed

$$u' \equiv \left.\frac{du}{dp}\right|_{(x,y,z)\text{ fixed}}$$

$$\dot{u} \equiv \left.\frac{du(X,Y,Z;p)}{dp}\right|_{(X,Y,Z)\text{ fixed}}$$

Spatial and Material Sensitivity (2)

compute $\dot{u}$ directly from u

$$u = u(x; p) = u(x(X, p); p)$$

$$\dot{u} = \left.\frac{du(x(X, p); p)}{dp}\right|_{X \text{ fixed}} = \left.\frac{du(x; p)}{dp}\right|_{x \text{ fixed}} + \frac{\partial u}{\partial x}\frac{dx}{dp} = u' + \frac{\partial u}{\partial x}\frac{dx}{dp}$$

$$3D: \dot{u} = u' + \left\{\frac{du}{dx} \quad \frac{du}{dy} \quad \frac{du}{dz}\right\}\left\{\frac{dx}{dp} \quad \frac{dy}{dp} \quad \frac{dz}{dp}\right\}^T$$

$$V = \left\{\frac{dx}{dp} \quad \frac{dy}{dp} \quad \frac{dz}{dp}\right\}^T \text{: design velocity}$$

$$\dot{u} = u' + \nabla u \cdot V$$

$u_V \equiv \nabla u \cdot V$: projected gradient

$\dot{u} = u' + u_V$: relating spatial and material sensitivities

Sensitivity of Spatial Gradient

spatial sensitivity: $\left(\nabla f\right)' \equiv \left.\dfrac{d\left(\nabla f\right)}{dp}\right|_{x\ \text{fixed}} = \nabla \left.\dfrac{df}{dp}\right|_{x\ \text{fixed}} \equiv \nabla f'$

material sensitivity: $\dot{\left(\nabla f\right)} = \left(\nabla f\right)' + \underbrace{\left(\nabla_H^2 f\right)}_{f_{,ij}} V = \nabla f' + \left(\nabla_H^2 f\right) V$

$$= \nabla \dot{f} - \left(\nabla V\right) \nabla f$$

$\nabla \dot{f} = \nabla\left(f' + \nabla f \cdot V\right) = \nabla f' + \nabla\left(\nabla f \cdot V\right)$: gradient of material sensitivity

$$\nabla\left(\nabla f \cdot V\right) = \left(f_{,i} V_i\right)_{,j} = \left(f_{,ij} V_i + f_{,i} V_{i,j}\right) = \left(\nabla_H^2 f\right) V + \left(\nabla V\right) \nabla f$$

$$\rightarrow \nabla \dot{f} = \nabla f' + \left(\nabla_H^2 f\right) V + \left(\nabla V\right) \nabla f$$

Example

– Find the material sensitivity of (1) spatial variable and (2) spatial gradient

$$\Omega_0 = [0 \quad 1] \text{ with } x = X(1+p)$$
$$u = 1 + x^2 p + x e^p$$

Example: Solution (1)

$$\Omega_0 = [0 \quad 1] \text{ with } x = X(1+p)$$

$$u = 1 + x^2 p + xe^p \xleftrightarrow{x=X(1+p)} u(X;p) = 1 + X^2(1+p)^2 p + X(1+p)e^p$$

$$u' \equiv \left.\frac{du}{dp}\right|_{(x,y,z)\text{ fixed}} = x^2 + xe^p$$

$[1]$ material representation $\rightarrow$ differentiate w.r.t. X $\rightarrow$ spatial representation

$$\dot{u}(X;p) = X^2\left[2(1+p)p + (1+p)^2\right] + X\left[e^p + (1+p)e^p\right]$$

$$\xrightarrow{X=\frac{x}{1+p}} \dot{u} = x^2\left[\frac{2p}{1+p} + 1\right] + xe^p\left[\frac{1}{1+p} + 1\right]$$

$[2]$ using design velocity

$$V(X;p) = x_{,p} = X = \frac{x}{1+p}$$

$$\nabla u(x;p) = 2xp + e^p$$

$$\dot{u} = u' + \nabla u \cdot V = \left(x^2 + xe^p\right) + \left(2xp + e^p\right)\frac{x}{1+p}$$

Example: Solution (2)

$$\Omega_0 = [0 \quad 1] \text{ with } x = X(1+p)$$

$$u = 1 + x^2 p + xe^p \xleftrightarrow{x=X(1+p)} u(X;p) = 1 + X^2(1+p)^2 p + X(1+p)e^p$$

$$\nabla u(x;p) = 2xp + e^p \rightarrow (\nabla u)' = 2x + e^p$$

$$u' \equiv \left.\frac{du}{dp}\right|_{(x,y,z)\text{ fixed}} = x^2 + xe^p \rightarrow \nabla(u') = 2x + e^p$$

$$[1]\begin{cases} \nabla u(x;p) = 2xp + e^p \xleftrightarrow{x=X(1+p)} \nabla u(X;p) = 2X(1+p)p + e^p \\ \dot{(\nabla u)}(X;p) = 2Xp + 2X(1+p) + e^p \xrightarrow{X=\frac{x}{1+p}} \dot{(\nabla u)}(x;p) = 2x + \dfrac{2xp}{1+p} + e^p \end{cases}$$

$$[2]\begin{cases} \dot{u} = u' + \nabla u \cdot V = \left(x^2 + xe^p\right) + \dfrac{\left(2xp + e^p\right)x}{1+p} \rightarrow \nabla\dot{u} = \left(2x + e^p\right) + \dfrac{4xp + e^p}{1+p} \\ \dot{(\nabla u)} = \nabla\dot{u} - (\nabla V)\nabla u = \left(2x + e^p\right) + \dfrac{4xp + e^p}{1+p} - \left(\dfrac{1}{1+p}\right)\left(2xp + e^p\right) = 2x + e^p + \dfrac{2xp}{1+p} \end{cases}$$

Volume Integral Transformation

- Sensitivity of a domain integral: derivative w.r.t. p
- Spatial domain Ω is dependent on p → change the domain of integration to the material domain Ω_0 that is independent of p

$$dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY + \frac{\partial x}{\partial Z} dZ \rightarrow \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} \rightarrow \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = J(X;p) \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix}$$

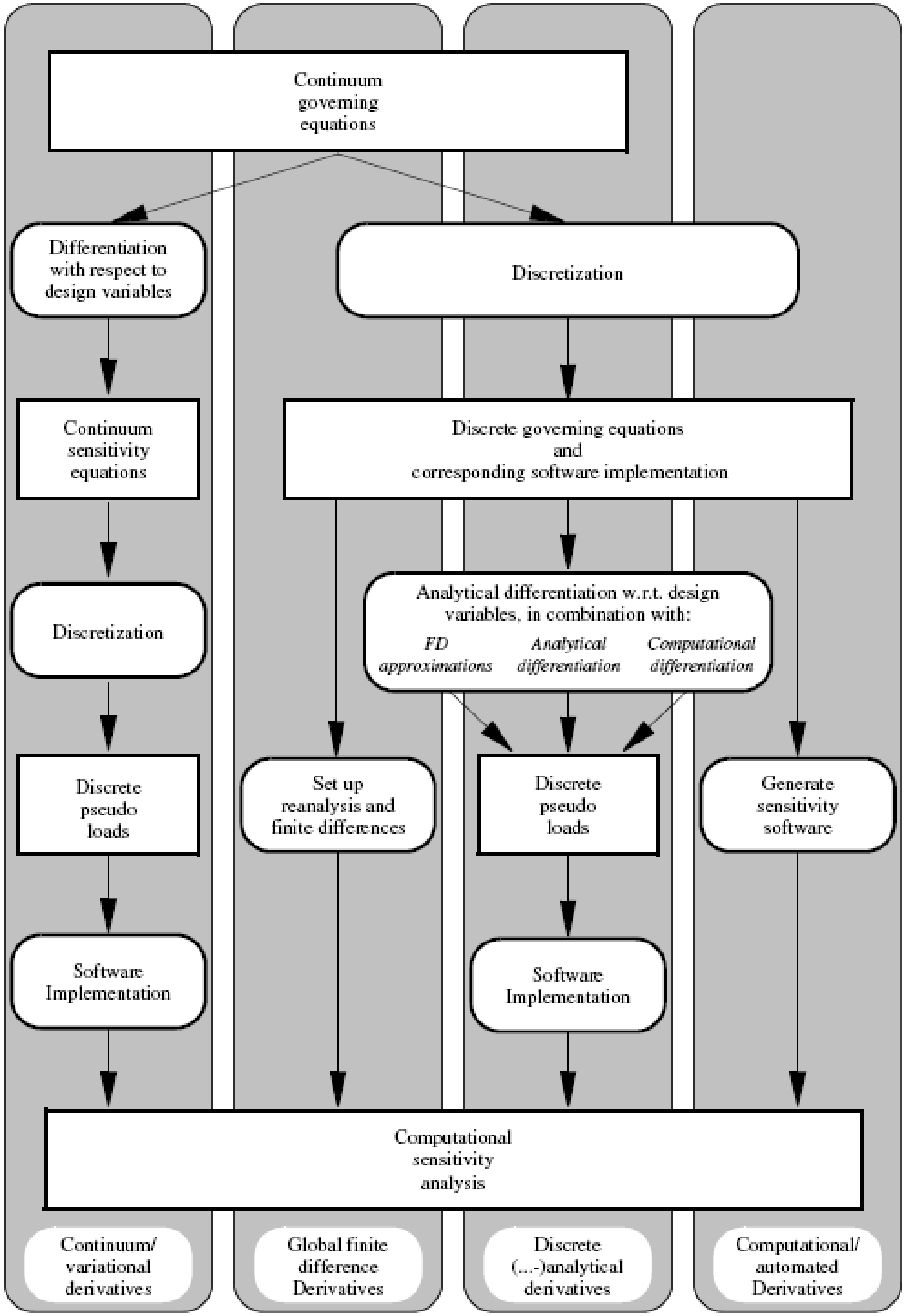
$$J(X;p) \equiv \frac{\partial(x,y,z)}{\partial(X,Y,Z)} \equiv \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} : \text{ Jacobian}$$

$$dxdydz = \underbrace{|J(X;p)|}_{\det J(X;p)} dXdYdZ \rightarrow d\Omega = |J(X;p)| d\Omega_0$$

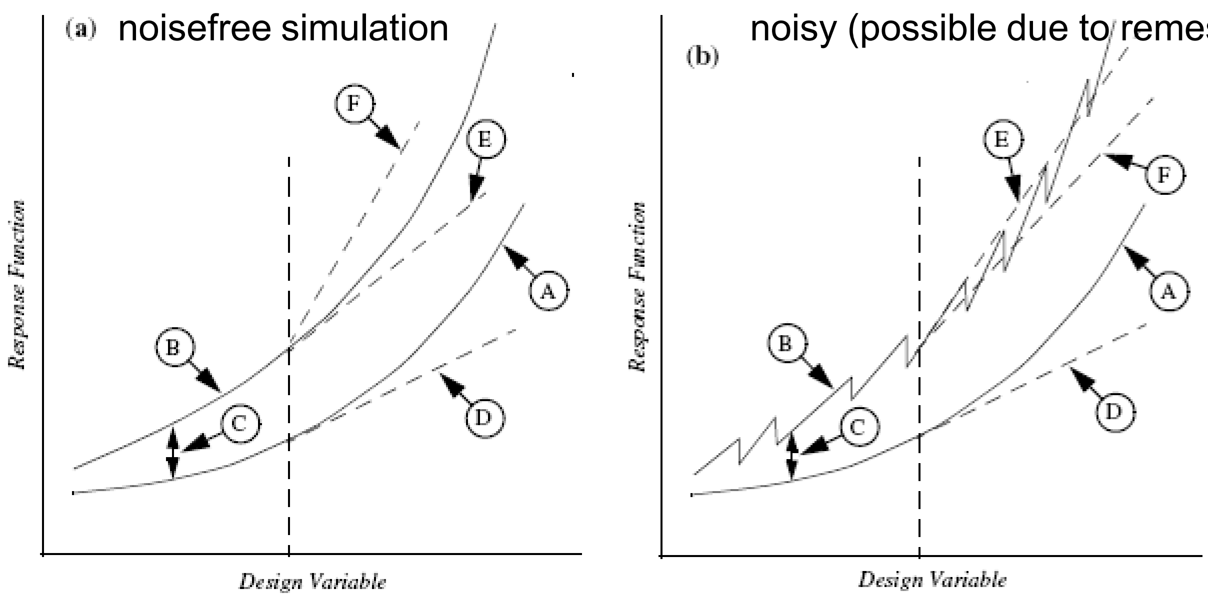
$$\int_\Omega f d\Omega = \int_{\Omega_0} f(X;p) |J(X;p)| d\Omega_0 \rightarrow \frac{d\left(\int_\Omega f d\Omega\right)}{dp}$$

Overview

F. Keulen, R.T. Haftka, N.H. Kim, Review on options for structural design sensitivity analysis. Part 1: Linear systems, Comput. Methods Appl. Mech. Engrg. 194 (2005) 3213-3243



- Categories
  - Overall finite differences
  - Direct derivatives
  - Continuum derivatives
  - Computational or automatic differentiation
- Criteria
  - Accuracy
  - Consistency
  - Computational cost and implementation effort



A: exact solution of the governing continuum equations
B: computational counterpart
C: modeling error
D: exact derivatives
E: consistent derivatives
F: non-exact and non-consistent computed derivatives

Approaches to Design Sensitivity Analysis

- Approximation approach
  - Forward finite difference
  - Central finite difference
- Discrete approach
  - Semi-analytical method
  - Analytical method
- Continuum approach
  - Continuum-discrete method
  - Continuum-continuum method

K. K. Choi and N.-H. Kim, Structural Sensitivity Analysis and Optimization 1 Linear Systems, Springer, 2005

Design Sensitivity Analysis Method

- Discrete Model

$$\boldsymbol{K}(\boldsymbol{b})\boldsymbol{z} = \boldsymbol{F}$$

↓

Algebraic Adjoint or State Sensitivity

↓

$$\psi' \approx \frac{\partial}{\partial \boldsymbol{b}}\left[\boldsymbol{K}(\boldsymbol{b})\boldsymbol{z}\right]\delta \boldsymbol{b}$$

↓

Evaluate Model Parameter Sensitivity

- Continuum Model

$$\iint_{\Omega} \boldsymbol{\sigma}^T \bar{\boldsymbol{\varepsilon}} d\Omega = \iint_{\Omega} \bar{\boldsymbol{z}}^T \boldsymbol{f}^B d\Omega + \int_{\Gamma} \bar{\boldsymbol{z}}^T \boldsymbol{f}^B d\Gamma$$

↓

Continuum Adjoint

↓

$$\psi' \approx \iint_{\Omega} h(\boldsymbol{z}, \boldsymbol{\lambda})\delta \boldsymbol{u} d\Omega$$

↓

$\boldsymbol{K\lambda} = \boldsymbol{G}$ or Other Analysis Methods

↓

Evaluate Design Sensitivity

Sensitivity of Discrete Systems

- Sensitivity: derivatives of response w.r.t. a design variable
  - Major computational cost of the optimization process
  - Statistical variation in the response of the structure

  - Discretization $\rightarrow$ differentiate (ch.7)
  - Differentiate the continuum equations $\rightarrow$ discretization (ch.8)

spatial discretization of the continuum equations

↓

algebraic equations (static response)
algebraic eigenvalue problems (vibration/buckling)
ordinary differential equations (transient response)

Finite Difference Approximation

- Popular, computationally expensive, easy to implement, accuracy problem
  - Forward-difference approximation

$$u(x+\Delta x)=u(x)+\Delta x u'(x)+\frac{(\Delta x)^2}{2!}u''(x)+\cdots \rightarrow u'(x)=\frac{u(x+\Delta x)-u(x)}{\Delta x}-\frac{\Delta x}{2}u''(x)-\cdots$$

$$\frac{\Delta u}{\Delta x}=\frac{u(x+\Delta x)-u(x)}{\Delta x}+O(\Delta x)$$

  - Central-difference approximation

$$\left.\begin{aligned} u(x+\Delta x)&=u(x)+\Delta x u'(x)+\frac{(\Delta x)^2}{2!}u''(x)+\frac{(\Delta x)^3}{3!}u'''(x)+\cdots \\ u(x-\Delta x)&=u(x)-\Delta x u'(x)+\frac{(\Delta x)^2}{2!}u''(x)-\frac{(\Delta x)^3}{3!}u'''(x)+\cdots \end{aligned}\right\}$$

$$\rightarrow u'(x)=\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}-\frac{(\Delta x)^2}{6}u'''(x)+\cdots$$

$$\frac{\Delta u}{\Delta x}=\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}+O\left[(\Delta x)^2\right]$$

Accuracy and Step Size

- Truncation error
  - Due to neglected terms in the Taylor series expansion

$$u\left(x+\Delta x\right)=u\left(x\right)+\Delta x\frac{du}{dx}\left(x\right)+\frac{\left(\Delta x\right)^2}{2!}\frac{d^2u}{dx^2}\left(x+\varsigma\Delta x\right),\quad 0\le\varsigma\le 1$$

$$\begin{cases}\text{forward difference: } e_T\left(\Delta x\right)=\dfrac{\Delta x}{2}\dfrac{d^2u}{dx^2}\left(x+\varsigma\Delta x\right)=\dfrac{\Delta x}{2}\left|s_b\right|,\quad 0\le\varsigma\le 1\\ \qquad s_b:\ \text{bound on the second derivative in } \left[x,x+\Delta x\right]\\ \text{central difference: } e_T\left(\Delta x\right)=\dfrac{\left(\Delta x\right)^2}{6}\dfrac{d^3u}{dx^3}\left(x+\varsigma\Delta x\right),\quad -1\le\varsigma\le 1\end{cases}$$

- Condition error
  - Round-off error, ill-conditioned numerical process

$$\text{forward−difference: } e_C(\Delta x)=\frac{2}{\Delta x}\varepsilon_u,\ \varepsilon_u\text{: absolute error bound } (10^{-16})$$

Optimal Step Size for Forward Difference

- Formula: $$\frac{\Delta u}{\Delta x} = \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

- Bound for total error: step size dilemma

$$e = \frac{\Delta x}{2}|s_b| + \frac{2}{\Delta x}\varepsilon_u$$

- Optimal step size

$$\frac{de}{d(\Delta x)} = \frac{|s_b|}{2} - \frac{2}{(\Delta x)^2}\varepsilon_u = 0 \rightarrow (\Delta x)_{opt} = 2\sqrt{\frac{\varepsilon_u}{|s_b|}}$$

Example 7.1.1

$$\left.\begin{aligned}101u + xv &= 10\\ xu + 100v &= 10\end{aligned}\right\} \rightarrow u = \frac{\begin{vmatrix}10 & x\\ 10 & 100\end{vmatrix}}{\begin{vmatrix}101 & x\\ x & 100\end{vmatrix}} = \frac{-10x + 1000}{10100 - x^2}$$

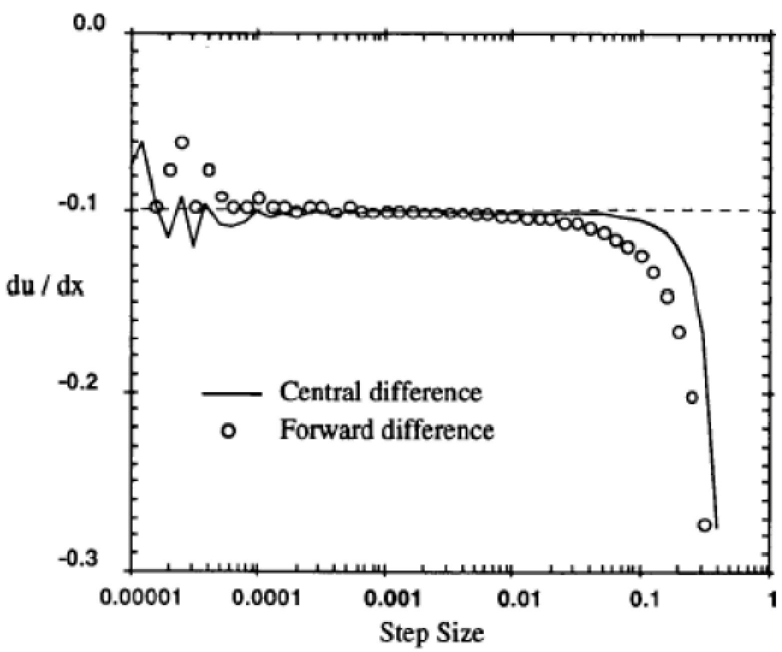
$\rightarrow$ derivative @ $x = 100$

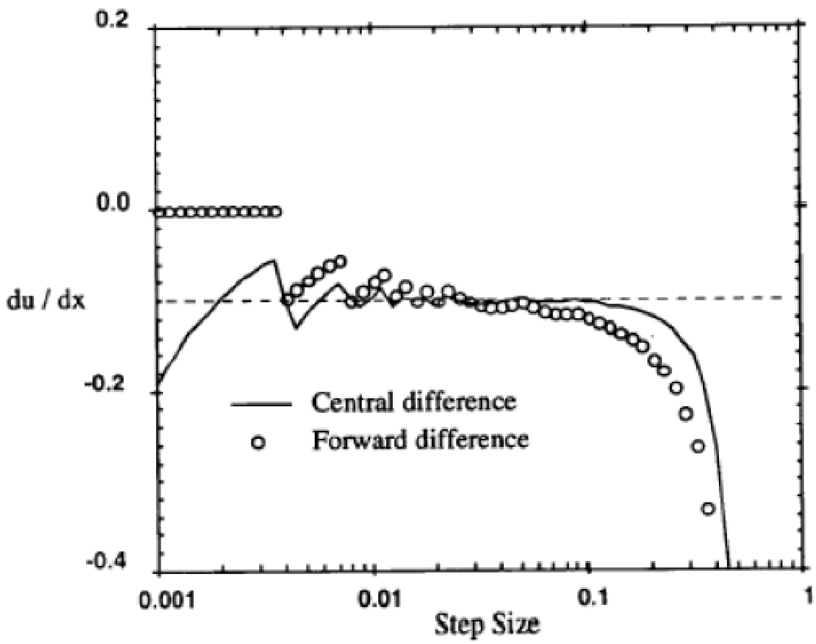
$$\frac{du}{dx} = \frac{30x^2 - 2000x - 101000}{\left(10100 - x^2\right)^2}$$

with poorer conditioning

$$\left.\begin{aligned}10001u + xv &= 1000\\ xu + 10000v &= 1000\end{aligned}\right\}$$

$\rightarrow$ derivative @ $x = 10{,}000$





Effect of derivative magnitude

- Large derivatives are easier to estimate than small ones
- What does it mean that derivative is large?
- Accuracy measure
  - Logarithmic derivative: percentage change in u due to a percent change in x

$$\frac{d_l u}{dx} = \frac{d(\log u)}{d(\log x)} = \frac{du/u}{dx/x} = \frac{du}{dx}\frac{x}{u} \xrightarrow{\text{change in sign}} \frac{d_{lm}u}{dx} = \frac{du/u_t}{dx/x_t}$$

$$\begin{cases} \dfrac{d_l u}{dx} > 1: \text{ the derivative to be large} \\ \dfrac{d_l u}{dx} < 1: \text{ the derivative to be small} \\ \qquad \rightarrow \text{difficult to evaluate it accurately using FDM} \end{cases}$$

Example: $y=10+(x-5)^2$

- compare the accuracy of forward difference derivatives with a step size of one at x=10 and x=6
  - What does it mean to have a logarithmic derivative equal to one?

$$y=10+(x-5)^2 \rightarrow \left\{\begin{array}{l}\text{exact: } \dfrac{dy}{dx}=2(x-5) \\ \text{forward difference: } \dfrac{\Delta y}{\Delta x}=\dfrac{y(x_0+1)-y(x_0)}{1}\end{array}\right\} \rightarrow \left.\frac{d_l y}{dx}\right|_{x=x_0}=\frac{dy}{dx}\frac{x_0}{y_0}$$

$$@x=10: \left\{\begin{array}{l}\text{exact: } \dfrac{dy}{dx}=2(x-5)=10 \\ \text{forward difference: } \dfrac{\Delta y}{\Delta x}=\dfrac{y(x_0+1)-y(x_0)}{1}=\dfrac{46-35}{1}=11\end{array}\right\} (10\% \text{ error}) \rightarrow \left.\frac{d_l y}{dx}\right|_{x=10}=10\frac{10}{35}=2.8$$

$$@x=6: \left\{\begin{array}{l}\text{exact: } \dfrac{dy}{dx}=2(x-5)=2 \\ \text{forward difference: } \dfrac{\Delta y}{\Delta x}=\dfrac{y(x_0+1)-y(x_0)}{1}=\dfrac{14-11}{1}=3\end{array}\right\} (50\% \text{ error}) \rightarrow \left.\frac{d_l y}{dx}\right|_{x=6}=2\frac{6}{11}=1.1$$

Sensitivity for Static Problem (1)

- Governing equations
  - K: normally large, banded, sparse and ill-conditioned, but positive definite

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \xrightarrow{\text{Cholesky decomposition}} \boldsymbol{L}\underbrace{\boldsymbol{D}\boldsymbol{L}^T\boldsymbol{u}}_{\boldsymbol{v}} = \boldsymbol{f}$$

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{f} \rightarrow \boldsymbol{D}\boldsymbol{L}^T\boldsymbol{u} = \boldsymbol{v}$$

- Analytical method
  - Direct method
  - Adjoint method
- Semi-analytical method

$$\frac{dg}{dx} = ?$$

Sensitivity for Static Problem (2)

- Direct method
  - Requires pseudo load calculation and solution for each design variable
  - Pseudo load can be calculated outside of finite element program
  - Requires only forward and backward substitution
  - Minimal effort with new constraints

equations for displacement vector: $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \xrightarrow{\text{differentiate}} \boldsymbol{K}\dfrac{d\boldsymbol{u}}{dx} = \underbrace{\dfrac{d\boldsymbol{f}}{dx} - \dfrac{d\boldsymbol{K}}{dx}\boldsymbol{u}}_{\text{pseudo load}}$

response quantity of interest:

$$g(\boldsymbol{u}, x) \geq 0 \xrightarrow{\text{differentiate}} \frac{dg}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial \boldsymbol{u}}\frac{d\boldsymbol{u}}{dx} \xrightarrow{z_i = \frac{\partial g}{\partial u_i}} \frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{z}^T\frac{d\boldsymbol{u}}{dx}$$

$\dfrac{d\boldsymbol{u}}{dx} \rightarrow \boldsymbol{z}^T\dfrac{d\boldsymbol{u}}{dx}$ (number of design variables)

Sensitivity for Static Problem (3)

- Adjoint method (dummy-load method)
  - Requires solution for each new g and pseudo load calculation for each design variable

$$\boldsymbol{K}\frac{d\boldsymbol{u}}{dx} = \underbrace{\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}}_{\text{pseudo load}} \rightarrow \frac{d\boldsymbol{u}}{dx} = \boldsymbol{K}^{-1}\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}\right)$$

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{z}^T\frac{d\boldsymbol{u}}{dx} \rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{z}^T\boldsymbol{K}^{-1}\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}\right)$$

$$\boldsymbol{z}^T\boldsymbol{K}^{-1} = \boldsymbol{\lambda}^T \rightarrow \boldsymbol{K\lambda} = \underbrace{\boldsymbol{z}}_{\text{dummy load}} \qquad \text{(number of constraints)}$$

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{z}^T\frac{d\boldsymbol{u}}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}\right)$$

Sensitivity for Static Problem (4)

- Adjoint method in general
  - Adding the derivative of the equations of equilibrium multiplied by a Lagrange multiplier to the derivative of the constraint

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \rightarrow \boldsymbol{K}\frac{d\boldsymbol{u}}{dx} = \frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}$$

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{z}^T\frac{d\boldsymbol{u}}{dx} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u} - \boldsymbol{K}\frac{d\boldsymbol{u}}{dx}\right) = \frac{\partial g}{\partial x} + \left(\boldsymbol{z}^T - \boldsymbol{\lambda}^T\boldsymbol{K}\right)\frac{d\boldsymbol{u}}{dx} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}\right)$$

$$\xrightarrow{\boldsymbol{K}\boldsymbol{\lambda}=\boldsymbol{z}} \frac{dg}{dx} = \frac{\partial g}{\partial x} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}}{dx} - \frac{d\boldsymbol{K}}{dx}\boldsymbol{u}\right)$$

Sensitivity for Static Problem (5)

- Semi-analytical method
  - Analytical derivatives of stiffness matrices are burdensome especially in commercial software
  - Most resort to finite difference calculation of derivatives of stiffness matrix and force vector
  - Accuracy problem? central difference approximation

$$\frac{d\boldsymbol{K}}{dx} \approx \frac{\boldsymbol{K}\left(x+\Delta x\right)-\boldsymbol{K}\left(x\right)}{\Delta x}$$

$$\frac{d\boldsymbol{f}}{dx} \approx \frac{\boldsymbol{f}\left(x+\Delta x\right)-\boldsymbol{f}\left(x\right)}{\Delta x}$$

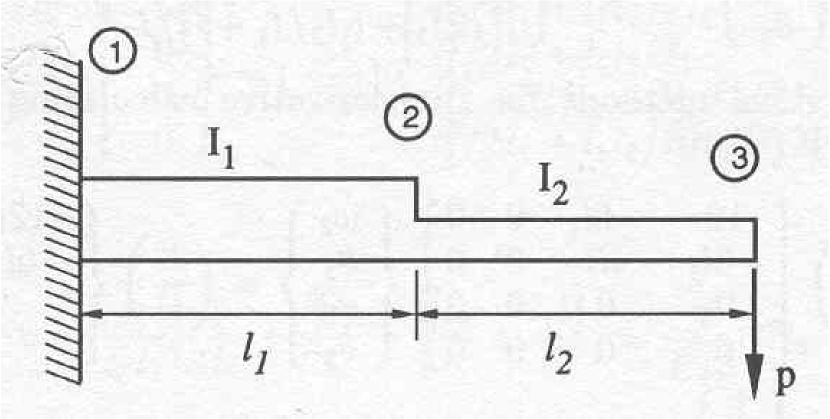
Example 7.2.1+7.2.2

- Sensitivity of a constraint on the tip displacement of a stepped cantilever beam w.r.t. the moment of inertia I_1 and length l_1
- Errors associated with the finite-difference method and the semi-analytical method

$$w_{tip} = \frac{p}{3EI_1}\left(l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2\right) + \frac{pl_2^3}{3EI_2}$$

$$g = c - w_{tip} \geq 0$$

$$\begin{cases} \dfrac{\partial g}{\partial I_1} = \dfrac{p}{3EI_1^2}\left(l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2\right) \\ \dfrac{\partial g}{\partial l_1} = -\dfrac{p}{3EI_1}\left(3l_1^2 + 6l_1 l_2 + 3l_2^2\right) = -\dfrac{p}{EI_1}\left(l_1 + l_2\right)^2 \end{cases}$$



$$U = \int_0^{l_1} \frac{\left[p\left(l_1 + l_2 - x\right)\right]^2}{2EI_1} dx + \int_{l_1}^{l_2} \frac{\left[p\left(l_1 + l_2 - x\right)\right]^2}{2EI_2} dx = \frac{p^2}{2EI_1} \int_0^{l_1} \left(l_1 + l_2 - x\right)^2 dx + \frac{p^2}{2EI_2} \int_{l_1}^{l_2} \left(l_1 + l_2 - x\right)^2 dx$$

$$= \frac{p^2}{2EI_1} \left(\frac{l_1^3}{3} + l_1^2 l_2 + l_1 l_2^2 \right) + \frac{p^2}{2EI_2} \left(\frac{l_2^3}{3} - \frac{l_1^3}{3} \right)$$

$$\delta = \frac{\partial U}{\partial p} = \frac{p}{3EI_1} \left(l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2 \right) + \frac{p}{3EI_2} \left(l_2^3 - l_1^3 \right)$$

$$\xrightarrow{???} w_{tip} = \frac{p}{3EI_1} \left(l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2 \right) + \frac{pl_2^3}{3EI_2}$$

$$g = c - w_{tip} \ge 0$$

$$\begin{cases} \dfrac{\partial g}{\partial I_1} = \dfrac{p}{3EI_1^2} \left(l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2 \right) \\ \dfrac{\partial g}{\partial l_1} = -\dfrac{p}{3EI_1} \left(3l_1^2 + 6l_1 l_2 + 3l_2^2 \right) = -\dfrac{p}{EI_1} \left(l_1 + l_2 \right)^2 \end{cases}$$

Solution for Displacement

$$\boldsymbol{K}_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \boldsymbol{u} = \begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}, \boldsymbol{f} = \begin{Bmatrix} 0 \\ 0 \\ p \\ 0 \end{Bmatrix}$$

$$\rightarrow E\begin{bmatrix} 12\frac{I_1}{l_1^3} & 6\frac{I_1}{l_1^2} & -12\frac{I_1}{l_1^3} & 6\frac{I_1}{l_1^2} & & \\ 6\frac{I_1}{l_1^2} & 4\frac{I_1}{l_1} & -6\frac{I_1}{l_1^2} & 2\frac{I_1}{l_1} & & \\ -12\frac{I_1}{l_1^3} & -6\frac{I_1}{l_1^2} & 12\frac{I_1}{l_1^3}+12\frac{I_2}{l_2^3} & -6\frac{I_1}{l_1^2}+6\frac{I_2}{l_2^2} & -12\frac{I_2}{l_2^3} & 6\frac{I_2}{l_2^2} \\ 6\frac{I_1}{l_1^2} & 2\frac{I_1}{l_1} & -6\frac{I_1}{l_1^2}+6\frac{I_2}{l_2^2} & 4\frac{I_1}{l_1}+4\frac{I_2}{l_2} & -6\frac{I_2}{l_2^2} & 2\frac{I_2}{l_2} \\ & & -12\frac{I_2}{l_2^3} & -6\frac{I_2}{l_2^2} & 12\frac{I_2}{l_2^3} & -6\frac{I_2}{l_2^2} \\ & & 6\frac{I_2}{l_2^2} & 2\frac{I_2}{l_2} & -6\frac{I_2}{l_2^2} & 4\frac{I_2}{l_2} \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \\ 0 \end{Bmatrix} \rightarrow \boldsymbol{u} = \begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \frac{p}{E}\begin{Bmatrix} \dfrac{2l_1^3+3l_1^2l_2}{6I_1} \\ \dfrac{l_1^2+2l_1l_2}{2I_1} \\ \dfrac{l_1^3+3l_1^2l_2+3l_1l_2^2}{3I_1}+\dfrac{l_2^3}{3I_2} \\ \dfrac{l_1^2+2l_1l_2}{2I_1}+\dfrac{l_2^2}{2I_2} \end{Bmatrix}$$

Direct Method

$$\frac{\partial \boldsymbol{u}}{\partial I_1} = \boldsymbol{K}^{-1}\left(\frac{\partial \boldsymbol{f}}{\partial I_1} - \frac{\partial \boldsymbol{K}}{\partial I_1}\boldsymbol{u}\right) \rightarrow \frac{\partial g}{\partial I_1} = -\frac{\partial w_3}{\partial I_1}$$

$$\frac{\partial \boldsymbol{K}}{\partial I_1}\boldsymbol{u} = \frac{E}{l_1^3}\begin{bmatrix} 12 & -6l_1 & 0 & 0 \\ -6l_1 & 4l_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \frac{E}{l_1^3}\begin{Bmatrix} 12w_2 - 6l_1\theta_2 \\ -6l_1 w_2 + 4l_1^2\theta_2 \\ 0 \\ 0 \end{Bmatrix} = \frac{p}{I_1}\begin{Bmatrix} 1 \\ l_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \frac{\partial \boldsymbol{u}}{\partial I_1} = \frac{\partial}{\partial I_1}\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \boldsymbol{K}^{-1}\begin{Bmatrix} \frac{p}{I_1} \\ \frac{pl_2}{I_1} \\ 0 \\ 0 \end{Bmatrix} = -\frac{p}{EI_1^2}\begin{Bmatrix} \frac{l_1^2 l_2}{2} + \frac{l_1^3}{3} \\ l_1 l_2 + \frac{l_1^2}{2} \\ l_1^2 l_2 + l_1 l_2^2 + \frac{l_1^3}{3} \\ l_1 l_2 + \frac{l_1^2}{2} \end{Bmatrix}$$

Adjoint Method

$$\boldsymbol{\lambda} = \boldsymbol{K}^{-1}\boldsymbol{z} = \boldsymbol{K}^{-1}\frac{\partial g}{\partial \boldsymbol{u}} = \boldsymbol{K}^{-1}\begin{Bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix} \rightarrow \frac{\partial g}{\partial I_1} = \frac{\partial g}{\partial I_1} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}}{dI_1} - \frac{d\boldsymbol{K}}{dI_1}\boldsymbol{u}\right) = -\boldsymbol{\lambda}^T\frac{\partial \boldsymbol{K}}{\partial I_1}\boldsymbol{u}$$

$$\boldsymbol{z}^T = -\frac{\partial w_{tip}}{\partial u} = \begin{Bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\lambda} = \boldsymbol{K}^{-1}\boldsymbol{z} = \boldsymbol{K}^{-1}\begin{Bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix} = -\frac{1}{E}\begin{Bmatrix} \dfrac{2l_1^3 + 3l_1^2 l_2}{6I_1} \\ \dfrac{l_1^2 + 2l_1 l_2}{2I_1} \\ \dfrac{l_1^3 + 3l_1^2 l_2 + 3l_1 l_2^2}{3I_1} + \dfrac{l_2^3}{3I_2} \\ \dfrac{l_1^2 + 2l_1 l_2}{2I_1} + \dfrac{l_2^2}{2I_2} \end{Bmatrix}$$

$$\rightarrow \frac{\partial g}{\partial I_1} = -\boldsymbol{\lambda}^T\frac{\partial \boldsymbol{K}}{\partial I_1}\boldsymbol{u} = \frac{p}{EI_1}\left(\frac{2l_1^3 + 3l_1^2 l_2}{6I_1} + \frac{l_1^2 + 2l_1 l_2}{2I_1}l_2\right) = \frac{p}{EI_1^2}\left(l_1^2 l_2 + l_1 l_2^2 + \frac{l_1^3}{3}\right)$$

Example 7.2.2

finite difference method:

$$\begin{cases} e_T\left(\Delta I_1\right) = \dfrac{\Delta I_1}{2}\dfrac{\partial^2 g}{\partial I_1^{\,2}}\left(I_1+\zeta\Delta I_1\right) = -\dfrac{\partial^2 w_{tip}}{\partial I_1^{\,2}}\dfrac{\Delta I_1}{2} = -\dfrac{p}{3EI_1^{\,3}}\left(l_1^{\,3}+3l_1^{\,2}l_2+3l_1l_2^{\,2}\right)\Delta I_1 = -\dfrac{\partial g}{\partial I_1}\dfrac{\Delta I_1}{I_1} \rightarrow \dfrac{e_T}{\dfrac{\partial g}{\partial I_1}} = -\dfrac{\Delta I_1}{I_1} \\ e_T\left(\Delta l_1\right) = -\dfrac{\partial^2 w_{tip}}{\partial l_1^{\,2}}\dfrac{\Delta l_1}{2} = -\dfrac{p}{EI_1}\left(l_1+l_2\right)\Delta l_1 = \dfrac{\partial g}{\partial l_1}\dfrac{\Delta l_1}{l_1+l_2} \rightarrow \dfrac{e_T}{\dfrac{\partial g}{\partial l_1}} = \dfrac{\Delta l}{l_1+l_2} \end{cases}$$

semi-analytical method: $\dfrac{\partial g}{\partial x} \approx \lambda^T \dfrac{K\left(x+\Delta x\right)-K\left(x\right)}{\Delta x}u$

$$\begin{cases} e_T\left(\Delta I_1\right) = \dfrac{\Delta I_1}{2}\dfrac{\partial^2 g}{\partial I_1^{\,2}} = \dfrac{\Delta I_1}{2}\lambda^T\dfrac{\partial^2 K}{\partial I_1^{\,2}}u = 0 \\ e_T\left(\Delta l_1\right) = \dfrac{\Delta l_1}{2}\dfrac{\partial^2 g}{\partial l_1^{\,2}} = \dfrac{\Delta l_1}{2}\lambda^T\dfrac{\partial^2 K}{\partial l_1^{\,2}}u = \dfrac{p\Delta l_1}{EI_1l_1}\left(3l_1^{\,2}+7l_1l_2+4l_2^{\,2}\right) \rightarrow \dfrac{e_T}{\dfrac{\partial g}{\partial l_1}} = \dfrac{3l_1^{\,2}+7l_1l_2+4l_2^{\,2}}{\left(l_1+l_2\right)^2}\dfrac{\Delta l}{l_1} \end{cases}$$

Sensitivity for Eigenvalue Problem (1)

- Undamped vibration and linear buckling

	$\boldsymbol{K u}-\mu \boldsymbol{M u}=0 \quad \left(\boldsymbol{u}^{T} \boldsymbol{M u}=1\right)$	
problem	Undamped vibration	Linear buckling
K	Stiffness	Stiffness
M	Mass	Geometric stiffness
u	Mode shape	Mode shape
μ	Square of the frequency	Buckling load factor

Sensitivity for Eigenvalue Problem (2)

(for the case of distinct eigenvalues)

$$
\begin{cases}
\boldsymbol{Ku} - \mu \boldsymbol{Mu} = 0 \rightarrow (\boldsymbol{K} - \mu \boldsymbol{M}) \dfrac{d\boldsymbol{u}}{dx} - \dfrac{d\mu}{dx} \boldsymbol{Mu} = -\left(\dfrac{d\boldsymbol{K}}{dx} - \mu \dfrac{d\boldsymbol{M}}{dx} \right) \boldsymbol{u} \rightarrow \dfrac{d\mu}{dx} = \dfrac{\boldsymbol{u}^T \left(\dfrac{d\boldsymbol{K}}{dx} - \mu \dfrac{d\boldsymbol{M}}{dx} \right) \boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{Mu}} \\
\boldsymbol{u}^T \boldsymbol{Wu} = 1 \rightarrow \boldsymbol{u}^T \boldsymbol{W} \dfrac{d\boldsymbol{u}}{dx} = -\dfrac{1}{2} \boldsymbol{u}^T \dfrac{d\boldsymbol{W}}{dx} \boldsymbol{u}
\end{cases}
$$

$$
\rightarrow \begin{bmatrix} \underline{\boldsymbol{K} - \mu \boldsymbol{M}} & -\boldsymbol{Mu} \\ \boldsymbol{u}^T \boldsymbol{W} & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{d\boldsymbol{u}}{dx} \\ \dfrac{d\mu}{dx} \end{Bmatrix} = \begin{Bmatrix} -\left(\dfrac{d\boldsymbol{K}}{dx} - \mu \dfrac{d\boldsymbol{M}}{dx} \right) \boldsymbol{u} \\ -\dfrac{1}{2} \boldsymbol{u}^T \dfrac{d\boldsymbol{W}}{dx} \boldsymbol{u} \end{Bmatrix}
$$

Sensitivity for Eigenvalue Problem (3)

- Nelson's method
  - Largest component of the eigenvector be equal to one

$$\boldsymbol{u}^T\boldsymbol{W}\boldsymbol{u} = 1 \rightarrow \boldsymbol{u} = u_m\bar{\boldsymbol{u}} \quad \left(\text{re-normalization}: \bar{u}_m = 1 \text{ and } \frac{d\bar{u}_m}{dx} = 0\right)$$

$$\frac{d\boldsymbol{u}}{dx} = \frac{du_m}{dx}\bar{\boldsymbol{u}} + u_m\frac{d\bar{\boldsymbol{u}}}{dx}$$

$$\boldsymbol{u}^T\boldsymbol{W}\frac{d\boldsymbol{u}}{dx} = -\frac{1}{2}\boldsymbol{u}^T\frac{d\boldsymbol{W}}{dx}\boldsymbol{u} \rightarrow \boldsymbol{u}^T\boldsymbol{W}\left(\frac{du_m}{dx}\bar{\boldsymbol{u}} + u_m\frac{d\bar{\boldsymbol{u}}}{dx}\right) = -\frac{1}{2}\boldsymbol{u}^T\frac{d\boldsymbol{W}}{dx}\boldsymbol{u}$$

$$\rightarrow \boldsymbol{u}^T\boldsymbol{W}\frac{\boldsymbol{u}}{u_m}\frac{du_m}{dx} = -u_m\boldsymbol{u}^T\boldsymbol{W}\frac{d\bar{\boldsymbol{u}}}{dx} - \frac{1}{2}\boldsymbol{u}^T\frac{d\boldsymbol{W}}{dx}\boldsymbol{u} \rightarrow \frac{du_m}{dx} = -u_m{}^2\boldsymbol{u}^T\boldsymbol{W}\frac{d\bar{\boldsymbol{u}}}{dx} - \frac{u_m}{2}\boldsymbol{u}^T\frac{d\boldsymbol{W}}{dx}\boldsymbol{u}$$

- Modal technique

$$\frac{d\boldsymbol{u}^k}{dx} = \sum_{j=1}^{l} c_{kj}\boldsymbol{u}^j \quad \text{where } c_{kj} = \frac{(\boldsymbol{u}^j)^T\left(\frac{d\boldsymbol{K}}{dx} - \mu_k\frac{d\boldsymbol{M}}{dx}\right)\boldsymbol{u}^k}{(\mu_k - \mu_j)(\boldsymbol{u}^j)^T\boldsymbol{M}\boldsymbol{u}^j}, \quad k \neq j$$

Derivatives of Repeated Eigenvalues

- Assume *m* repeated eigenvectors
- To find eigenvalue derivatives need to solve a second eigenvalue problem

$$\boldsymbol{u} = \sum_{i=1}^{m} q_i \boldsymbol{u}_i = U\boldsymbol{q}$$

$$\boldsymbol{Ku} - \mu\boldsymbol{Mu} = 0 \rightarrow \left(\boldsymbol{K} - \mu\boldsymbol{M}\right)\frac{d\boldsymbol{u}}{dx} - \frac{d\mu}{dx}\boldsymbol{Mu} = -\left(\frac{d\boldsymbol{K}}{dx} - \mu\frac{d\boldsymbol{M}}{dx}\right)\boldsymbol{u}$$

$$U^T\left[\left(\boldsymbol{K} - \mu\boldsymbol{M}\right)\frac{d\left(U\boldsymbol{q}\right)}{dx} - \frac{d\mu}{dx}\boldsymbol{M}\left(U\boldsymbol{q}\right)\right] = -U^T\left[\left(\frac{d\boldsymbol{K}}{dx} - \mu\frac{d\boldsymbol{M}}{dx}\right)\left(U\boldsymbol{q}\right)\right]$$

$$U^T\left(\frac{d\boldsymbol{K}}{dx} - \mu\frac{d\boldsymbol{M}}{dx}\right)U\boldsymbol{q} - \frac{d\mu}{dx}U^T\boldsymbol{M}U\boldsymbol{q} = 0$$

$$\xrightarrow{U^T\left(\frac{d\boldsymbol{K}}{dx} - \mu\frac{d\boldsymbol{M}}{dx}\right)U = \boldsymbol{A},\ U^T\boldsymbol{M}U = \boldsymbol{B}} \left(\boldsymbol{A} - \frac{d\mu}{dx}\boldsymbol{B}\right)\boldsymbol{q} = 0$$

Constraints on Transient Response

$$g(\boldsymbol{u}, x, t) \geq 0, \quad 0 \leq t \leq t_f \rightarrow g_i(\boldsymbol{u}, x, t_i) \geq 0, \quad i = 1, \ldots, n_t$$

- Remove the time dependence of the constraint
  - Equivalent integrated constraint
    - Blurring effect

$$\bar{g}(\boldsymbol{u}, x) = \left[\frac{1}{t_f} \int_0^{t_f} \langle -g(\boldsymbol{u}, x, t) \rangle^2 dt \right]^{1/2}$$

$$\bar{g}(\boldsymbol{u}, x) = -\frac{1}{\rho} \ln\left[\sum_{i=1}^{n_t} e^{-\rho g_i} dt \right] \rightarrow \bar{g} = g_{\min} - \frac{1}{\rho} \ln\left[\sum_{i=1}^{n_t} e^{-\rho(g_i - g_{\min})} dt \right]$$

  - Critical point constraint

$$g(\boldsymbol{u}, x, t_{mi}) \geq 0, \quad i = 1, 2, \cdots \rightarrow \frac{dg(t_{mi})}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial \boldsymbol{u}} \frac{d\boldsymbol{u}}{dx} + \frac{\partial g}{\partial t} \frac{dt_{mi}}{dx}$$

Sensitivity of Constraints

$$\bar{g}(\boldsymbol{u},x)=\int_0^{t_f} p(\boldsymbol{u},x,t)dt \geq 0 \quad \text{where} \quad p(\boldsymbol{u},x,t)=g(\boldsymbol{u},x,t)\delta(t-t_{mi})$$

$$\rightarrow \frac{d\bar{g}}{dx}=\int_0^{t_f}\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial \boldsymbol{u}}\boxed{\frac{d\boldsymbol{u}}{dx}}\right)dt \qquad \boxed{\boldsymbol{A}\dot{\boldsymbol{u}}=f(\boldsymbol{u},x,t),\quad \boldsymbol{u}(0)=\boldsymbol{u}_0}$$

- Direct method

$$\boldsymbol{A}\frac{d\dot{\boldsymbol{u}}}{dx}=\boldsymbol{J}\frac{d\boldsymbol{u}}{dx}-\frac{d\boldsymbol{A}}{dx}\dot{\boldsymbol{u}}+\frac{\partial f}{\partial x},\quad \frac{d\boldsymbol{u}(0)}{dx}=0 \quad \left(J_{ij}=\frac{\partial f_i}{\partial u_j}\right)$$

- Adjoint method

$$\frac{d\bar{g}}{dx}=\int_0^{t_f}\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial \boldsymbol{u}}\frac{d\boldsymbol{u}}{dx}\right)dt+\int_0^{t_f}\boldsymbol{\lambda}^T\left(\boldsymbol{A}\frac{d\dot{\boldsymbol{u}}}{dx}-\boldsymbol{J}\frac{d\boldsymbol{u}}{dx}-\frac{\partial f}{\partial x}+\frac{d\boldsymbol{A}}{dx}\dot{\boldsymbol{u}}\right)dt$$

$$=\int_0^{t_f}\left\{\frac{\partial p}{\partial x}-\boldsymbol{\lambda}^T\left(\frac{\partial f}{\partial x}-\frac{d\boldsymbol{A}}{dx}\dot{\boldsymbol{u}}\right)+\left[\frac{\partial p}{\partial \boldsymbol{u}}-\boldsymbol{\lambda}^T\left(\dot{\boldsymbol{A}}+\boldsymbol{J}\right)-\left(\dot{\boldsymbol{\lambda}}\right)^T\boldsymbol{A}\right]\frac{d\boldsymbol{u}}{dx}\right\}dt+\boldsymbol{\lambda}^T\boldsymbol{A}\frac{d\boldsymbol{u}}{dx}\bigg|_0^{t_f}$$

$$\rightarrow \frac{d\bar{g}}{dx}=\int_0^{t_f}\left[\frac{\partial p}{\partial x}-\boldsymbol{\lambda}^T\left(\frac{\partial f}{\partial x}-\frac{d\boldsymbol{A}}{dx}\dot{\boldsymbol{u}}\right)\right]dt$$

$$\frac{\partial p}{\partial \boldsymbol{u}}-\boldsymbol{\lambda}^T\left(\dot{\boldsymbol{A}}+\boldsymbol{J}\right)-\left(\dot{\boldsymbol{\lambda}}\right)^T\boldsymbol{A}=0\rightarrow \boldsymbol{A}^T\dot{\boldsymbol{\lambda}}+\left(\boldsymbol{J}^T+\dot{\boldsymbol{A}}^T\right)\boldsymbol{\lambda}=\left(\frac{\partial p}{\partial \boldsymbol{u}}\right)^T,\quad \boldsymbol{\lambda}(t_f)=0$$