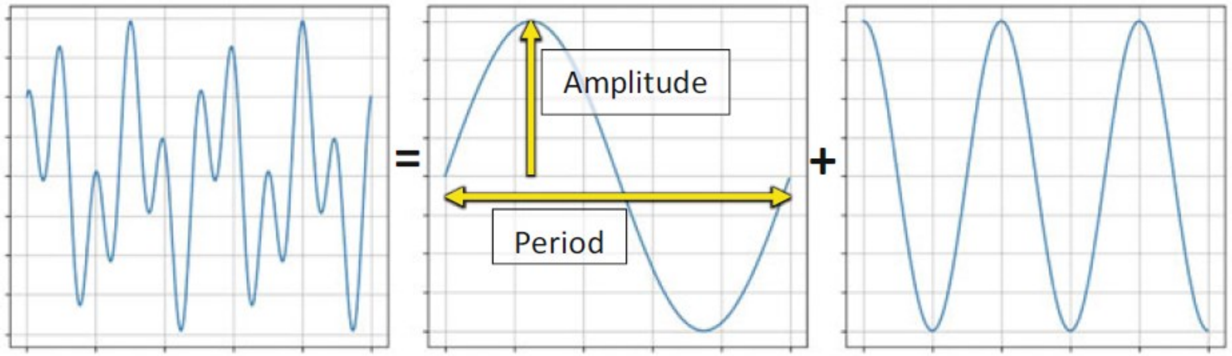
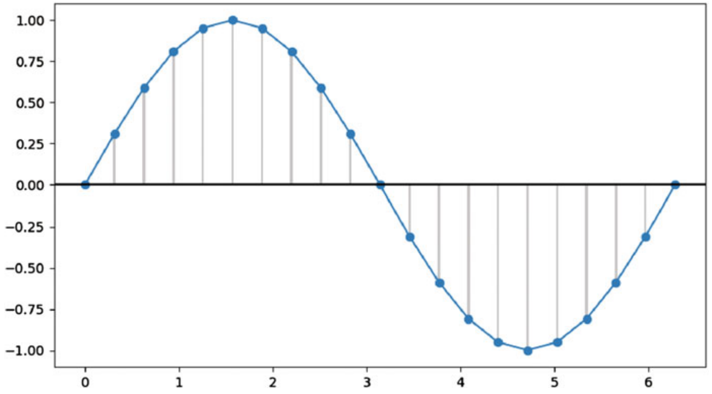
Fourier Transform (FT)

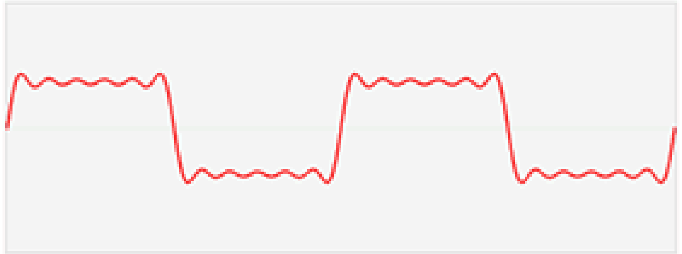
- everything from sounds to photographs can be described in terms of waves
  - it is assumed that there is a periodic (continuously repeating) pattern
  - Amplitude, period, frequency





$$
\begin{cases}
g(t) = a_0 + \sum_{i=1}^{n} a_i \sin(i\omega t) + \sum_{i=1}^{n} b_i \cos(i\omega t) \\
a_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} g(t) \sin(n\omega t)\, dt \\
b_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} g(t) \cos(n\omega t)\, dt
\end{cases}
\rightarrow
\begin{cases}
g(t) = A_0 + \sum_{i=1}^{n} A_i \sin(i\omega t + \theta_i) \\
{A_n}^2 = {a_n}^2 + {b_n}^2 \\
\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)
\end{cases}
$$

Fourier Series and Transform



$$\begin{cases} F(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + \cdots \\ \int_{-\pi}^{\pi} F(x) e^{-ikx} dx = 2\pi c_k \rightarrow c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-ikx} dx \end{cases}$$

$$F(x) \xrightarrow[\text{transfrom}]{\text{Fourier}} c_k \xrightarrow[\text{transfrom}]{\text{inverse Fourier}} F(x)$$

space domain — amplitude, k: frequency, frequency domain — space domain

$$\begin{cases} S(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where} \quad S(x+2\pi) = S(x),\ S(-x) = -S(x) \\ \int_0^{\pi} S(x) \sin kx dx = b_k \frac{\pi}{2} \rightarrow b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} S(x) \sin kx dx \end{cases}$$

$$\begin{cases} C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where} \quad C(-x) = C(x) \\ \int_0^{\pi} C(x) dx = a_0 \pi \rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(x) dx \\ \int_0^{\pi} C(x) \cos kx dx = a_k \frac{\pi}{2} \rightarrow a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} C(x) \cos kx dx \end{cases}$$

$$\begin{cases} F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = C(x) + S(x) \\ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx \\ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx dx \\ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin kx dx \end{cases}$$

Fourier Transform: Discrete and Continuous

	apply	basis
Real Fourier series	real periodic functions	$\cos nx, \sin nx$
Complex Fourier series	complex periodic functions	e^{inx} for $n = 0, \pm 1, \cdots$
Fourier integral transform	complex functions $f(x)$ for $-\infty < x < \infty$	e^{ikx} for $-\infty < k < \infty$
Discrete Fourier series	complex vectors $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$	$\begin{cases} N \text{ basis vectors } b_k \text{ with} \\ (b_k)_j = e^{2\pi ijk/N} = \left(e^{2\pi i/N}\right)^{jk} \end{cases}$

What is the "transform"?

It is the rule that connects f to its coefficients in these combinations $a_k, b_k, c_k, \hat{f}(k)$ of basis functions:

Real series: $f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos x + b_2 \sin x + \cdots$

Complex series: $f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots$

Fourier integrals: $f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$

Discrete series: $\mathbf{f} = c_0 \mathbf{b}_0 + c_1 \mathbf{b}_1 + \cdots + c_{N-1} \mathbf{b}_{N-1} = (\text{Fourier matrix } \mathbf{F})$ times $\mathbf{c}$

Each Fourier transform takes f in "x-space" to its coefficients in "frequency space". $(\mathbf{f} \to \mathbf{c}:$ analysis, fft$)$

The inverse transform starts with the coefficients and reconstructs the original function. $(\mathbf{c} \to \mathbf{f}:$ synthesis, ifft$)$

Fourier Matrix (F) and DFT Matrix (Ω)

- N x N, symmetric (but complex)
- Same columns but the order of columns is different

$$\left.\begin{array}{l}\mathbf{F}_N \text{ contains powers of } w = e^{2\pi i/N} \\ \Omega_N \text{ contains powers of the complex conjugate } \bar{w} = \omega = e^{-2\pi i/N}\end{array}\right\} \rightarrow \bar{\mathbf{F}}_N = \Omega_N$$

$$N = 4: \; w = e^{2\pi i/4} = i, \; \omega = e^{-2\pi i/N} = -i$$

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, \; \Omega_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\text{orthogonality check: } (\text{col } 0)^T (\text{col } 1), \; (\text{col } 1)^T (\text{col } 3), \; \left\|(\text{col } 1)\right\|^2$$

$$\frac{1}{4}\mathbf{F}_4\Omega_4 = \left(\frac{1}{2}\mathbf{F}_4\right)\left(\frac{1}{2}\Omega_4\right) = \mathbf{I}$$

$$\Omega_4 = \mathbf{F}_4\mathbf{P}, \; \mathbf{P}^2 = \mathbf{I} \rightarrow \mathbf{F}_4{}^2\mathbf{P} = \mathbf{F}_4\Omega_4 = 4\mathbf{I} \rightarrow \mathbf{F}_N{}^4 = \Omega_N{}^4 = N^2\mathbf{I}$$

Fourier Matrix (F) and DFT Matrix (Ω)

$$\mathbf{F}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \rightarrow \begin{cases} \mathbf{F}_{jk} = w^{jk} = \left(e^{2\pi i/N}\right)^{jk} = e^{2\pi ijk/N} \\ \mathbf{F}_N \Omega_N = N\mathbf{I} \rightarrow \text{proof} \\ \mathbf{F}_N^{-1} = \frac{1}{N}\Omega_N = \frac{1}{N}\overline{\mathbf{F}}_N \\ \left(\frac{1}{\sqrt{N}}\mathbf{F}_N\right)\left(\frac{1}{\sqrt{N}}\Omega_N\right) = \left(\frac{1}{\sqrt{N}}\mathbf{F}_N\right)\left(\frac{1}{\sqrt{N}}\overline{\mathbf{F}}_N\right) = \mathbf{I} \end{cases}$$

$$1,\ w, \cdots,\ w^{N-1} \rightarrow \begin{cases} \text{magnitude } 1 \\ \text{equally spaced around the unit circle in the complex plane} \\ N \text{ solutions of } z^N = 1 \end{cases}$$

- Unitary matrices
  - Two matrices are inverse and also complex conjugate
  - Complex version of orthogonal matrices

$$\begin{array}{c} \text{real} \leftrightarrow \text{complex} \\ \text{Symmetric}\left(\mathbf{S}^T = \mathbf{S}\right) \leftrightarrow \text{Hermitian}\left(\overline{\mathbf{S}}^T = \mathbf{S}\right) \\ \text{Orthogonal}\left(\mathbf{Q}^T = \mathbf{Q}^{-1}\right) \leftrightarrow \text{Unitary}\left(\overline{\mathbf{Q}}^T = \mathbf{Q}^{-1}\right) \end{array}$$

Discrete Fourier Transform

$$\begin{cases} F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \rightarrow f_j = \sum_{k=0}^{N-1} c_k w^{jk} \rightarrow \mathbf{f} = \mathbf{F}_N \mathbf{c} \\ w^{jk} \text{ is the same as } e^{ikx} \text{ at the } j\text{th point } x_j = j2\pi/N \\ c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-ikx} dx \rightarrow c_k = \frac{1}{N} = \sum_{j=0}^{N-1} f_j \bar{w}^{jk} \rightarrow \mathbf{c} = \mathbf{F}_N^{-1}\mathbf{f} \\ c_0 = \left(f_0 + \cdots + f_{N-1}\right)/N \end{cases}$$

Discrete series: $\mathbf{f} = c_0\mathbf{b}_0 + c_1\mathbf{b}_1 + \cdots + c_{N-1}\mathbf{b}_{N-1} = \left(\text{Fourier matrix } \mathbf{F}\right)$ times $\mathbf{c}$

$$\mathbf{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{b}_0 & \cdots & \mathbf{b}_{N-1} \end{bmatrix}}_{\text{Fourier basis vectors}} \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \mathbf{F}_N\mathbf{c} \rightarrow \begin{cases} \text{forward transform (analysis step): } \mathbf{c} = \mathbf{F}_N^{-1}\mathbf{f} = \frac{1}{N}\Omega_N\mathbf{f} = fft(\mathbf{f}) \\ \text{inverse transform (synthesis step): } \mathbf{f} = \mathbf{F}_N\mathbf{c} = ifft(\mathbf{c})\left[= ifft\left(fft(\mathbf{f})\right)\right] \end{cases}$$

$$\mathbf{F}_N\Omega_N = \mathbf{I} \rightarrow \Omega_N = \mathbf{F}_N^{-1}\mathbf{I} = fft(\mathbf{I}) = conj(\mathbf{F}_N)$$

Example: tranform of $\mathbf{f} = (1,\ 0, \ldots,\ 0)$ is $\mathbf{c} = \frac{1}{N}(1,\ 1, \ldots,\ 1)$.

Example: tranform of sine function.

Example: A shifted in the delta vector to $\mathbf{f} = (0,\ 1,\ 0, \ldots,\ 0)$ produces a "modulation" in its transform.

Fast Fourier Transform (1)

- Multiply F times c as quick as possible?
- Fourier matrix has no zeros!
- Factorize in a way that produces many zeros and recursion → FFT
- Idea: connect F_N with the half-size Fourier matrix $F_{N/2}$

$$w^4 = e^{2\pi i/4} = i = \left(w^8\right)^2$$

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \leftrightarrow \begin{bmatrix} \mathbf{F}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i \end{bmatrix} \to \mathbf{F}_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & 1 & \\ & 1 & & i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i & & \\ & & 1 & 1 \\ & & 1 & i \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

Cooley and Tukey

N is a power of 2, for example, $N = 2^{10} = 1024 \to \mathbf{F}_{1024} = \begin{bmatrix} \mathbf{I}_{512} & \mathbf{D}_{512} \\ \mathbf{I}_{512} & -\mathbf{D}_{512} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{512} & \\ & \mathbf{F}_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$

final count for size $N = 2^l$: $N^2 \to \frac{1}{2} Nl = \frac{1}{2} N \log_2 N$

$l = 10: \ (1024)^2 \xrightarrow{\text{factor of 200}} (5)(1024)$

Fast Fourier Transform (2)

$$M = \frac{N}{2}$$

$$\mathbf{y} = \mathbf{F}_N \mathbf{c} = \begin{bmatrix} \mathbf{I}_M & \mathbf{D}_M \\ \mathbf{I}_M & -\mathbf{D}_M \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & \\ & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix} \mathbf{c} = \begin{bmatrix} \mathbf{I}_M & \mathbf{D}_M \\ \mathbf{I}_M & -\mathbf{D}_M \end{bmatrix} \begin{bmatrix} \mathbf{y}' \\ \mathbf{y}'' \end{bmatrix} = \begin{bmatrix} \mathbf{I}_M \mathbf{y}' + \mathbf{D}_M \mathbf{y}'' \\ \mathbf{I}_M \mathbf{y}' - \mathbf{D}_M \mathbf{y}'' \end{bmatrix}, \quad \begin{bmatrix} \mathbf{y}' \\ \mathbf{y}'' \end{bmatrix} = \begin{bmatrix} \mathbf{F}_M \mathbf{c}' \\ \mathbf{F}_M \mathbf{c}'' \end{bmatrix}$$

$$\rightarrow \begin{cases} y_j = y'_j + \left(w_N\right)^j y''_j, \ \ j = 0, \ldots, M-1 \\ y_j = y'_j - \left(w_N\right)^j y''_j, \ \ j = 0, \ldots, M-1 \end{cases}$$

$$y_j = \sum_{j=0}^{N-1} w^{jk} c_k = \sum_{j=0}^{M-1} w^{j(2k)} c_{2k} + \sum_{j=0}^{M-1} w^{j(2k+1)} c_{2k+1} = \sum_{j=0}^{M-1} \left(w_M\right)^{jk} c'_k + \left(w_N\right)^j \sum_{j=0}^{M-1} \left(w_M\right)^{jk} c''_k = y'_j + \left(w_N\right)^j y''_j$$

$$\text{split } \mathbf{c} \text{ into} \begin{cases} \mathbf{c}' = \left(c_0, c_2, \ldots\right) \\ \mathbf{c}'' = \left(c_1, c_3, \ldots\right) \end{cases} \text{ and } \begin{cases} \left(w_N\right)^2 = w_M \\ \left(w_N\right)^{2jk} = \left(w_M\right)^{jk} \end{cases}$$

Shift Matrices and Circulant Matrices

upward shift cyclic permutation

$$\mathbf{Px} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}, \ \mathbf{P}^2\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\left(\mathbf{P}^3\right)\mathbf{P} = \mathbf{P}^4 = \mathbf{I}$$

Circulant matrix: $\mathbf{C} = c_0\mathbf{I} + c_1\mathbf{P} + c_2\mathbf{P}^2 + c_3\mathbf{P}^3 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$

eigenvalues and eigenvectors of $\mathbf{P}$ and $\mathbf{C}$

$$\begin{cases} \mathbf{Px} = \lambda\mathbf{x} \to \lambda\left(\mathbf{P}\right) = 1, w, \ldots w^{N-1}, \ \mathbf{q}_k \text{ is the column of Fourier matrix} \\ \mathbf{Cq}_k = \lambda\mathbf{q}_k \to \lambda\left(\mathbf{C}\right) = \mathbf{Fc} \end{cases}$$

Convolution

$\mathbf{CD} = \mathbf{DC}$

$$\mathbf{CD} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 4 & 5 & 0 \\ 0 & 4 & 5 \end{bmatrix}$$

When we multiply N by N circulant matrices **C** and **D**, we take the cyclic convolution of the vectors $(c_0, c_1, \ldots, c_{N-1})$ and $(d_0, d_1, \ldots, d_{N-1})$.

Ordinary convolution finds the coefficients when we multiply $\left(c_0\mathbf{I} + c_1\mathbf{P} + \cdots + c_{N-1}\mathbf{P}^{N-1}\right)$ times $\left(d_0\mathbf{I} + d_1\mathbf{P} + \cdots + d_{N-1}\mathbf{P}^{N-1}\right)$.

crucial fact: $\mathbf{P}^N = \mathbf{I}$

$$\begin{cases} \text{convolution: } (1,2,3) * (5,0,4) = (5,10,19,8,12) \\ \text{cyclic convolution: } (1,2,3) * (5,0,4) = (13,22,19) \rightarrow \text{quick check?} \end{cases}$$

$(0,1,0) * (d_0, d_1, d_2)$

$(1,1,1) * (d_0, d_1, d_2)$

$(c_0, c_1, c_2) * (d_0, d_1, d_2)$

Convolution Rule

$\text{top row of } \mathbf{CD} = \text{cyclic convolution} = \mathbf{c} * \mathbf{d} \rightarrow \lambda(\mathbf{CD}) = \mathbf{F}(\mathbf{c} * \mathbf{d})$

$\lambda(\mathbf{C}) = \mathbf{Fc},\ \lambda(\mathbf{D}) = \mathbf{Fd},\ \mathbf{q}_k$ are the same for $\mathbf{C}$ and $\mathbf{D}$

$\lambda(\mathbf{CD}) = \lambda(\mathbf{C})\lambda(\mathbf{D}) = (\mathbf{Fc}).*(\mathbf{Fd})$

$.*(\text{Hadamard product})$: component-by-component multiplication

convolution rule: $\mathbf{F}(\mathbf{c} * \mathbf{d}) = (\mathbf{Fc}).*(\mathbf{Fd})$

$\Lambda(\mathbf{C})\Lambda(\mathbf{D}) = (\mathbf{F}^{-1}\mathbf{CF})(\mathbf{F}^{-1}\mathbf{DF}) = \mathbf{F}^{-1}(\mathbf{CD})\mathbf{F} = \Lambda(\mathbf{CD})$

Example: $N = 2,\ \mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \mathbf{F}(\mathbf{c} * \mathbf{d}) = (\mathbf{Fc}).*(\mathbf{Fd})$

Convolution of Functions

periodic function f and g $\left(-\pi \le x \le \pi\right)$, and their Fourier series are infinite $\left(k = 0, \pm 1, \ldots\right)$

what are the Fourier coefficients of $f\left(x\right)g\left(x\right)$?

$$f\left(x\right)g\left(x\right) = \left(\sum_{k=-\infty}^{\infty} c_k e^{ikx}\right)\left(\sum_{m=-\infty}^{\infty} d_m e^{imx}\right) = \sum_{n=-\infty}^{\infty} h_n e^{inx}$$

The coefficients of h_n combines all products of $c_k d_m$ with $k + m = n$

$h_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}$ is convolution $\mathbf{h} = \mathbf{c} * \mathbf{d}$ for infinite vectors.

convolution of 2π-periodic functions: $\left(f * g\right)\left(x\right) = \int_{t=-\pi}^{\pi} f\left(t\right) g\left(x - t\right) dt$

convolution rule for periodic functions: Fourier coefficients of $f * g$ are $2\pi c_k d_k$

check with $f\left(x\right) = \sin x$