EXERCISE 2.3 Start from (2.9), which is reproduced here:

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad d = \bar{u}_{xj} - \bar{u}_{xi}, \tag{E2.5}$$

in which F and d are connected by (2.8), namely $F = (EA/L)d$. Express the foregoing two equations in matrix form bringing up also the $\bar{y}$ nodal displacements by introducing zeros as appropriate in the vectors:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} F = \frac{EA}{L} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} d, \quad d = \bar{u}_{xj} - \bar{u}_{xi} = [\,-1 \quad 0 \quad 1 \quad 0\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}. \tag{E2.6}$$

and combine theses two as matrix product:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} [\,-1 \quad 0 \quad 1 \quad 0\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}, \tag{E2.7}$$

which is equation (2.10). This "Mechanics of Materials" matrix-flavored technique is used in Chapter 5 to derive other structural elements.

EXERCISE 2.7 From (E2.2) one gets $e = d\bar{u}/d\bar{x} = (u_{xj} - u_{xi})/L = d/L$. Hooke's law gives $\sigma = Ee = Ed/L$. Both e and σ do not depend on $\bar{x}$. Substitution into (E2.3) gives

$$\Pi(d) = \tfrac{1}{2} \int_0^L \frac{EA}{L^2} d^2\, d\bar{x} - Fd = \frac{EA}{2L^2} d^2 \int_0^L d\bar{x} - Fd = \frac{EA}{2L} d^2 - Fd. \tag{E2.12}$$

Applying the MPE yields

$$\frac{\partial \Pi}{\partial d} = \frac{EA}{L} d - F = 0, \tag{E2.13}$$

whence $F = (EA/L)\, d$ follows.

EXERCISE 3.3 For the example truss, joint forces may be also recovered from consideration of joint equilibrium, because the structure is statically determinate. Once the joint displacements (3.17) are known, the joint forces in the *local* system of member e and the internal (axial) force f^e may be recovered from the generic-member equilibrium relation

$$\bar{\mathbf{f}}^e = \begin{bmatrix} \bar{f}^e_{xi} \\ \bar{f}^e_{yi} \\ \bar{f}^e_{xj} \\ \bar{f}^e_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} f^e = \bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{K}}^e \mathbf{T}^e \mathbf{u}^e. \tag{E3.3}$$

Carrying out the operations for the example truss we get

$$\bar{\mathbf{f}}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{f}}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{f}}^{(3)} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix}, \tag{E3.4}$$

from which it follows that the member axial forces $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ are 0, -1 (compression) and $+2\sqrt{2}$ (tension), respectively. See Figure E3.3 for physical interpretation. *This method is applicable only to statically determinate structures.*

EXERCISE 3.6

(a) Recall that the generic member stiffness matrix in global coordinates is

$$\mathbf{K}^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \tag{E3.9}$$

where $c = \cos\varphi^e$, $s = \sin\varphi^e$, E^e, A^e and L^e are the elastic modulus, cross-section area and length, respectively. For member (element) (1): $E^{(1)} = 1000$, $A^{(1)} = 2$, $L^{(1)} = \sqrt{4^2+3^2} = 5$, $\varphi^{(1)} = \arctan(3/4)$, $c = 4/5 = 0.8$, $s = 3/5 = 0.6$. Therefore

$$\mathbf{K}^{(1)} = \frac{1000\times 2}{5}\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 \\ 192 & 144 & -192 & -144 \\ -256 & -192 & 256 & 192 \\ -192 & -144 & 192 & 144 \end{bmatrix} \tag{E3.10}$$

For member (element) (2): $E^{(2)} = 1000$, $A^{(2)} = 4$, $L^{(2)} = \sqrt{4^2+(-3)^2} = 5$, $\varphi^{(2)} = \arctan(-3/4)$, $c = 4/5 = 0.8$, $s = -3/5 = -0.6$. Therefore

$$\mathbf{K}^{(2)} = \frac{1000\times 4}{5}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} = \begin{bmatrix} 512 & -384 & -512 & 384 \\ -384 & 288 & 384 & -288 \\ -512 & 384 & 512 & -384 \\ 384 & -288 & -384 & 288 \end{bmatrix} \tag{E3.11}$$

Next the member stiffness equations are augmented by adding zero rows and columns as appropriate to complete the force and displacement vectors. Compatibility is used to drop the member index on the displacements. For member (1):

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 256 & 192 & 0 & 0 \\ -192 & -144 & 192 & 144 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.12}$$

For member (2):

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 512 & -384 & -512 & 384 \\ 0 & 0 & -384 & 288 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.13}$$

Adding the two equations and using the force equilibrium condition $\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} = (\mathbf{K}^{(1)} + \mathbf{K}^{(2)})\mathbf{u} = \mathbf{K}\mathbf{u}$, we arrive at the master stiffness equations

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.14}$$

(b) We apply the displacement boundary conditions:

$$u_{x1} = u_{y1} = u_{x3} = u_{y3} = 0,\ f_{x2} = P = 12, \text{ and } f_{y2} = 0, \tag{E3.16}$$

by removing equations 1, 2, 5, and 6 from the system. This is done by deleting rows and columns 1, 2, 5, and 6 from **K**, and the corresponding components from **f** and **u**. The reduced two-equation system is

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \tag{E3.17}$$

Solving this linear system by any method gives

$$u_{x2} = \frac{9}{512} \quad \text{and} \quad u_{y2} = \frac{1}{128}. \tag{E3.18}$$

(c) To recover all the joint forces note that the complete displacement vector is $\mathbf{u} = [\,0\ \ 0\ \ 9/512\ \ 1/128\ \ 0\ \ 0\,]^T$. Using the original master stiffness equations (E3.14):

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \mathbf{Ku} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 9/512 \\ 1/128 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -9/2 \\ 12 \\ 0 \\ -6 \\ 9/2 \end{bmatrix}. \tag{E3.19}$$

Overall equilibrium is verified by summing the forces in the $\{x, y\}$ directions and taking z-moments about any joint:

$$\begin{aligned} \sum f_x &\Rightarrow f_{x1} + f_{x2} + f_{x3} = -6 + 12 - 6 = 0, \\ \sum f_y &\Rightarrow f_{y1} + f_{y2} + f_{y3} = -\frac{9}{2} + 0 + \frac{9}{2} = 0, \\ \sum M_{wrt1} &\Rightarrow 4 f_{y2} - 3 f_{x2} + 8 f_{y3} = 0 - 36 + 36 = 0, \\ \sum M_{wrt2} &\Rightarrow -3 f_{x1} + 4 f_{y1} - 3 f_{x3} + 4 f_{y3} = -18 + 18 - 18 + 18 = 0, \\ \sum M_{wrt3} &\Rightarrow 8 f_{y1} - 4 f_{y2} - 3 f_{x2} = 36 - 0 - 36 = 0. \end{aligned} \tag{E3.20}$$

(d) Using the method described in §3.4.2 we proceed as follows. For member (1), $c = 4/5$, $s = 3/5$, $\mathbf{u}^{(1)} = [\,0\ \ 0\ \ 9/512\ \ 1/128\,]^T$. The local joint displacements are recovered by

$$\overline{\mathbf{u}}^{(1)} = \begin{bmatrix} \overline{u}_{xi}^{(1)} \\ \overline{u}_{yi}^{(1)} \\ \overline{u}_{xj}^{(1)} \\ \overline{u}_{yj}^{(1)} \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 & 0 & 0 \\ -3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 4/5 & 3/5 \\ 0 & 0 & -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 9/512 \\ 1/128 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/160 \\ -11/2560 \end{bmatrix} \tag{E3.21}$$

The elongation is $d^{(1)} = 3/160 - 0 = 3/160$, from which $F^{(1)} = 1000 \times 2 \times (3/160)/5 = 15/2 = 7.5$. The positive sign indicates that member (1) is in tension.

For member (2), $c = 4/5$, $s = -3/5$, $\mathbf{u}^{(2)} = [\,9/512\ \ 1/128\ \ 0\ \ 0\,]^T$. The local joint displacements are recovered by

$$\overline{\mathbf{u}}^{(2)} = \begin{bmatrix} \overline{u}_{xi}^{(2)} \\ \overline{u}_{yi}^{(2)} \\ \overline{u}_{xj}^{(2)} \\ \overline{u}_{yj}^{(2)} \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 & 0 & 0 \\ 3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 4/5 & -3/5 \\ 0 & 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 9/512 \\ 1/128 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/320 \\ 43/2560 \\ 0 \\ 0 \end{bmatrix} \tag{E3.22}$$

The elongation is $d^{(2)} = 0 - 3/320 = -3/320$, from which $F^{(2)} = 1000 \times 4 \times (-3/320)/5 = -15/2 = -7.5$. The negative sign indicates that member (2) is in compression.

EXERCISE 3.7

(a) The master stiffness matrix is the same as in the previous Exercise:

$$\mathbf{K} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \tag{E3.23}$$

However, now we have a prescribed displacement, $u_{y3} = -\frac{1}{2}$:

$$\begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ u_{y2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 12 \\ 0 \\ f_{x3} \\ f_{y3} \end{bmatrix} \tag{E3.24}$$

(b) For hand computation, reduce the system by removing rows 1, 2, 5 and 6 that pertain to the prescribed displacements:

$$\begin{bmatrix} -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ u_{y2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \tag{E3.25}$$

Next, columns 1, 2, 5 and 6 are removed by tranferring all known terms from the left to the right hand side:

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} - \begin{bmatrix} (-256)(0) + (-192)(0) + (-512)(0) + (384)(-\frac{1}{2}) \\ (-192)(0) + (-144)(0) + (384)(0) + (-288)(-\frac{1}{2}) \end{bmatrix} \tag{E3.26}$$

which gives

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} 204 \\ -144 \end{bmatrix} \tag{E3.27}$$

Solution by Gausss elimination yields

$$\boxed{u_{x2} = \frac{105}{512} \quad \text{and} \quad u_{y2} = -\frac{31}{128}} \tag{E3.28}$$

(c) To recover all the joint forces we complete the node displacement vector with the known values:

$$\boxed{\mathbf{u}^T = [\,0 \quad 0 \quad \tfrac{105}{512} \quad -\tfrac{31}{128} \quad 0 \quad -\tfrac{1}{2}\,]} \tag{E3.29}$$

and use $\mathbf{f} = \mathbf{K}\mathbf{u}$ to get

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{105}{512} \\ -\frac{31}{128} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -6 \\ -\frac{9}{2} \\ 12 \\ 0 \\ -6 \\ \frac{9}{2} \end{bmatrix} \tag{E3.30}$$

Horizontal force equilibrium is verified by

$$\boxed{f_{x1} + f_{x2} + f_{x3} = -6 + 12 - 6 = 0} \tag{E3.31}$$

and likewise for y-force and moment equilibrium.

EXERCISE 7.7 Symmetry and antisymmetry lines are identified on Figure E7.11. Problem domains may be reduced to the darker regions. Appropriate supports to realize these symmetry and antisymmetry conditions as well as actual supports (if given) are depicted in Figure E7.11.

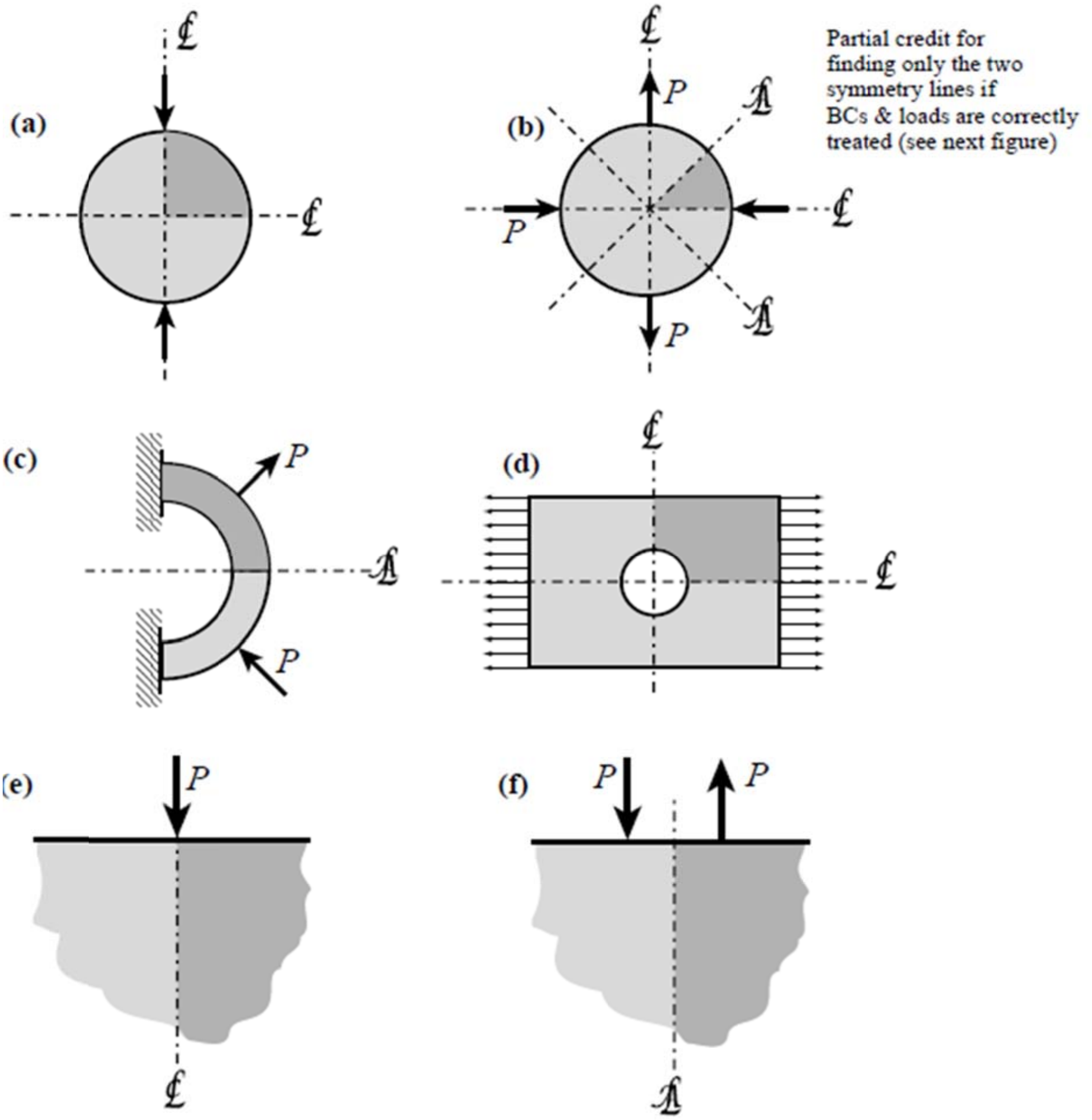

FIGURE E7.11. Symmetry and antisymmetry lines in problems of Exercise 7.4.