Problem 1

$$\begin{bmatrix} 1+p_1 & -p_2 \\ -p_2 & 2-p_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 3-p_1 \\ 1+p_1+4p_2 \end{Bmatrix}$$

(1) $u'@p = \{1,1\}^T$?

(2) $\varphi = (u_1)^2 + u_2 + 6p_1(u_1 + u_2) \rightarrow \varphi'$ using both direct and adjoint methods

Problem 1: Direct Method

$$\begin{bmatrix} 1+p_1 & -p_2 \\ -p_2 & 2-p_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 3-p_1 \\ 1+p_1+4p_2 \end{Bmatrix} \rightarrow u' @ p = \{1,1\}^T \text{ ?}$$

$$Au = f \xrightarrow{(Au)'=A'u+Au'} Au' = f' - A'u \rightarrow Au^{(k)\prime} = f^{(k)\prime} - A^{(k)\prime}u$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 14 \end{Bmatrix}$$

$$A^{(1)\prime} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f^{(1)\prime} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} u^{(1)\prime} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} 8 \\ 14 \end{Bmatrix} = \begin{Bmatrix} -9 \\ 15 \end{Bmatrix} \rightarrow u^{(1)\prime} = \begin{Bmatrix} 6 \\ 21 \end{Bmatrix}$$

$$A^{(2)\prime} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, f^{(1)\prime} = \begin{Bmatrix} 0 \\ 4 \end{Bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} u^{(2)\prime} = \begin{Bmatrix} 0 \\ 4 \end{Bmatrix} - \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} 8 \\ 14 \end{Bmatrix} = \begin{Bmatrix} 14 \\ 12 \end{Bmatrix} \rightarrow u^{(2)\prime} = \begin{Bmatrix} 26 \\ 38 \end{Bmatrix}$$

Problem 1: Adjoint Method

$$\varphi = (u_1)^2 + u_2 + 6p_1(u_1 + u_2) \rightarrow \varphi' ?$$

$$[1]\varphi\big(p, u(p)\big) \rightarrow \varphi' = \nabla_p \varphi + (\nabla_u \varphi) u' \rightarrow \varphi^{(k)\prime} = \nabla_{p_k} \varphi + (\nabla_u \varphi) u^{(k)\prime}$$

$$\varphi^{(1)\prime} = 6(u_1 + u_2) + \begin{Bmatrix} 2u_1 + 6p_1 \\ 1 + 6p_1 \end{Bmatrix}^T \underbrace{u^{(1)\prime}} = 6(8 + 14) + \begin{Bmatrix} 22 \\ 7 \end{Bmatrix}^T \begin{Bmatrix} 6 \\ 21 \end{Bmatrix} = 411$$

$$\varphi^{(2)\prime} = 0 + \begin{Bmatrix} 2u_1 + 6p_1 \\ 1 + 6p_1 \end{Bmatrix}^T \underbrace{u^{(2)\prime}} = 0 + \begin{Bmatrix} 22 \\ 7 \end{Bmatrix}^T \begin{Bmatrix} 26 \\ 38 \end{Bmatrix} = 838$$

$$[2] A^T \lambda = (\nabla_u \varphi)^T \rightarrow \varphi^{(k)\prime} = \nabla_{p_k} \varphi + \lambda^T \left(f^{(k)\prime} - A^{(k)\prime} u \right)$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \lambda = \begin{Bmatrix} 22 \\ 7 \end{Bmatrix} \rightarrow \lambda = \begin{Bmatrix} 29 \\ 36 \end{Bmatrix}$$

$$\varphi^{(1)\prime} = 6(8 + 14) + \begin{Bmatrix} 29 \\ 36 \end{Bmatrix}^T \left(\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} 8 \\ 14 \end{Bmatrix} \right) = 411$$

$$\varphi^{(2)\prime} = 0 + \begin{Bmatrix} 29 \\ 36 \end{Bmatrix}^T \left(\begin{Bmatrix} 0 \\ 4 \end{Bmatrix} - \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} 8 \\ 14 \end{Bmatrix} \right) = 838$$

Problem 2

– Consider the function $g = U_3$ where $\boldsymbol{U}$ is determined from $\boldsymbol{K}(\boldsymbol{x})\boldsymbol{U} = \boldsymbol{F}$ as:

$$\begin{bmatrix} 5x_1 & -5x_1 & 0 \\ -5x_1 & 5x_1 + 10x_2 + 5x_3 & -5x_3 \\ 0 & -5x_3 & 5x_3 + 5x_4 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 15 \end{Bmatrix}$$

– Given $\boldsymbol{x}^0 = [1.0, 1.0, 1.0, 1.0]^T$, determine the gradient $dg/d\boldsymbol{x}$ using (1) the direct method, and (2) the adjoint method.

Problem 2: solution

$$\boldsymbol{K}(\boldsymbol{x}^0)\boldsymbol{U}^0 = \boldsymbol{F}^0 \rightarrow \begin{bmatrix} 5 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 10 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 15 \end{Bmatrix} \rightarrow \boldsymbol{U}^0 = \begin{Bmatrix} 6.2 \\ 2.2 \\ 2.6 \end{Bmatrix}$$

direct method

$$\boldsymbol{K}(\boldsymbol{x}^0)\frac{d\boldsymbol{U}}{dx_i} + \frac{d\boldsymbol{K}}{dx_i}\boldsymbol{U}^0 = \frac{d\boldsymbol{F}}{dx_i} \rightarrow \boldsymbol{K}(\boldsymbol{x}^0)\frac{d\boldsymbol{U}}{dx_i} = -\frac{d\boldsymbol{K}}{dx_i}\boldsymbol{U}^0$$

$$\frac{\partial \boldsymbol{K}}{\partial x_1} = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{\partial \boldsymbol{K}}{\partial x_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{\partial \boldsymbol{K}}{\partial x_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix}, \frac{\partial \boldsymbol{K}}{\partial x_4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\frac{\partial(\boldsymbol{K}\boldsymbol{U}^0)}{\partial x_1} = \begin{Bmatrix} 20 \\ -20 \\ 0 \end{Bmatrix}, \frac{\partial(\boldsymbol{K}\boldsymbol{U}^0)}{\partial x_2} = \begin{Bmatrix} 0 \\ -22 \\ 0 \end{Bmatrix}, \frac{\partial(\boldsymbol{K}\boldsymbol{U}^0)}{\partial x_3} = \begin{Bmatrix} -2 \\ 2 \\ 0 \end{Bmatrix}, \frac{\partial(\boldsymbol{K}\boldsymbol{U}^0)}{\partial x_4} = \begin{Bmatrix} 0 \\ 0 \\ 13 \end{Bmatrix}$$

$$\frac{d\boldsymbol{U}}{dx_1} = \begin{Bmatrix} -4 \\ 0 \\ 0 \end{Bmatrix}, \frac{d\boldsymbol{U}}{dx_2} = \begin{Bmatrix} -1.76 \\ -1.76 \\ -0.88 \end{Bmatrix}, \frac{d\boldsymbol{U}}{dx_3} = \begin{Bmatrix} 0.08 \\ 0.08 \\ -0.16 \end{Bmatrix}, \frac{d\boldsymbol{U}}{dx_4} = \begin{Bmatrix} -0.52 \\ -0.52 \\ -1.56 \end{Bmatrix}$$

$$\nabla f = (0, -0.88, -0.16, -1.56)^T$$

adjoint method

$$\boldsymbol{K}(\boldsymbol{x}^0)\boldsymbol{\lambda} = z = \frac{df}{d\boldsymbol{U}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \rightarrow \boldsymbol{\lambda} = \begin{Bmatrix} 0.04 \\ 0.04 \\ 0.12 \end{Bmatrix}$$

$$\frac{df}{dx_i} = \frac{\partial f}{\partial x_i} + \frac{df}{d\boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{F}}{dx_i} - \frac{d\boldsymbol{K}}{dx_i}\boldsymbol{U}^0\right) = -\boldsymbol{\lambda}^T\frac{d\boldsymbol{K}}{dx_i}\boldsymbol{U}^0$$

$$\nabla f = (0, -0.88, -0.16, -1.56)^T$$

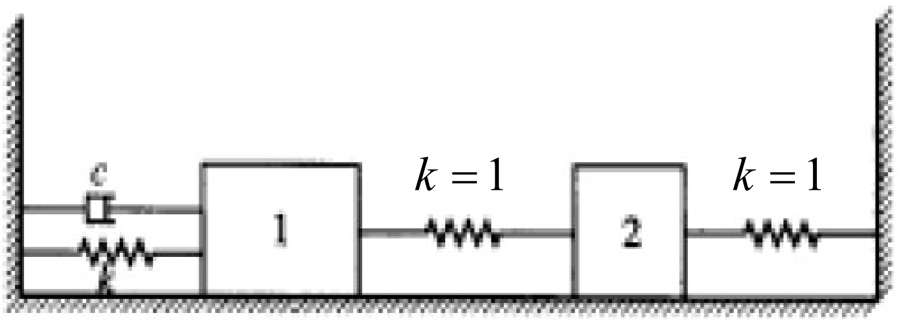
Problem 3

$c = 0, m_1 = m_2 = 1, k = 1 \rightarrow \frac{d\mu}{dx} = ?, \frac{d\mathbf{u}}{dx} = ?$

using (1) direct method and (2) Nelson's method

normalization $\rightarrow \begin{cases} \mathbf{u}^T\mathbf{W}\mathbf{u} = 1 \\ \bar{u}_m = 1 \end{cases}$

$\mathbf{K} = \begin{bmatrix} 1+k & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Problem 3: solution

$$\left.\begin{aligned}-ku_1+(u_2-u_1)=m_1\ddot{u}_1\\-(u_2-u_1)-u_2=m_2\ddot{u}_2\end{aligned}\right\}\to\left\{\begin{aligned}&m_1\ddot{u}_1+(1+k)u_1-u_2=0\\&m_2\ddot{u}_2-u_1+2u_2=0\end{aligned}\right\}\to\begin{bmatrix}m_1&0\\0&m_2\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\\ddot{u}_2\end{Bmatrix}+\begin{bmatrix}1+k&-1\\-1&2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\begin{cases}\text{kinetic energy: } T=\dfrac{1}{2}m_1\dot{u}_1{}^2+\dfrac{1}{2}m_2\dot{u}_2{}^2\\[2ex]\text{elastic energy: } E=\dfrac{1}{2}ku_1{}^2+\dfrac{1}{2}(u_2-u_1)^2+\dfrac{1}{2}u_2{}^2\end{cases}$$

$$\boldsymbol{K}=\begin{bmatrix}1+k&-1\\-1&2\end{bmatrix},\ \boldsymbol{M}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\to\boldsymbol{K}'=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \boldsymbol{M}'=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$k=1:\boldsymbol{Ku}-\mu\boldsymbol{Mu}=0\to\begin{bmatrix}2-\mu&-1\\-1&2-\mu\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\to\mu_{1,2}=1,3$$

Problem 3: Direct method

$$\mu = 1: \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow u_1 = u_2 \rightarrow \boldsymbol{u}^T \boldsymbol{M}\boldsymbol{u} = 1 \rightarrow \boldsymbol{u} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{bmatrix} \boldsymbol{K} - \mu \boldsymbol{M} & -\boldsymbol{M}\boldsymbol{u} \\ \boldsymbol{u}^T \boldsymbol{W} & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{d\boldsymbol{u}}{dx} \\ \dfrac{d\mu}{dx} \end{Bmatrix} = \begin{Bmatrix} -\left(\dfrac{d\boldsymbol{K}}{dx} - \mu \dfrac{d\boldsymbol{M}}{dx} \right) \boldsymbol{u} \\ -\dfrac{1}{2} \boldsymbol{u}^T \dfrac{d\boldsymbol{W}}{dx} \boldsymbol{u} \end{Bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -\dfrac{1}{\sqrt{2}} \\ -1 & 1 & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{du_1}{dx} \\ \dfrac{du_2}{dx} \\ \dfrac{d\mu}{dx} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{\sqrt{2}} \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} \dfrac{du_1}{dx} \\ \dfrac{du_2}{dx} \\ \dfrac{d\mu}{dx} \end{Bmatrix} = \begin{Bmatrix} -0.1768 \\ 0.1768 \\ 0.5000 \end{Bmatrix}$$

Problem 3: Nelson's method

$$\boldsymbol{u} = \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\1\end{Bmatrix} = u_m\bar{\boldsymbol{u}} \rightarrow u_2 = \frac{1}{\sqrt{2}}, \bar{\boldsymbol{u}} = \begin{Bmatrix}1\\1\end{Bmatrix} \rightarrow \bar{u}_2 = 1, \frac{d\bar{u}_2}{dx} = 0$$

$$\left(\boldsymbol{K} - \mu\boldsymbol{M}\right)\boldsymbol{u} = 0 \rightarrow \left(\boldsymbol{K} - \mu\boldsymbol{M}\right)\frac{d\bar{\boldsymbol{u}}}{dx} - \frac{d\mu}{dx}\boldsymbol{M}\bar{\boldsymbol{u}} = -\left(\frac{d\boldsymbol{K}}{dx} - \mu\frac{d\boldsymbol{M}}{dx}\right)\bar{\boldsymbol{u}}$$

$$\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}\frac{d\bar{u}_1}{dx}\\\frac{d\bar{u}_2}{dx}\end{Bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix} = -\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix} \rightarrow \begin{Bmatrix}\frac{d\bar{u}_1}{dx} - \frac{d\bar{u}_2}{dx} = -0.5\\-\frac{d\bar{u}_1}{dx} + \frac{d\bar{u}_2}{dx} = 0.5\\\frac{d\bar{u}_2}{dx} = 0\end{Bmatrix} \rightarrow \begin{Bmatrix}\frac{d\bar{u}_1}{dx}\\\frac{d\bar{u}_2}{dx}\end{Bmatrix} = \begin{Bmatrix}-0.5\\0\end{Bmatrix}$$

$$\frac{du_2}{dx} = -u_2{}^2\boldsymbol{u}^T\boldsymbol{M}\frac{d\bar{\boldsymbol{u}}}{dx} - \frac{u_2}{2}\boldsymbol{u}^T\frac{d\boldsymbol{M}}{dx}\boldsymbol{u} = -\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}-0.5\\0\end{Bmatrix} = \frac{\sqrt{2}}{8}$$

$$\frac{d\boldsymbol{u}}{dx} = \frac{du_2}{dx}\bar{\boldsymbol{u}} + u_2\frac{d\bar{\boldsymbol{u}}}{dx} = \frac{\sqrt{2}}{8}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{1}{\sqrt{2}}\begin{Bmatrix}-0.5\\0\end{Bmatrix} = \frac{\sqrt{2}}{8}\begin{Bmatrix}-1\\1\end{Bmatrix}$$

$$\mu' = \left(\omega^2\right)' = \boldsymbol{u}^T\boldsymbol{K}'\boldsymbol{u} = 0.5, \bar{\boldsymbol{u}}' = \begin{Bmatrix}-0.5\\0\end{Bmatrix} \rightarrow \boldsymbol{u}' = \frac{\sqrt{2}}{8}\begin{Bmatrix}-1\\1\end{Bmatrix}$$