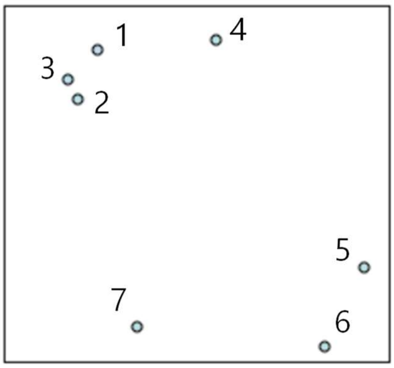
Midterm Exam

05/03/2023

1. Build a linear regression model for these data. (20 pts)

Height (cm)	166	176	171	173	169
Weight (kg)	58	75	62	70	60

2. (1) Perform K-means on the dataset below. Circles are data points and there are two initial cluster centers at data points 5 and 7. Draw the cluster centers (as squares) and the decision boundaries that define each cluster. If no points belong to a particular cluster, assume its center does not change. Use as many of the pictures as you need for convergence. (5 pts)



(2) Give advantages of hierarchical clustering over K-means clustering, and advantages of K-means clustering over hierarchical clustering. (5 pts)

3. Suppose $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of a matrix $\mathbf{A}$, and $\mathbf{v}_1^T = [1 \quad 0]$ and $\mathbf{v}_2^T = [1 \quad 1]$ are the corresponding eigenvectors.

(1) Calculate the matrix $\mathbf{A}$. (5 pts)

(2 Calculate the matrix $\mathbf{A}^8$, its eigenvalues and its eigenvectors. (5 pts)

(3) Let $u(t) = [u_1(t) \quad u_2(t)]^T$ satisfy $u'(t) = \mathbf{A}u(t)$ with $u(0) = [0 \quad 1]^T$. Solve for $u(t)$. (10 pts)

4. All entries in the factorization of $\mathbf{F}_6$ involve powers of $\omega = \text{sixth root of } 1$. $\mathbf{F}_6 = \begin{bmatrix} \mathbf{I} & \mathbf{D} \\ \mathbf{I} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{F}_3 & \\ & \mathbf{F}_3 \end{bmatrix} \begin{bmatrix} \text{even} \\ \text{odd} \end{bmatrix}$

Write down these factors and multiply to check. (10 pts)

5. Find a closet rank-1 approximation to these matrices (L^2 or Frobenius norm): (10 pts)

$$\mathbf{A}=\begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0&3\\2&0\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

6. (1) What multiple of $\mathbf{a}=\begin{bmatrix}1&1\end{bmatrix}^T$ should be subtracted from $\mathbf{b}=\begin{bmatrix}4&0\end{bmatrix}^T$ to make the result $\mathbf{A}_2$ orthogonal to $\mathbf{a}$? Sketch a figure to show $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{A}_2$. (5 pts)

   (2) Complete the Gram-Schmidt process by factoring into QR. (5 pts)

7. You are given a design matrix $\mathbf{X}=\begin{bmatrix}6&-4\\-3&5\\-2&6\\7&-3\end{bmatrix}$. Let's use PCA to reduce the dimension from 2 to 1.

(1) Compute the covariance matrix for the sample points. (Warning: observe that $\mathbf{X}$ is not centered.) Then compute the unit eigenvectors, and the corresponding eigenvalues, of the covariance matrix. Hint: If you graph the points, you can probably guess the eigenvectors. (10 pts)

(2) Suppose we use PCA to project the sample points onto a one-dimensional space. What one-dimensional subspace are we projecting onto? For each of the four sample points in $\mathbf{X}$, write the coordinate (in principal coordinate space, not in $\mathbb{R}^2$) that the point is projected to. (5 pts)

(3) Given a design matrix $\mathbf{X}$ that is taller than it is wide, prove that every right singular vector of $\mathbf{X}$ with singular value $\boldsymbol{\sigma}$ is an eigenvector of the covariance matrix with eigenvalue $\boldsymbol{\sigma}^2$. (5 pts)