Midterm Exam

05/03/2023

1. 데이터 평균이용 실제 계산값 유도가 핵심, 방법만 제시한 경우 10점

	Height	Weight	X	Y
	166	58	-5	-7
	176	75	5	10
	171	62	0	-3
	173	70	2	5
	169	60	-2	-5
AVG	171	65	0	0

$$y_n^* = w_0 + w_1 x_n \approx y_n \quad \text{for } n = 1,\ldots,5 \xrightarrow[Y_n = y_n - \bar{y}]{X_n = x_n - \bar{x}} Y_n^* = W_0 + W_1 X_n \approx Y_n$$

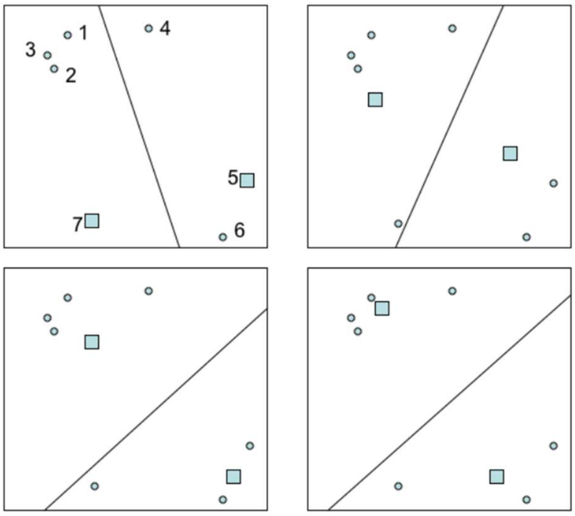
$$L = \sum_{n=1}^{5} (W_0 + W_1 X_n - Y_n)^2 = (W_0 - 5W_1 + 7)^2 + (W_0 + 5W_1 - 10)^2 + (W_0 - 0W_1 + 3)^2 + (W_0 + 2W_1 - 5)^2 + (W_0 - 2W_1 + 5)^2$$

$$= 5W_0^2 + 58W_1^2 - 210W_1 + 208$$

$$\text{minimize } L \to W_0 = 0,\ W_1 = \frac{210}{116}$$

$$y_n - 65 = 0 + \frac{210}{116}(x_n - 171) \to y_n = \frac{210}{116}x_n + \left(65 - \frac{210}{116}(171)\right)$$

2. (1) (**5** pts)



(2) (**5** pts)

Advantages of hierarchical clustering:

- Don't need to know how many clusters you're after
- Can cut hierarchy at any level to get any number of clusters
- Easy to interpret hierarchy for particular applications
- Can deal with long stringy data

Advantages of K-means clustering:

- Can be much faster than hierarchical clustering, depending on data
- Nice theoretical framework
- Can incorporate new data and reform clusters easily

3. **(5점 , 5점 , 10점)**

$$(1)\ \mathbf{A}=\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\0&2\end{bmatrix}$$

$$(2)\ \mathbf{A}^8=\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\right)^8=\mathbf{V}\boldsymbol{\Lambda}^8\mathbf{V}^{-1}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1^8&0\\0&2^8\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}1&256\\0&256\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}1&255\\0&256\end{bmatrix}$$

$$(3)\ u'(t)=\mathbf{A}u(t)=\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}u(t)$$

$$w(t)=\mathbf{V}^{-1}u(t)\rightarrow w'(t)=\mathbf{V}^{-1}\mathbf{A}u(t)=\mathbf{V}^{-1}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}u(t)=\boldsymbol{\Lambda}w(t)=\begin{bmatrix}c_1e^t\\c_2e^{2t}\end{bmatrix}$$

$$u(t)=\mathbf{V}w(t)==\begin{bmatrix}c_1e^t+c_2e^{2t}\\c_2e^{2t}\end{bmatrix}\xrightarrow{u(0)=\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}c_1+c_2\\c_2\end{bmatrix}}u(t)=\begin{bmatrix}e^t\\0\end{bmatrix}$$

4. (10 pts)

$$w=\exp\left(\frac{2\pi i}{6}\right)$$

$$\mathbf{F}_6=\begin{bmatrix}1&1&1&1&1&1\\1&w&w^2&w^3&w^4&w^5\\1&w^2&w^4&w^6&w^8&w^{10}\\1&w^3&w^6&w^9&w^{12}&w^{15}\\1&w^4&w^8&w^{12}&w^{16}&w^{20}\\1&w^5&w^{10}&w^{15}&w^{20}&w^{25}\end{bmatrix}=\begin{bmatrix}1&&&1&&\\&1&&&w&\\&&1&&&w^2\\1&&&-1&&\\&1&&&-w&\\&&1&&&-w^2\end{bmatrix}\begin{bmatrix}1&1&1&&&\\1&w&w^2&&&\\1&w^2&w^4&&&\\&&&1&1&1\\&&&1&w&w^2\\&&&1&w^2&w^4\end{bmatrix}\begin{bmatrix}1&&&&&\\&&1&&&\\&&&&1&\\&1&&&&\\&&&1&&\\&&&&&1\end{bmatrix}$$

5. (10 pts)

The closest rank 1 approximations are $A=\begin{bmatrix}3&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $\quad A=\begin{bmatrix}0&3\\0&0\end{bmatrix}$ $\quad A=\frac{3}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$

6. (5 pts, 5 pts)

$$(1)\ \mathbf{q}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{A}_2 = \mathbf{b} - \left(\mathbf{b}^T\mathbf{q}_1\right)\mathbf{q}_1 = \begin{bmatrix}4\\0\end{bmatrix} - \frac{4}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix},\ \mathbf{q}_2 = \frac{\mathbf{A}_2}{\|\mathbf{A}_2\|} = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\-2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$(2)\ \begin{bmatrix}1 & 4\\1 & 0\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2\end{bmatrix}\begin{bmatrix}\mathbf{a}^T\mathbf{q}_1 & \mathbf{A}_2{}^T\mathbf{q}_1\\0 & \mathbf{A}_2{}^T\mathbf{q}_2\end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{\mathbf{Q}}\underbrace{\begin{bmatrix}\sqrt{2} & 2\sqrt{2}\\0 & 2\sqrt{2}\end{bmatrix}}_{\mathbf{R}}$$

7. (10 pts, 5 pts, 5 pts)

$$(1)\ \begin{cases}\overline{x} = 2\\ \overline{y} = 1\end{cases} \to \mathbf{B} = \mathbf{X} - \overline{\mathbf{X}} = \begin{bmatrix}4 & -5\\-5 & 4\\-4 & 5\\5 & -4\end{bmatrix},\ \mathbf{C} = \frac{1}{n-1}\mathbf{B}^T\mathbf{B} = \frac{1}{3}\begin{bmatrix}82 & -80\\-80 & 82\end{bmatrix} \to \begin{cases}\lambda_1 = 162,\ \mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\\ \lambda_2 = 2,\ \mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\end{cases}$$

(2) project onto the subspace spanned by $\mathbf{q}_1$ (larger eigenvalue) $:(x,y) \to \frac{x-y}{\sqrt{2}}$

$$(6,-4) \to \frac{10}{\sqrt{2}},\ (-3,5) \to \frac{-8}{\sqrt{2}},\ (-2,6) \to \frac{-8}{\sqrt{2}},\ (7,-3) \to \frac{10}{\sqrt{2}}$$

(3) if v is a right singular vector (column of $\mathbf{V}$) of $\mathbf{X}$, $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^T \to \mathbf{X}^T\mathbf{X} = \mathbf{V}\mathbf{D}^T\mathbf{U}^T\left(\mathbf{U}\mathbf{D}\mathbf{V}^T\right) = \mathbf{V}\mathbf{D}^2\mathbf{V}^T$