4.5 True value: $f(3) = 554$.

zero order:

$$f(3) = f(1) = -62 \qquad \varepsilon_t = \left|\frac{554-(-62)}{554}\right| \times 100\% = 111.191\%$$

first order:

$$f(3) = -62 + f'(1)(3-1) = -62 + 70(2) = 78 \qquad \varepsilon_t = 85.921\%$$

second order:

$$f(3) = 78 + \frac{f''(1)}{2}(3-1)^2 = 78 + \frac{138}{2}4 = 354 \qquad \varepsilon_t = 36.101\%$$

third order:

$$f(3) = 354 + \frac{f^{(3)}(1)}{6}(3-1)^3 = 354 + \frac{150}{6}8 = 554 \qquad \varepsilon_t = 0\%$$

Thus, the third-order result is perfect because the original function is a third-order polynomial.

4.6 True value:

$$f'(x) = 75x^2 - 12x + 7$$

$$f'(2) = 75(2)^2 - 12(2) + 7 = 283$$

function values:

$x_{i-1} = 1.8$	$f(x_{i-1}) = 50.96$
$x_i = 2$	$f(x_i) = 102$
$x_{i+1} = 2.2$	$f(x_{i+1}) = 164.56$

forward:

$$f'(2) = \frac{164.56 - 102}{0.2} = 312.8 \qquad \varepsilon_t = \left|\frac{283 - 312.8}{283}\right| \times 100\% = 10.53\%$$

backward:

$$f'(2) = \frac{102 - 50.96}{0.2} = 255.2 \qquad \varepsilon_t = \left|\frac{283 - 255.2}{283}\right| \times 100\% = 9.823\%$$

centered:

$$f'(2) = \frac{164.56 - 50.96}{2(0.2)} = 284 \qquad \varepsilon_t = \left|\frac{283 - 284}{283}\right| \times 100\% = 0.353\%$$

Both the forward and backward have errors that can be approximated by (recall Eq. 4.15),

$$|E_t| \approx \frac{f''(x_i)}{2} h$$

$$f''(2) = 150x - 12 = 150(2) - 12 = 288$$

$$|E_t| \approx \frac{288}{2} 0.2 = 28.8$$

This is very close to the actual error that occurred in the approximations

forward: $|E_t| \approx |283 - 312.8| = 29.8$

backward: $|E_t| \approx |283 - 255.2| = 27.8$

The centered approximation has an error that can be approximated by,

$$E_t \approx -\frac{f^{(3)}(x_i)}{6} h^2 = -\frac{150}{6} 0.2^2 = -1$$

which is exact: $E_t = 283 - 284 = -1$. This result occurs because the original function is a cubic equation which has zero fourth and higher derivatives.

4.7 True value:

$$f''(x) = 150x - 12$$
$$f''(2) = 150(2) - 12 = 288$$

$\underline{h = 0.25:}$

$$f''(2) = \frac{f(2.25) - 2f(2) + f(1.75)}{0.25^2} = \frac{182.1406 - 2(102) + 39.85938}{0.25^2} = 288$$

$\underline{h = 0.125:}$

$$f''(2) = \frac{f(2.125) - 2f(2) + f(1.875)}{0.125^2} = \frac{139.6738 - 2(102) + 68.82617}{0.125^2} = 288$$

Both results are exact because the errors are a function of 4^{th} and higher derivatives which are zero for a 3^{rd}-order polynomial.

25.19 The two differential equations to be solved are

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

$$\frac{dx}{dt} = -v$$

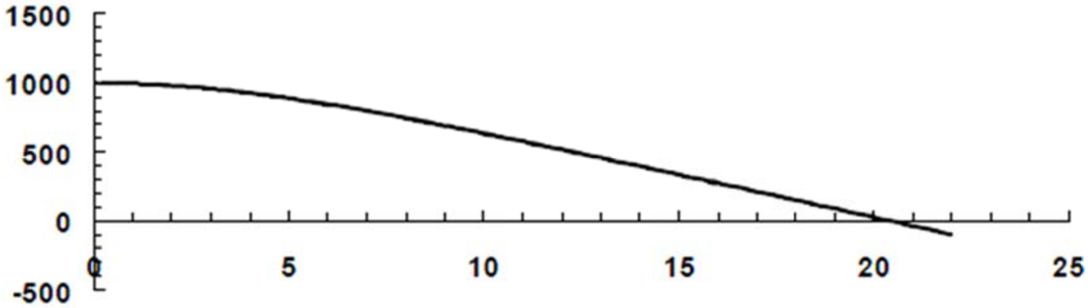
(a) Here are the first few steps of Euler's method with a step size of $h = 0.2$.

t	*x*	*v*	*dx/dt*	*dv/dt*
0	1000	0	0	9.81
0.2	1000	1.962	-1.962	9.800376
0.4	999.6076	3.922075	-3.92208	9.771543
0.6	998.8232	5.876384	-5.87638	9.72367
0.8	997.6479	7.821118	-7.82112	9.657075
1	996.0837	9.752533	-9.75253	9.57222

(b) Here are the results of the first few steps of the 4th-order RK method with a step size of $h = 0.2$.

t	*x*	*v*
0	1000	0
0.2	999.8038	1.961359
0.4	999.2157	3.918875
0.6	998.2368	5.868738
0.8	996.869	7.807195
1	995.1149	9.730582

The results for x of both methods are displayed graphically on the following plots. Because the step size is sufficiently small the results are in close agreement. Both indicate that the parachutist would hit the ground at a little after 20 s. The more accurate 4th-order RK method indicates that the solution reaches the ground between t = 20.2 and 20.4 s.



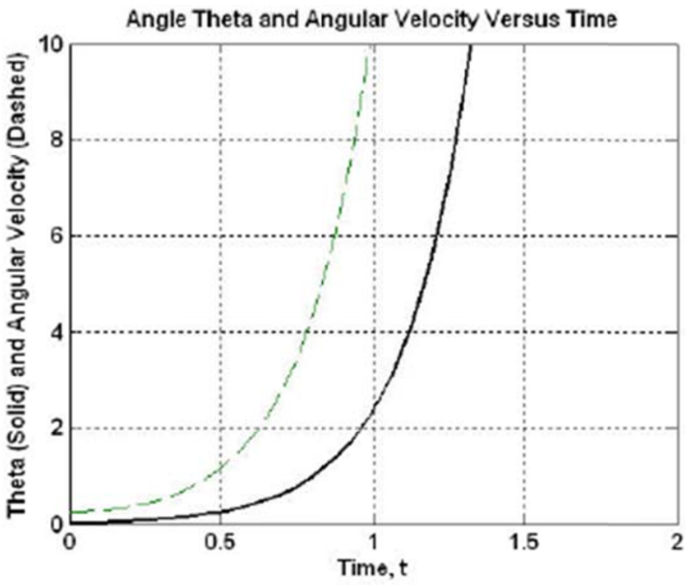
26.13 The second-order equation can be composed into a pair of first-order equations as

$$\frac{d\theta}{dt} = x \qquad \frac{dx}{dt} = \frac{g}{l}\theta$$

We can use MATLAB to solve this system of equations.

```
tspan=[0,5]';
x0=[0,0.25]';
[t,x]=ode45('dxdt',tspan,x0);
plot(t,x(:,1),t,x(:,2),'--')
grid
title('Angle Theta and Angular Velocity Versus Time')
xlabel('Time, t')
ylabel('Theta (Solid) and Angular Velocity (Dashed)')
axis([0 2 0 10])
zoom

function dx=dxdt(t,x)
 dx=[x(2);(9.81/0.5)*x(1)];
```



26.15 (a) Analytic solution:

$$y = \frac{1}{999}\left(1000e^{-x} - e^{-1000x}\right)$$

(b) The second-order differential equation can be expressed as the following pair of first-order ODEs,

$$\frac{dy}{dx} = w$$

$$\frac{dw}{dx} = -1000y - 1001w$$

where $w = y'$. Using the same approach as described in Sec. 26.1, the following simultaneous equations need to be solved to advance each time step,

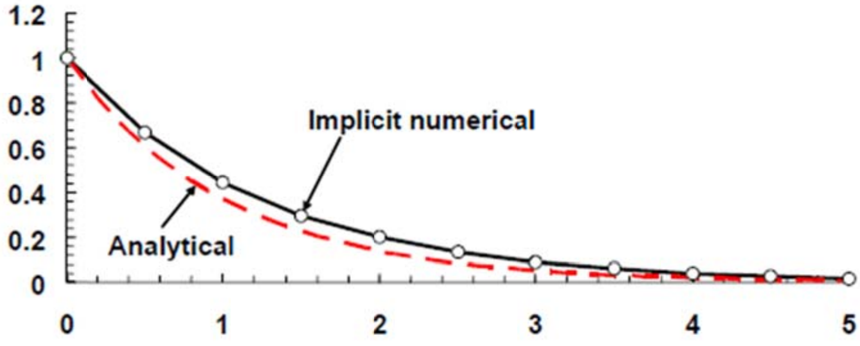
$$y_{i+1} - hw_{i+1} = y_i$$

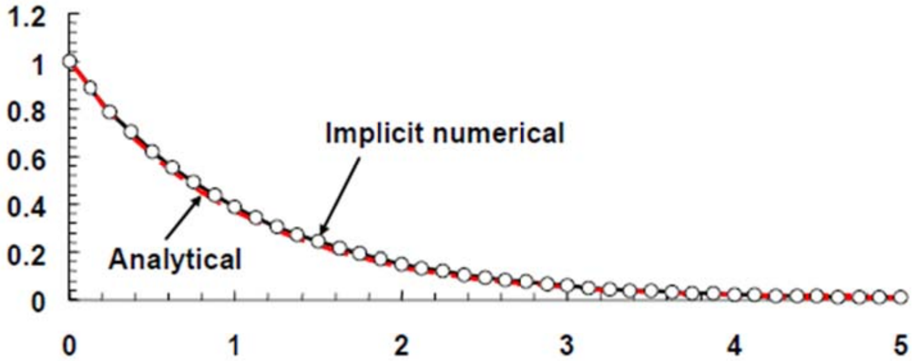
$$1000hy_{i+1} + 1001hw_{i+1} = w_i$$

If these are implemented with a step size of 0.5, the following values are simulated

x	*y*	*w*
0	1	0
0.5	0.667332	-0.66534
1	0.444889	-0.44489
1.5	0.296593	-0.29659
2	0.197729	-0.19773
2.5	0.131819	-0.13182
3	0.087879	-0.08788
3.5	0.058586	-0.05859
4	0.039057	-0.03906
4.5	0.026038	-0.02604
5	0.017359	-0.01736

The results for y along with the analytical solution are displayed below:





Finally, we can also solve this problem using one of the MATLAB routines expressly designed for stiff systems. To do this, we first develop a function to hold the pair of ODEs,

```
function dy = dydx(x, y)
dy = [y(2);-1000*y(1)-1001*y(2)];
```

Then the following session generates a plot of both the analytical and numerical solutions. As can be seen, the results are indistinguishable.

```
x=[0:.1:5];
y=1/999*(1000*exp(-x)-exp(-1000*x));
xspan=[0 5];
x0=[1 0];
[xx,yy]=ode23s(@dydx,xspan,x0);
plot(x,y,xx,yy(:,1),'o')
grid
xlabel('x')
ylabel('y')
```