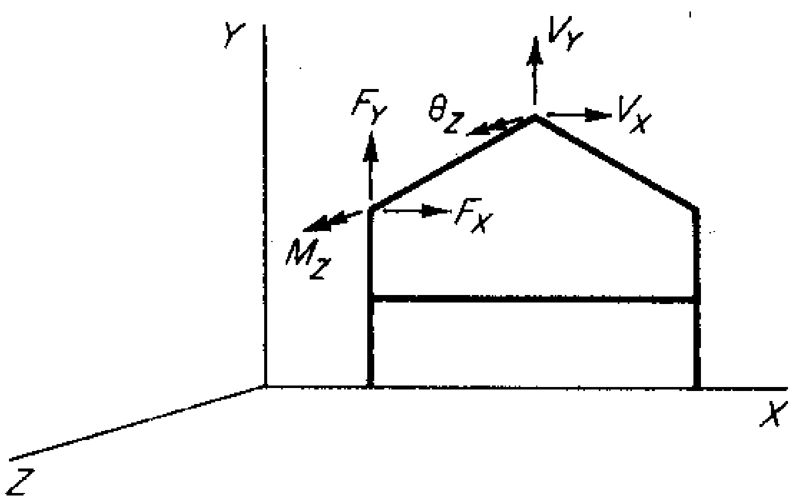
Transformations

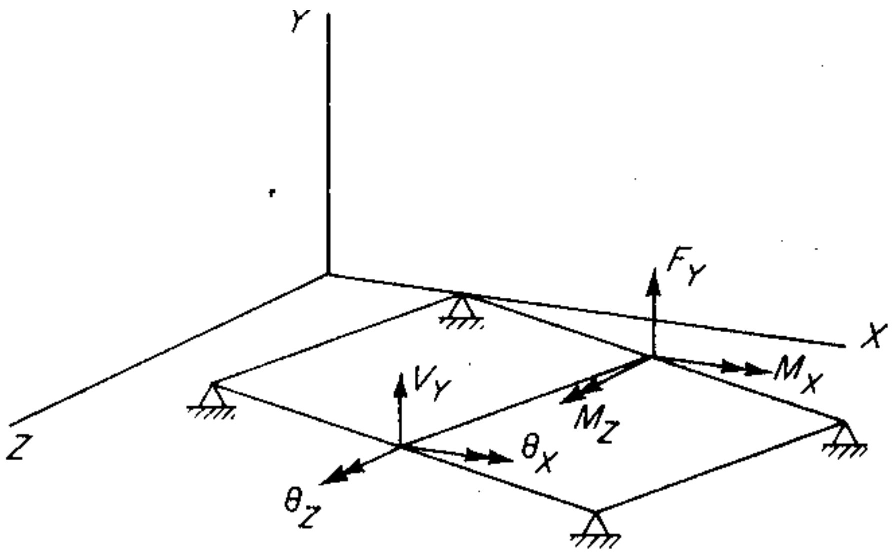
- Force and displacement
  - Components for rotation of coordinate axes in 2D
  - Vectors for coordinate axes in 3D
  - Components from one set of coordinate axes to a parallel set
- Inertia matrix
  - Cartesian tensor transformations for rotation of axes
  - Parallel-axes theorem
- Stress tensor
  - 2D stress state and rotation of axes

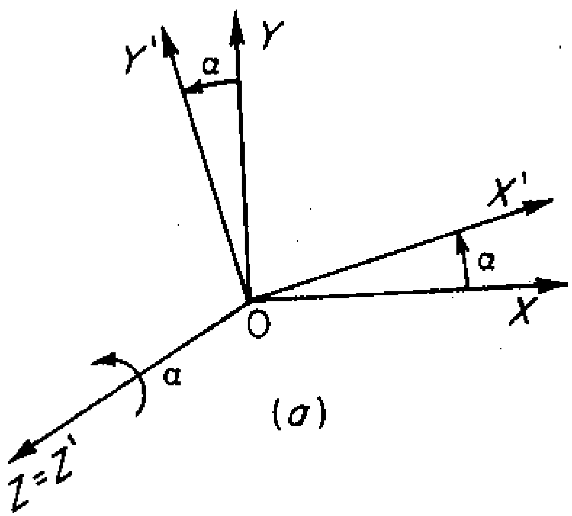
Planar Structures

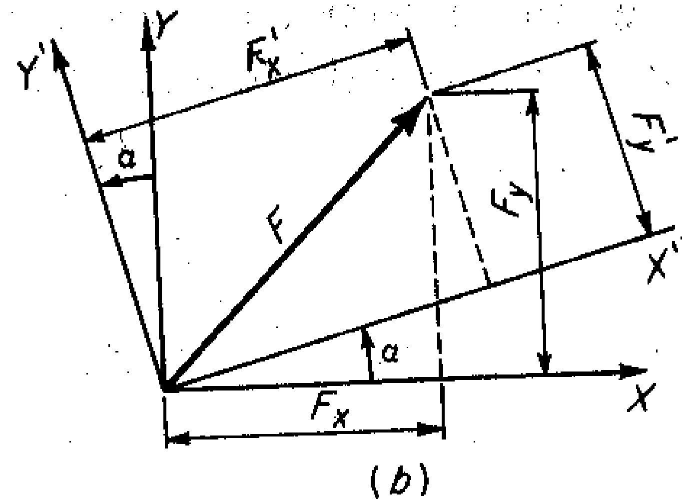
- Plane frame

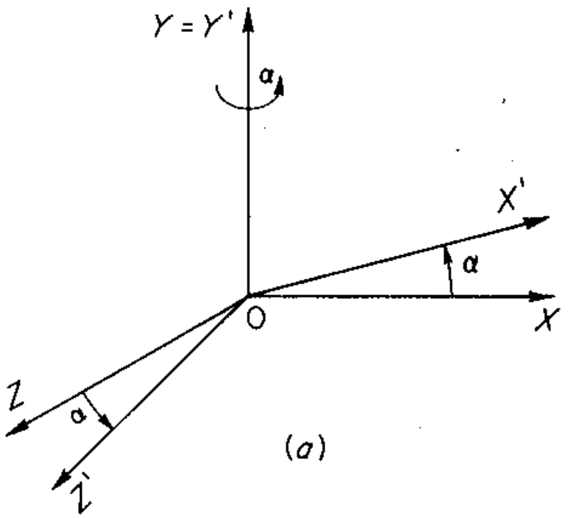


- Grid structure

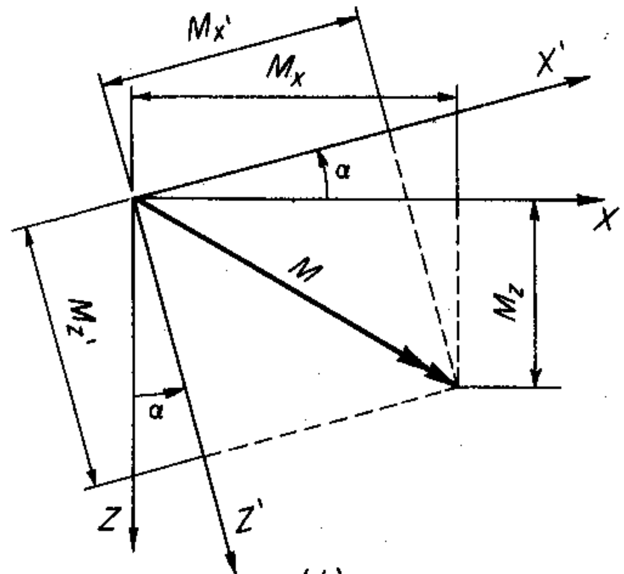



Y'
α
Y
X'
α
X
O
α
Z=Z'
(a)


Y'
Y
F'x
α
F
F'y
Fy
X'
α
X
Fx
(b)

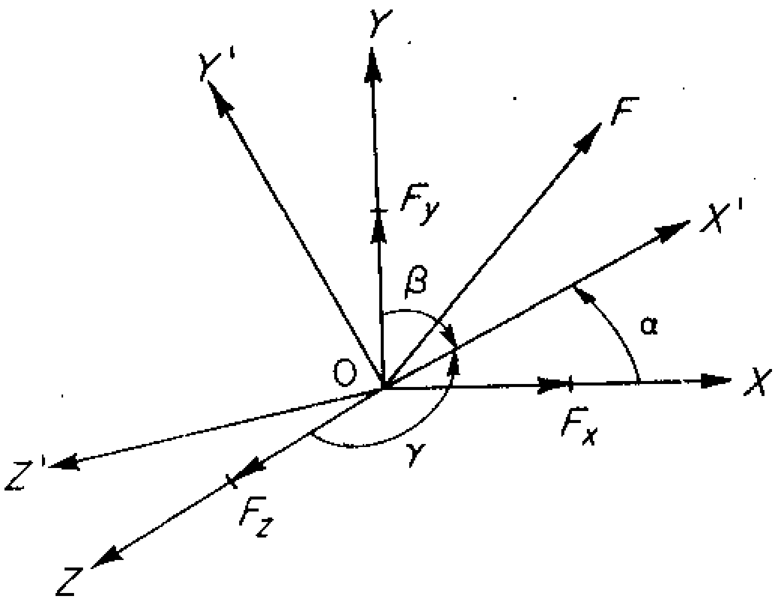


(a)

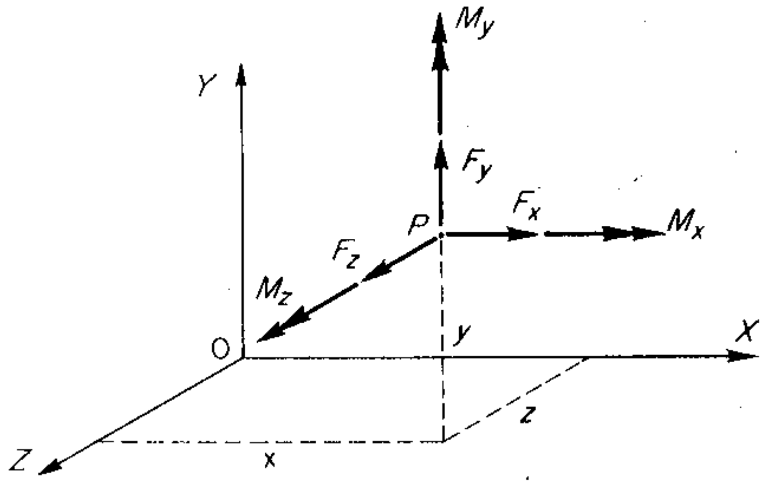


(b)

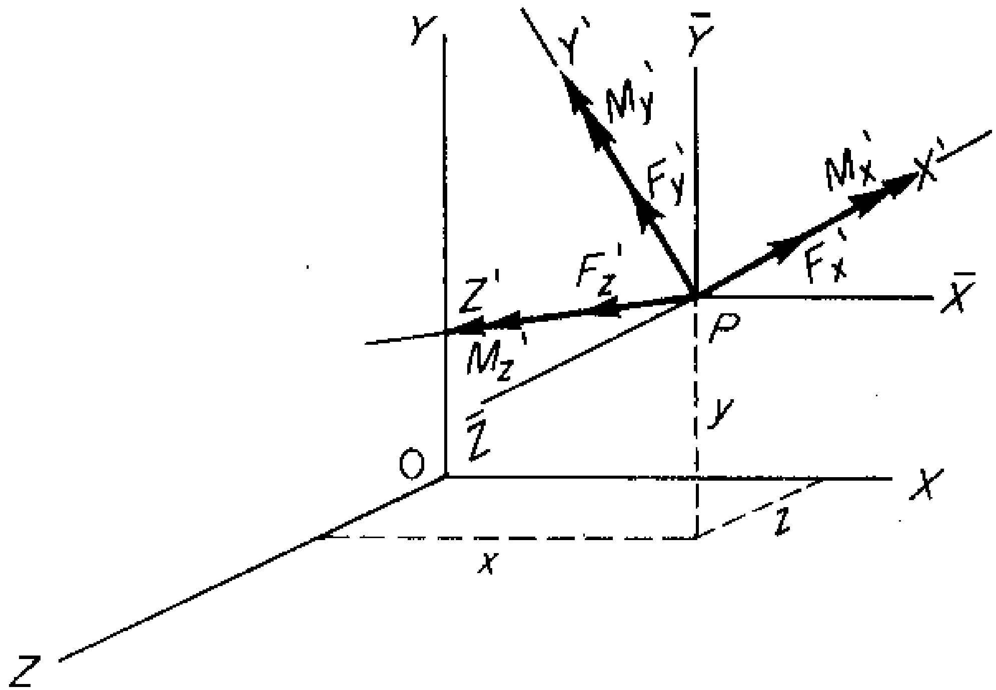
3D Rotation



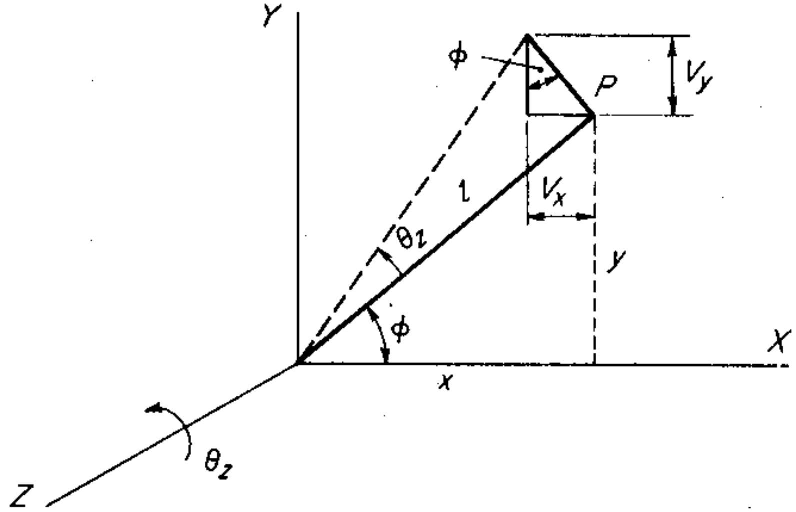
Force Translation

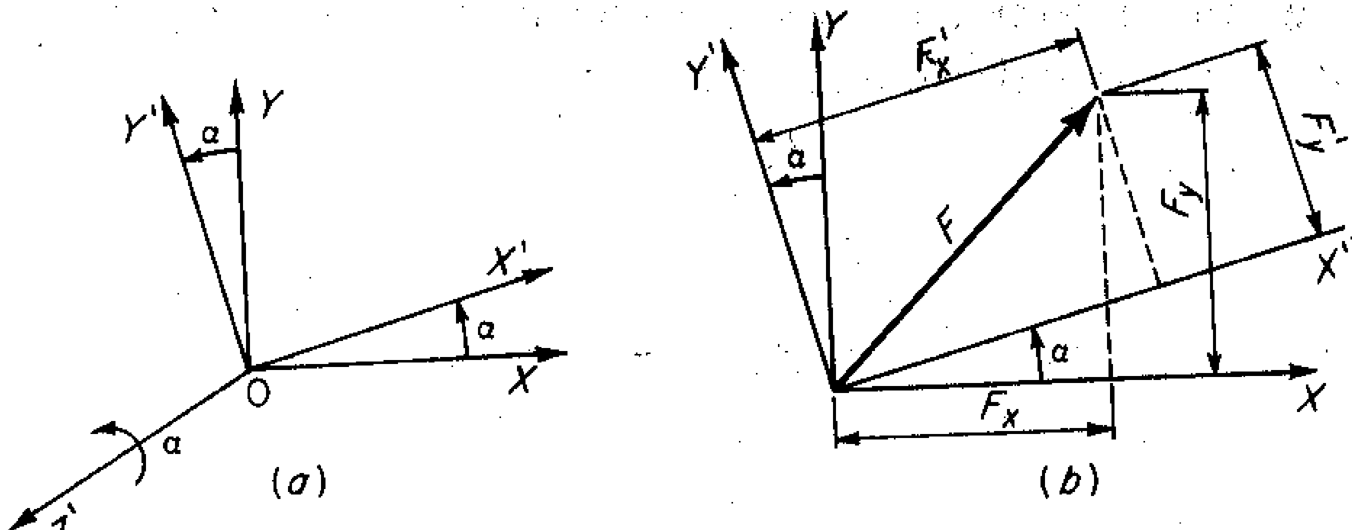


Force Transformation



Displacement



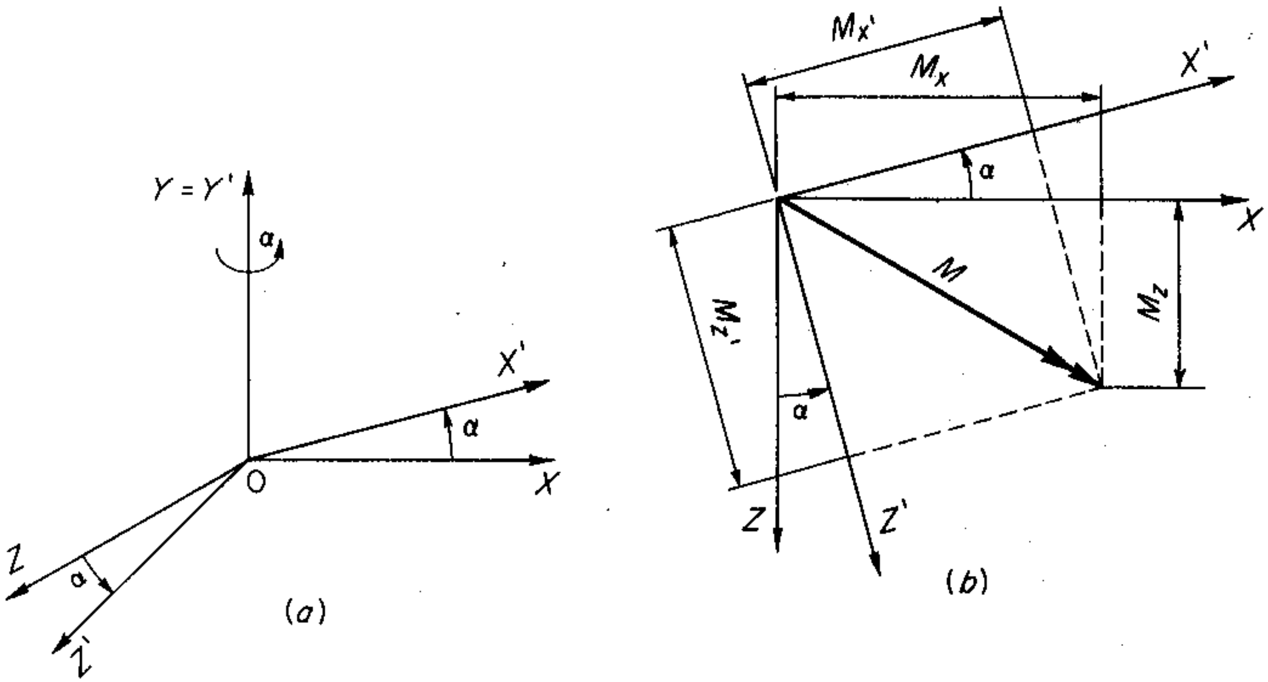


$$\left.\begin{aligned} F_x' &= F_x\cos\alpha + F_y\sin\alpha \\ F_y' &= -F_x\sin\alpha + F_y\cos\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} F_x' \\ F_y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\text{direction cosines of } \left\{\begin{aligned} OX' &: \langle l_{11} \quad l_{12} \rangle = \langle \cos\alpha \quad \sin\alpha \rangle \\ OY' &: \langle l_{21} \quad l_{22} \rangle = \langle -\sin\alpha \quad \cos\alpha \rangle \end{aligned}\right\} \text{with respect to the } X,\ Y \text{ axes}$$

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = L^T \rightarrow \begin{bmatrix} F_x' \\ F_y' \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \rightarrow F' = L^T F$$

$$\left.\begin{aligned} {l_{11}}^2 + {l_{12}}^2 &= 1 \\ {l_{21}}^2 + {l_{22}}^2 &= 1 \\ l_{11}l_{21} + l_{12}l_{22} &= 0 \end{aligned}\right\} \xrightarrow{\text{orthogonal}} L^T L = I \rightarrow L^T = L^{-1} \rightarrow F = LF'$$

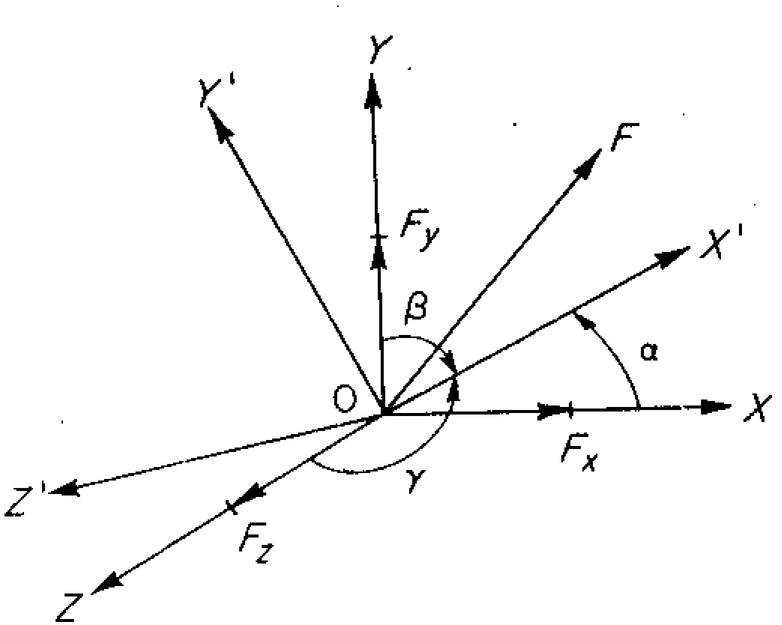


$$\left.\begin{aligned} M'_x &= M_x \cos\alpha - M_z \sin\alpha \\ M'_y &= M_x \sin\alpha + M_z \cos\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} M'_x \\ M'_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} M_x \\ M_z \end{bmatrix}$$

rotation about the OY axis $\rightarrow$ rotation about the OZ axis

$$M' = L^T M \rightarrow M = LM'$$

3D Rotation



$$l_{ij} : \begin{cases} i: \text{ axes } X',\ Y',\ Z' \\ j: \text{ axes } X,\ Y,\ Z \end{cases}$$

$$l_{11} = \cos\alpha,\quad l_{12} = \cos\beta,\quad l_{13} = \cos\gamma$$

$$\begin{bmatrix} F'_x \\ F'_y \\ F'_z \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \rightarrow F' = L^T F$$

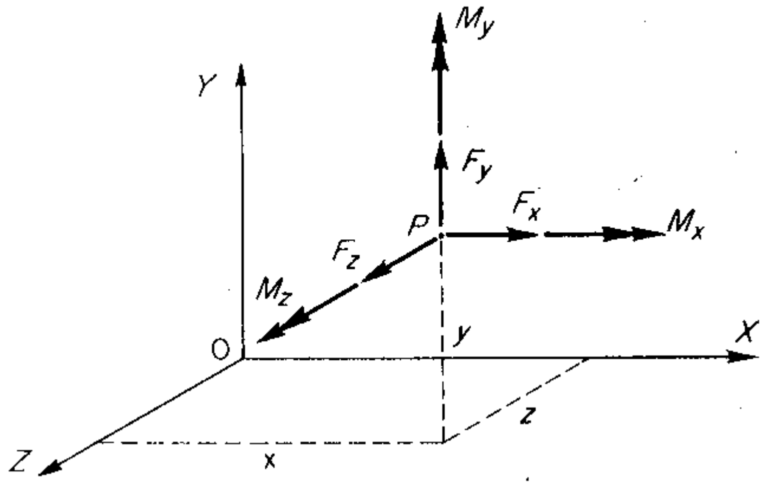
$$\left.\begin{aligned} &\sum_{j=1,3} l_{ij} l_{ij} = 1,\quad i = 1,2,3 \\ &\sum_{j=1,3} l_{ij} l_{kj} = 1,\quad \begin{cases} i = 1,2,3 \\ k = 1,2,3 \\ i \neq k \end{cases} \end{aligned}\right\} \rightarrow l_{ij} l_{kj} = \delta_{ik}$$

$$\rightarrow \begin{cases} i = k = 1 : l_{1j} l_{1j} = l_{11}{}^2 + l_{12}{}^2 + l_{13}{}^2 = \delta_{11} = 1 \\ i = 1, k = 2 : l_{1j} l_{2j} = l_{11} l_{21} + l_{12} l_{22} + l_{13} l_{23} = \delta_{12} = 0 \end{cases}$$

$$R' = \begin{bmatrix} F' \\ M' \end{bmatrix} = \begin{bmatrix} L^T & 0 \\ 0 & L^T \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix} = L_D^T R$$

$$r' = \begin{bmatrix} v' \\ \theta' \end{bmatrix} = \begin{bmatrix} L^T & 0 \\ 0 & L^T \end{bmatrix} \begin{bmatrix} v \\ \theta \end{bmatrix} = L_D^T r$$

Force Translation

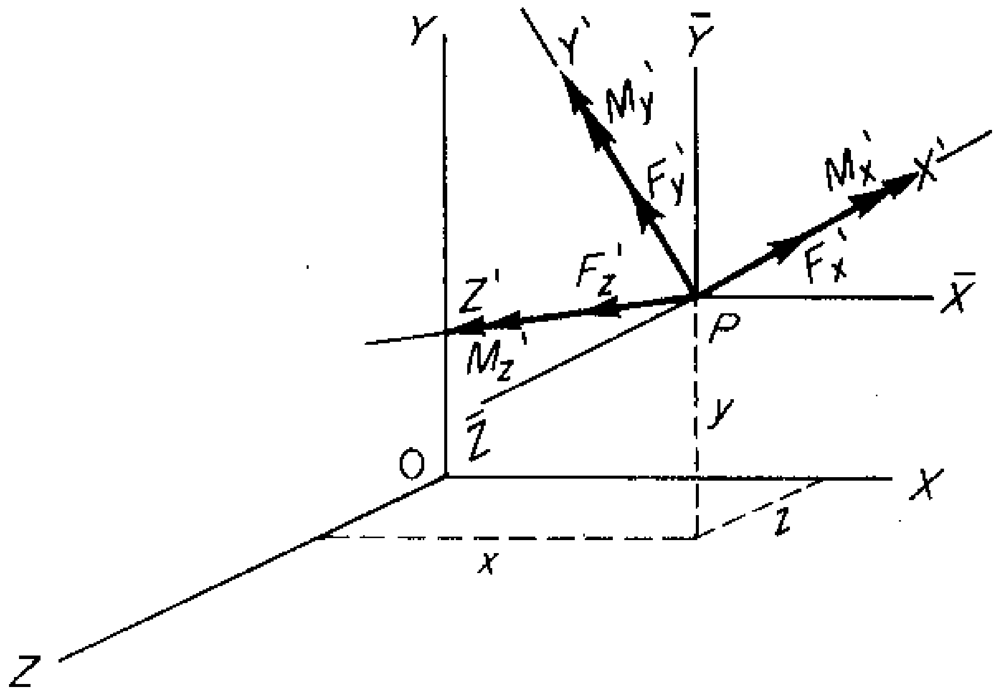


$$
\left.\begin{aligned}
F_{xO} &= F_{xP} \\
F_{yO} &= F_{yP} \\
F_{zO} &= F_{zP} \\
M_{xO} &= M_{xP} - F_y z + F_z y \\
M_{yO} &= M_{yP} + F_x z - F_z x \\
M_{zO} &= M_{zP} - F_x y + F_y x
\end{aligned}\right\} \rightarrow
\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}_O =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -z & y & 1 & 0 & 0 \\
z & 0 & -x & 0 & 1 & 0 \\
-y & x & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}_P
$$

$$R_O = TR_P \rightarrow R_P = T^{-1} R_O$$

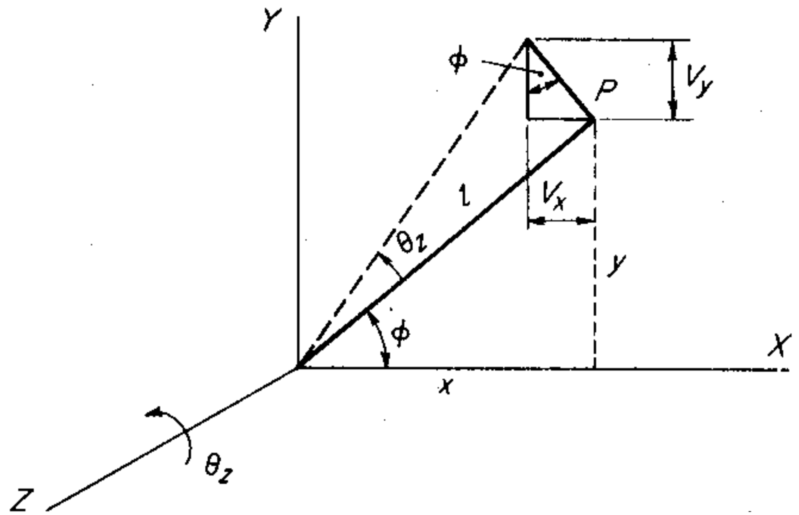
$$T = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \rightarrow TT^{-1} = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} I & 0 \\ W^T & I \end{bmatrix}$$

Force Transformation



$$(X',Y',Z') \xrightarrow[\substack{R_P'=L_D^T R_P \\ R_P=\left(L_D^T\right)^{-1} R_P'=L_D R_P'}]{\text{rotation}} (\bar{X},\bar{Y},\bar{Z}) \xrightarrow[R_O=TR_P=TL_D R_P'=bR_P']{\text{translation}} (X,Y,Z)$$

Displacement



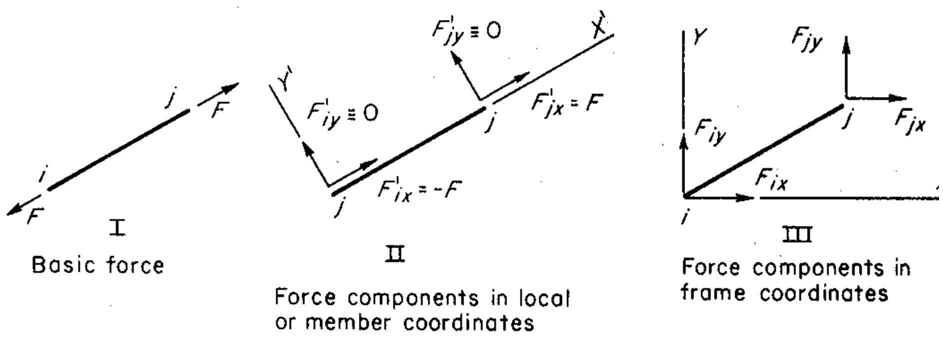
$$\left.\begin{aligned} v_{xP} &= v_{xO} + \theta_y z - \theta_z y \\ v_{yP} &= v_{yO} - \theta_x z + \theta_z x \\ v_{zP} &= v_{zO} + \theta_x y - \theta_y x \\ \theta_{xP} &= \theta_{xO} \\ \theta_{yP} &= \theta_{yO} \\ \theta_{zP} &= \theta_{zO} \end{aligned}\right\} \rightarrow \begin{bmatrix} v_x \\ v_y \\ v_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}_P = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}_O$$

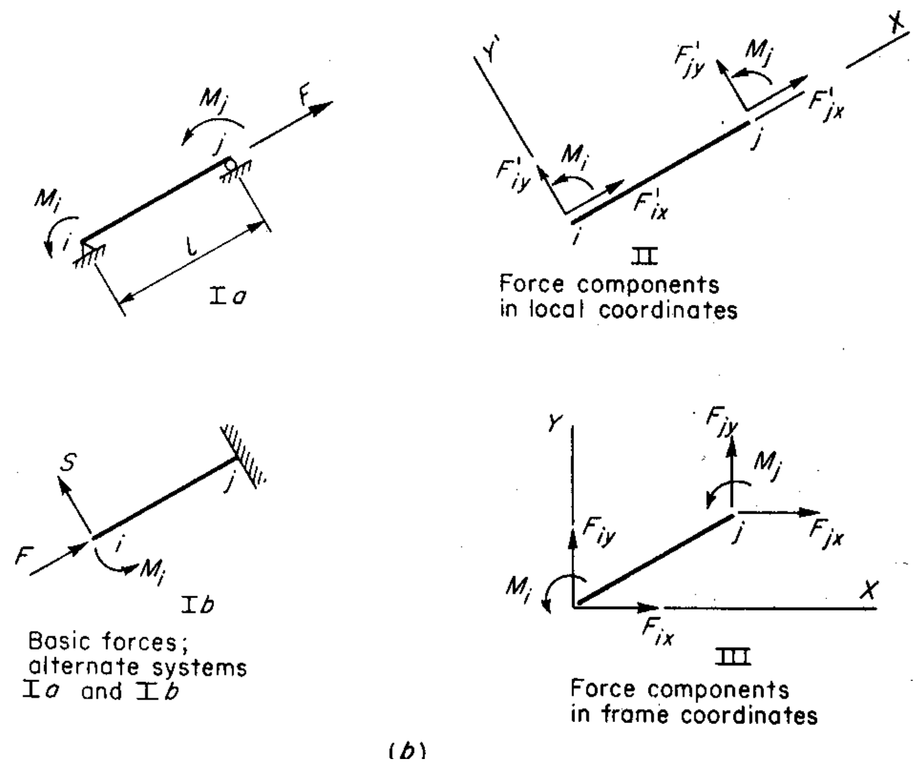
$$r_P = T^T r_O \xrightarrow{r'_P = L_D^T r_P} r'_P = \underline{L_D^T T^T \, r_O} = a r_O \rightarrow r_O = \left(L_D^T T^T\right)^{-1} r'_P = \left(T^T\right)^{-1} L_D r'_P$$

$$a = b^T, \left(T^T\right)^{-1} = \left(T^{-1}\right)^T, \left(T^T\right)^{-1} = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix}$$

Force Systems

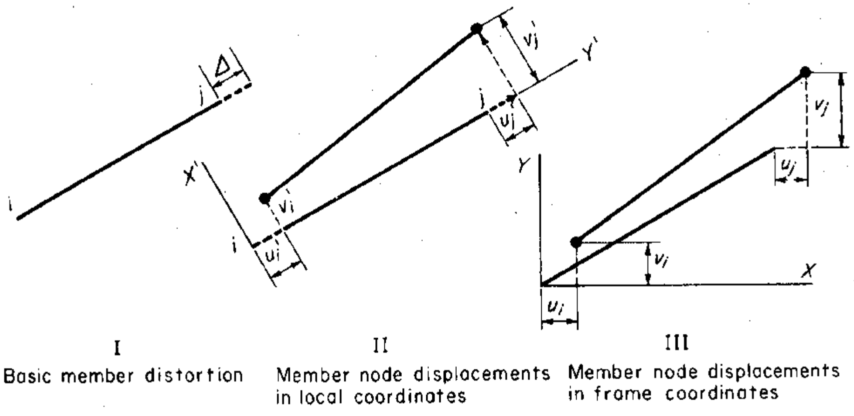
- Pin-jointed truss member
- Frame or beam member

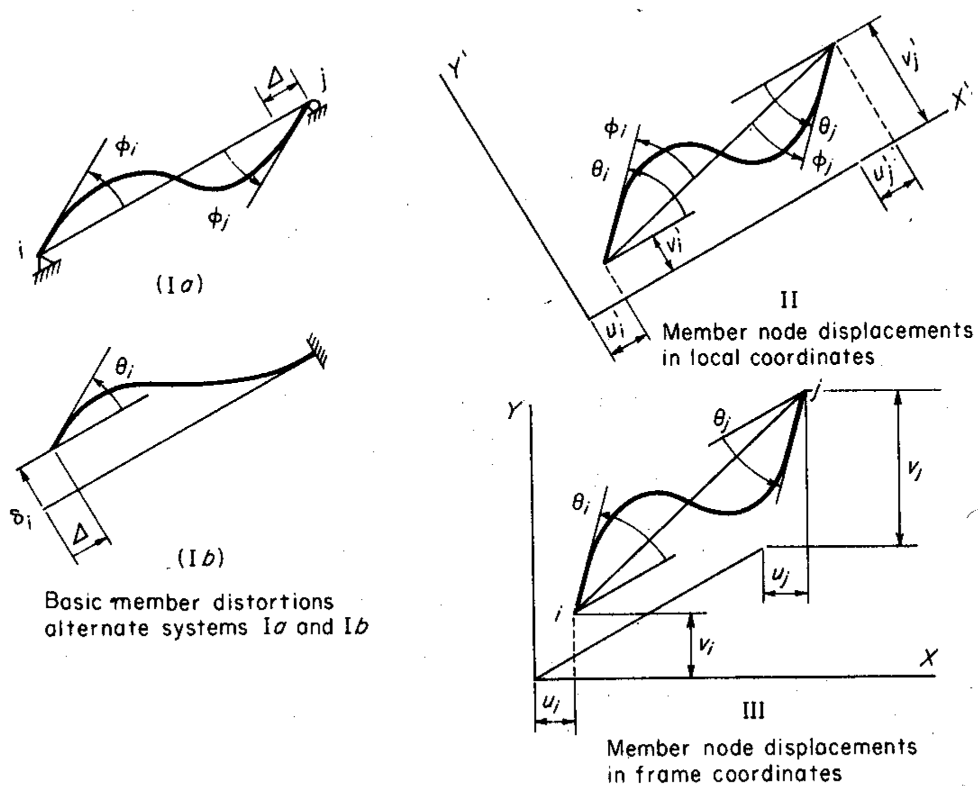




Member Distortions & Joint Displacements

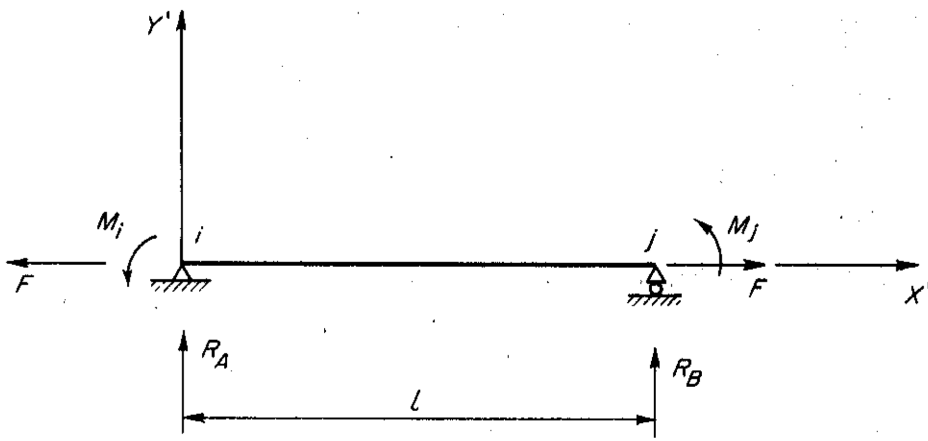
- Pin-jointed truss member
- Frame or beam member

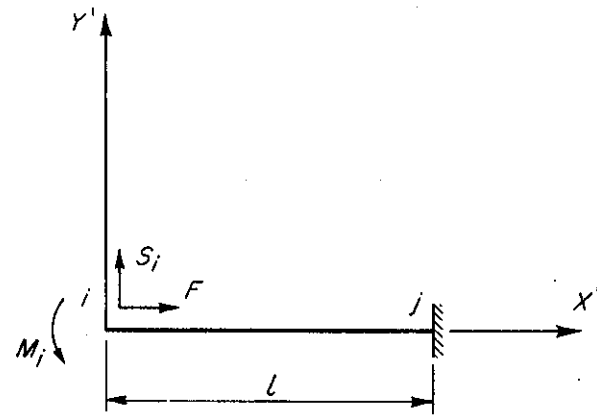




Force Transformations

- Pin-jointed truss member
- Frame member
  - Type Ia
  - Type Ib
  - Type Ia: intermediate hinge





$$\begin{bmatrix} F'_{ix} \\ F'_{iy} \\ F'_{jx} \\ F'_{jy} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} F \to \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix} = L_D \begin{bmatrix} F'_{ix} \\ F'_{iy} \\ F'_{jx} \\ F'_{jy} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} F = \begin{bmatrix} -\cos\alpha \\ -\sin\alpha \\ \cos\alpha \\ \sin\alpha \end{bmatrix} F$$

$$\left.\begin{aligned} R_A &= \frac{M_i + M_j}{l} \\ R_B &= -\frac{M_i + M_j}{l} \end{aligned}\right\} \to \begin{bmatrix} F'_{ix} \\ F'_{iy} \\ M'_i \\ F'_{jx} \\ F'_{jy} \\ M'_j \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1/l & 1/l & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1/l & -1/l & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_i \\ M_j \\ F \end{bmatrix} \to F' = TS$$

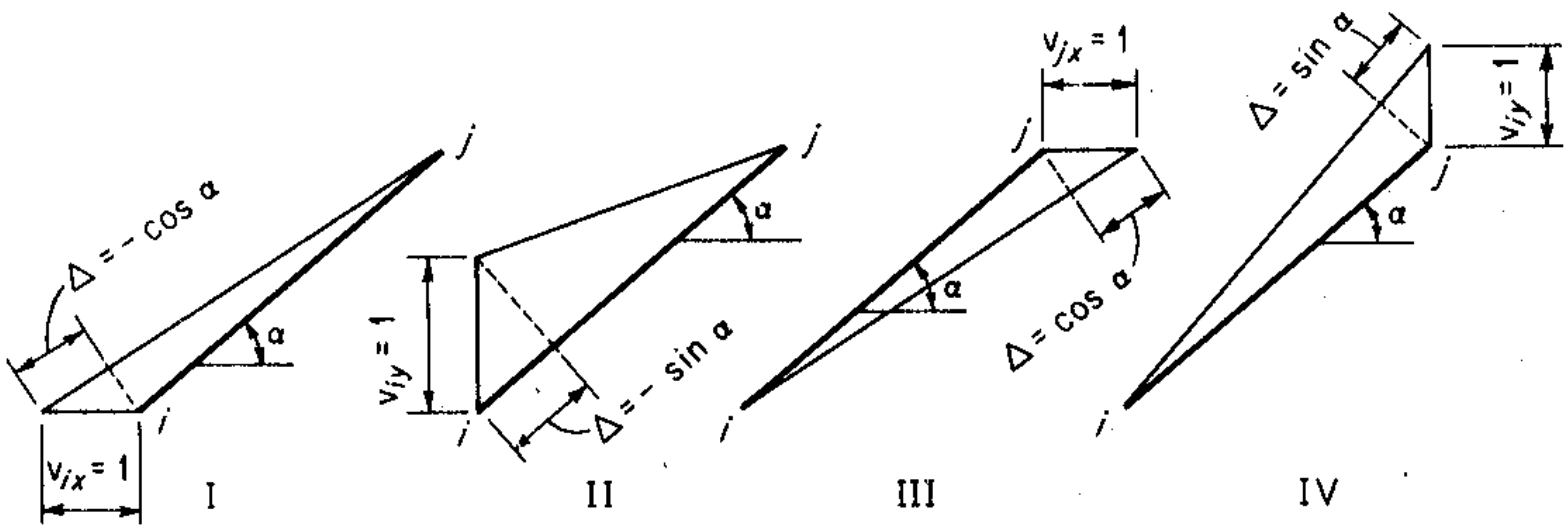
$$F = L_D F' = L_D TS \quad \text{where} \quad L_D = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & 0 & 0 & 0 & 0 \\ l_{12} & l_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{11} & l_{21} & 0 \\ 0 & 0 & 0 & l_{12} & l_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} S_j &= -S_i \\ M_j &= -M_i + S_i l \end{aligned}\right\} \rightarrow \begin{bmatrix} F'_{ix} \\ F'_{iy} \\ M'_i \\ F'_{jx} \\ F'_{jy} \\ M'_j \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & l & 0 \end{bmatrix} \begin{bmatrix} M_i \\ S_i \\ F \end{bmatrix} \rightarrow F' = T_1 S \rightarrow F = L_D F' = L_D T_1 S$$

$$\left. M_j = \frac{B}{A} M_i \right\} \rightarrow \begin{bmatrix} F'_{ix} \\ F'_{iy} \\ M'_i \\ F'_{jx} \\ F'_{jy} \\ M'_j \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1/A & 0 \\ 1 & 0 \\ 0 & 1 \\ -1/A & 0 \\ B/A & 0 \end{bmatrix} \begin{bmatrix} M_i \\ F \end{bmatrix}$$

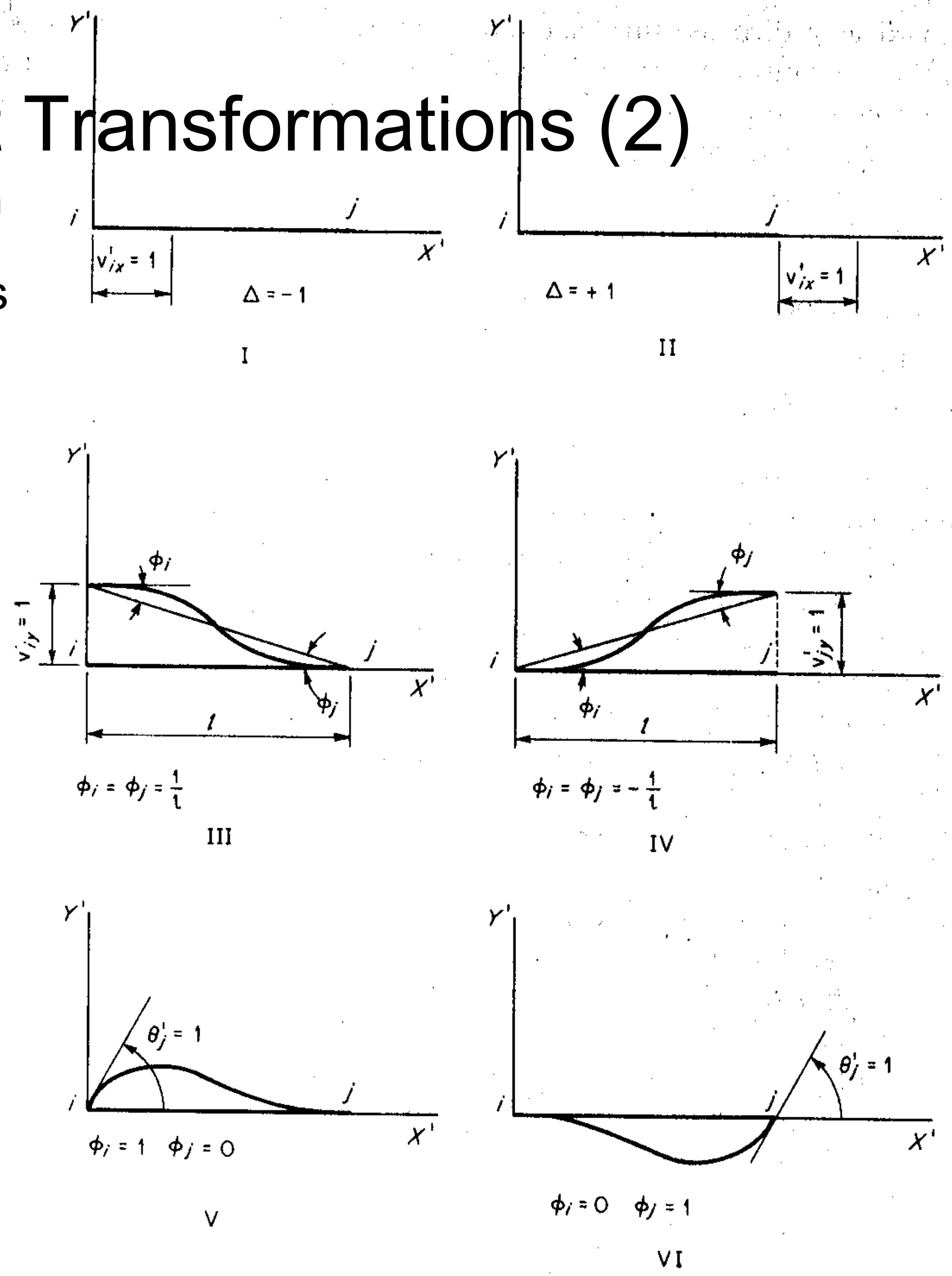
Displacement Transformations (1)

- Pin-jointed truss member



Displacement Transformations (2)

- Frame or Beam members



$$\Delta = \begin{bmatrix} -\cos\alpha & -\sin\alpha & \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} v_{ix} \\ v_{iy} \\ v_{jx} \\ v_{jy} \end{bmatrix}$$

$$\left. \begin{aligned} v^T &= \begin{bmatrix} \phi_i & \phi_j & \Delta \end{bmatrix} \\ r^T &= \begin{bmatrix} v_{ix} & v_{iy} & \theta_i & v_{jx} & v_{jy} & \theta_j \end{bmatrix} \end{aligned} \right\} \rightarrow v = T^T L_D^T r$$

$$\begin{bmatrix} \phi_i \\ \phi_j \\ \Delta \end{bmatrix} = \begin{bmatrix} 0 & 1/l & 1 & 0 & -1/l & 0 \\ 0 & 1/l & 0 & 0 & -1/l & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v'_{ix} \\ v'_{iy} \\ \theta_i \\ v'_{jx} \\ v'_{jy} \\ \theta_j \end{bmatrix} \rightarrow v = T^T r' \rightarrow v = T^T L_D^T r$$

transformation	Force	displacement
member	S	v
node	R	r
	S = bR	v = ar, R = a^TS
	b=$(a^T)^{-1}$	

$$S = bR$$

$$r = b^T v$$

$$R_i = 1 \rightarrow \begin{Bmatrix} W_e = 1 \times r_i \\ W_i = S_i^T v \end{Bmatrix} \rightarrow r_i = S_i^T v = b_i^T v$$

$$\left.\begin{matrix} v = fS = fbR \\ r = b^T v \end{matrix}\right\} \rightarrow r = b^T fbR = FR$$

$$v = ar$$

$$R - a^T S = 0 \rightarrow R = a^T S$$

$$S = \left(a^T\right)^{-1} R = bR \rightarrow\rightarrow b = \left(a^T\right)^{-1}$$

$$S = kv \rightarrow R = a^T kv = a^T kar$$

Member Stiffness

- Prismatic member of length l and area A with Young's modulus of elasticity E

$$f = \frac{l}{AE} \rightarrow k = f^{-1} = \frac{EA}{l}$$

- Beam element that is prismatic, of length l and of bending stiffness EI

$$\begin{bmatrix} \phi_i \\ \phi_j \end{bmatrix} = \frac{1}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} M_i \\ M_j \end{bmatrix} \rightarrow f = \frac{1}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow k = f^{-1} = \frac{2EI}{l}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} M_i \\ M_j \\ F \end{bmatrix} = \frac{2EI}{l}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \frac{A}{2I} \end{bmatrix}\begin{bmatrix} \phi_i \\ \phi_j \\ \delta \end{bmatrix}$$

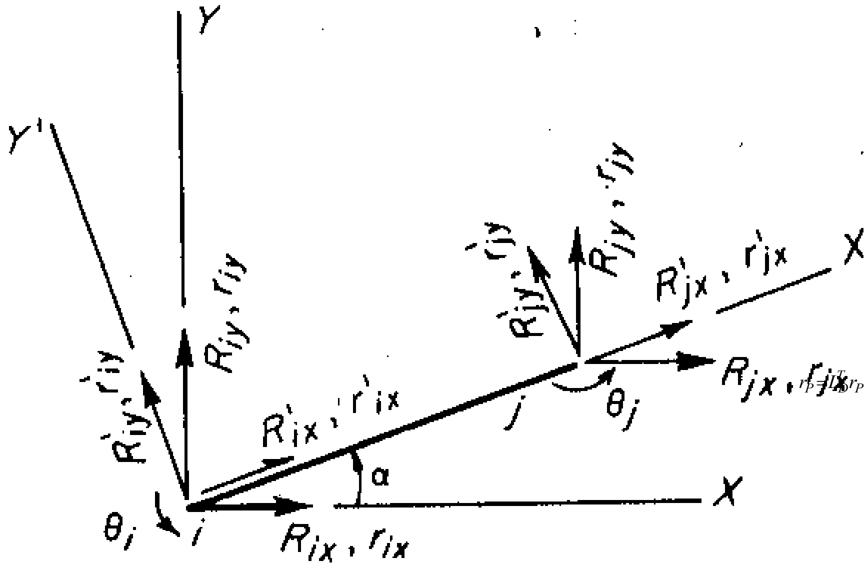
$$r_P = T^T r_O$$

$$r'_P = L^T r_P = L^T T^T r_O$$

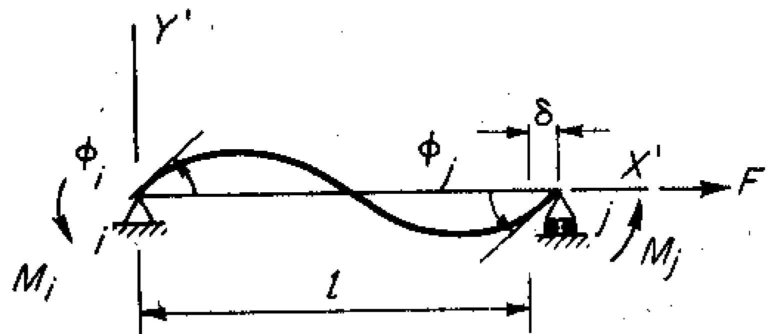
$$v_N = U r'_P = U L^T T^T r_O \leftrightarrow v_N = a_N r_O$$

$$\begin{bmatrix} \phi_{iz} \\ \phi_{jz} \\ \phi_{ij} \\ \phi_{jy} \\ \theta_x \\ \delta \end{bmatrix}_N = \begin{bmatrix} 0 & -1/l & 0 & 0 & 0 & 0 \\ 0 & -1/l & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/l & 0 & 0 & 0 \\ 0 & 0 & 1/l & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}'_P \quad k_N = \begin{bmatrix} \frac{4EI_1}{l} & \frac{2EI_1}{l} & 0 & 0 & 0 & 0 \\ \frac{2EI_1}{l} & \frac{4EI_1}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4EI_2}{l} & \frac{2EI_2}{l} & 0 & 0 \\ 0 & 0 & \frac{2EI_2}{l} & \frac{4EI_2}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_T & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} \end{bmatrix} \text{ where } \begin{cases} I_1 = \int y^2 dA \\ I_2 = \int z^2 dA \end{cases}$$

Plane Frame Member



(a)



(b)

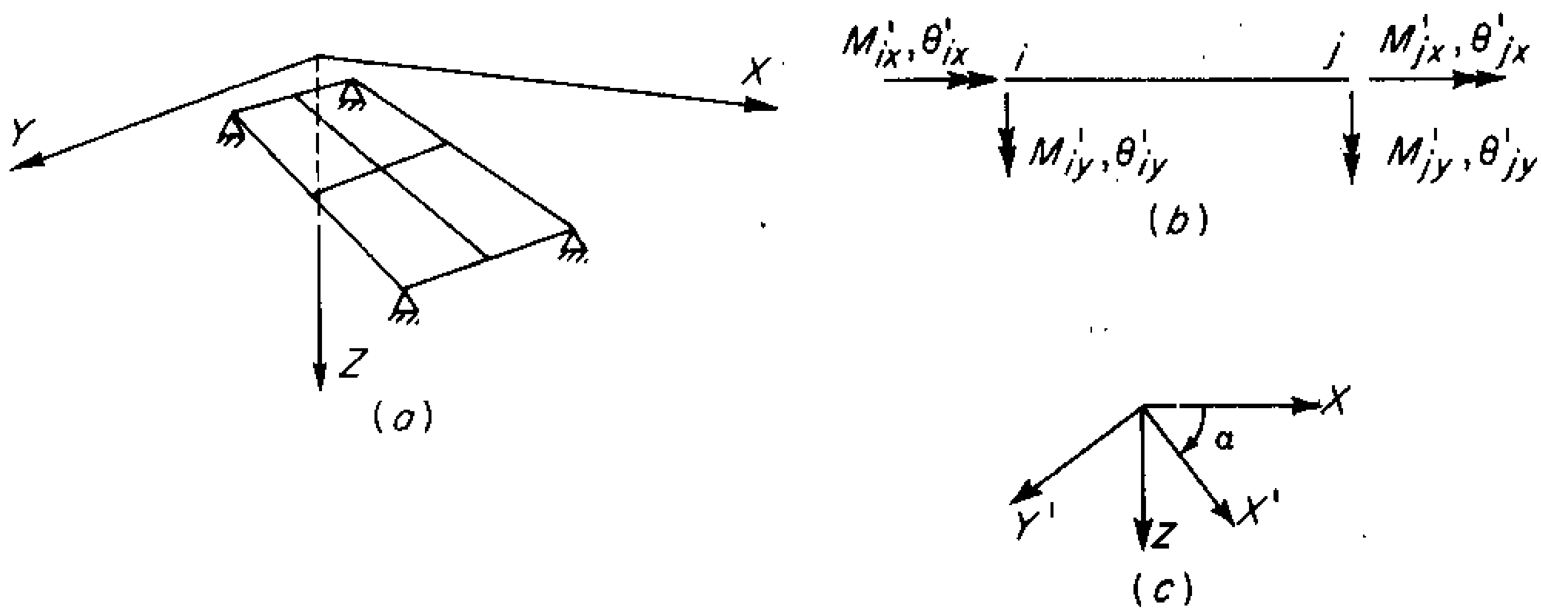
$$\begin{bmatrix} M_i \\ M_j \\ F \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_j \\ \delta \end{bmatrix} \rightarrow S = k'v \text{ where } k' = \frac{2EI}{l} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \dfrac{A}{2I} \end{bmatrix}$$

$$\begin{bmatrix} \phi_i \\ \phi_j \\ \delta \end{bmatrix} = \begin{bmatrix} 0 & 1/l & 1 & 0 & -1/l & 0 \\ 0 & 1/l & 0 & 0 & -1/l & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r'_{ix} \\ r'_{iy} \\ \theta_i \\ r'_{jx} \\ r'_{jy} \\ \theta_j \end{bmatrix} \rightarrow v = T^T r' \xrightarrow{r' = L_D{}^T r} v = T^T L_D{}^T r$$

$$R = L_D T S = L_D T k' v = L_D T k' T^T L_D{}^T r = kr \rightarrow k = L_D T k' T^T L_D{}^T$$

$$S = k' T^T L_D{}^T r$$

Grid Member



X
Y
Z
(a)
M'_{ix}, θ'_{ix}
i
j
M'_{jx}, θ'_{jx}
M'_{iy}, θ'_{iy}
M'_{jy}, θ'_{jy}
(b)
X
α
Y'
Z
X'
(c)

$$\begin{bmatrix} M_{iy} \\ M_{jy} \\ M_T \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} \phi_{iy} \\ \phi_{jy} \\ \theta_T \end{bmatrix} \rightarrow S = k'v$$

$$\begin{bmatrix} \phi_{iy} \\ \phi_{jy} \\ \theta_T \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/l & 0 & 0 & 1/l \\ 0 & 0 & -1/l & 0 & 1 & 1/l \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{ix} \\ \theta_{iy} \\ r_{iz} \\ \theta_{jx} \\ \theta_{jy} \\ r_{jz} \end{bmatrix} \rightarrow v = T^T r' \xrightarrow{r' = L_D{}^T r} v = T^T L_D{}^T r$$

$$R^T = \begin{bmatrix} M_{iX} & M_{iY} & R_{iZ} & M_{jX} & M_{jY} & R_{jZ} \end{bmatrix}$$

$$R = L_D T S = L_D T k' v = L_D T k' T^T L_D{}^T r = kr \rightarrow k = L_D T k' T^T L_D{}^T$$

$$S = k' T^T L_D{}^T r$$

Prismatic Member in 3D

$$\begin{bmatrix} \delta \\ \phi_{iz} \\ \phi_{iy} \\ \theta_T \\ \phi_{jz} \\ \phi_{jy} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/l & 0 & 0 & 0 & 1 & 0 & -1/l & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/l & 0 & 1 & 0 & 0 & 0 & 1/l & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/l & 0 & 0 & 0 & 0 & 0 & -1/l & 0 & 0 & 0 & 1 \\ 0 & 0 & -1/l & 0 & 0 & 0 & 0 & 0 & 1/l & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{ix} \\ r_{iy} \\ r_{iz} \\ \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ r_{jx} \\ r_{jy} \\ r_{jz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \end{bmatrix}' \rightarrow v = T^T r' \xrightarrow{r' = L_D{}^T r} v = T^T L_D{}^T r$$

$$S^T = \begin{bmatrix} F & M_{iz} & M_{iy} & M_T & M_{jz} & R_{jy} \end{bmatrix}$$

$$S = k'v \text{ where } k' = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4EI_z}{l} & 0 & 0 & \frac{2EI_z}{l} & 0 \\ 0 & 0 & \frac{4EI_y}{l} & 0 & 0 & \frac{2EI_y}{l} \\ 0 & 0 & 0 & \frac{GI_p}{l} & 0 & 0 \\ 0 & \frac{2EI_z}{l} & 0 & 0 & \frac{4EI_z}{l} & 0 \\ 0 & 0 & \frac{2EI_y}{l} & 0 & 0 & \frac{4EI_y}{l} \end{bmatrix}$$

$$R = L_D TS = L_D Tk'v = L_D Tk'T^T {L_D}^T r = kr \rightarrow k = L_D Tk'T^T {L_D}^T$$

$$S = k'T^T {L_D}^T r$$

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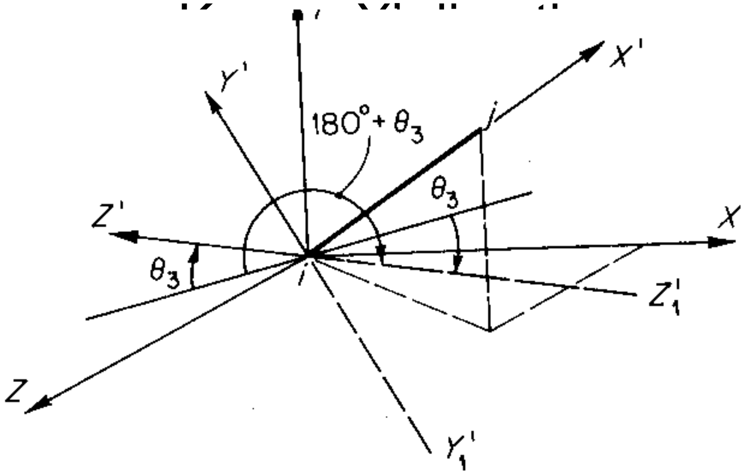
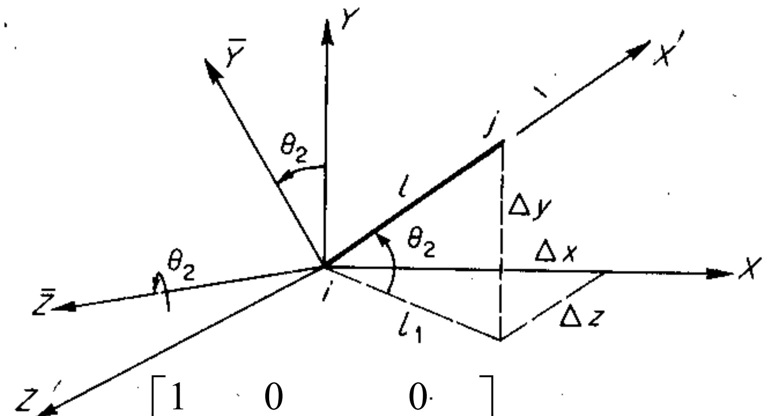
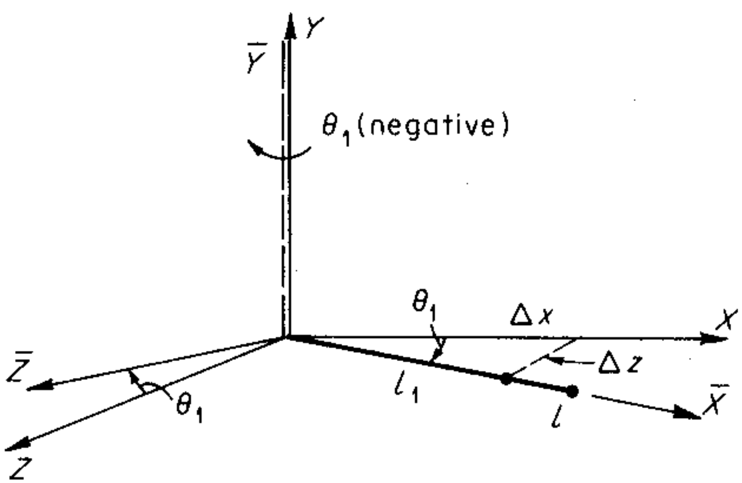
ve)





$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix} (\text{about } iX')$$



$$L_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{about } i\bar{Z})$$

$$L_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ 0 & 1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} (\text{about } iY)$$

$$L = L_1 L_2 L_3 = \begin{bmatrix} \cos\theta_1\cos\theta_2 & 0 & -\sin\theta_1 \\ \sin\theta_2 & 1 & 0 \\ -\sin\theta_1\cos\theta_2 & 0 & \cos\theta_1 \end{bmatrix}$$

Summary

- Solution to the structures problem by the displacement method

1. $v = ar$ (displacement transformation)
2. $S = kv$ (member forces in terms of member displacements)
3. $R = a^T S$ (joint equilibrium equations)
4. $r = \left(a^T ka\right)^{-1} R$ (solution for joint displacements)
5. $S = kar$ (member forces)

$$K_e^{truss} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_e^{beam-v} = \frac{EI_y}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 6L & 0 & 0 & -12 & 0 & 0 & 6L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6L & 0 & 0 & 4L^2 & 0 & 0 & -6L & 0 & 0 & 2L^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & -6L & 0 & 0 & 12 & 0 & 0 & -6L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6L & 0 & 0 & 2L^2 & 0 & 0 & -6L & 0 & 0 & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_e^{beam-w} = \frac{EI_z}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 6L & 0 & 0 & -12 & 0 & 0 & 6L \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6L & 0 & 0 & 4L^2 & 0 & 0 & -6L & 0 & 0 & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 & -6L & 0 & 0 & 12 & 0 & 0 & -6L \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6L & 0 & 0 & 2L^2 & 0 & 0 & -6L & 0 & 0 & 4L^2 \end{bmatrix}$$

$$K_e^{torsion} = \frac{GJ}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element Stiffness

$$K_e = T^T\left(K_e^{truss} + K_e^{beam-v} + K_e^{beam-w} + K_e^{torsion}\right)T$$

$$= T^T \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_y}{L^3} & 0 & 0 & \frac{6EI_y}{L^2} & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & \frac{12EI_z}{L^3} & 0 & 0 & \frac{6EI_z}{L^2} & 0 & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & \frac{6EI_y}{L^2} & 0 & 0 & \frac{4EI_y}{L} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & \frac{2EI_y}{L} & 0 \\
0 & 0 & \frac{6EI_z}{L^2} & 0 & 0 & \frac{4EI_z}{L} & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_y}{L^3} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & \frac{12EI_y}{L^3} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & \frac{12EI_z}{L^3} & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & \frac{6EI_y}{L^2} & 0 & 0 & \frac{2EI_y}{L} & 0 & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & \frac{4EI_y}{L} & 0 \\
0 & 0 & \frac{6EI_z}{L^2} & 0 & 0 & \frac{2EI_z}{L} & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix} T$$