EXERCISE 11.3

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g A \ell \int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} d\zeta = \rho g A \ell \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \tfrac{1}{2} \rho g A \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad \text{(E11.6)}$$

This is the same answer given by element-by-element load lumping.

EXERCISE 11.4

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g \ell \int_0^1 \begin{bmatrix} \big[A_i(1-\zeta) + A_j\zeta\big](1-\zeta) \\ \big[A_i(1-\zeta) + A_j\zeta\big]\zeta \end{bmatrix} d\zeta = \rho g \ell \begin{bmatrix} \frac{1}{3}A_i + \frac{1}{6}A_j \\ \frac{1}{6}A_i + \frac{1}{3}A_j \end{bmatrix}. \qquad \text{(E11.7)}$$

Obviously if $A_i = A_j = A$ we recover (E11.6).

EXERCISE 12.1 A *Mathematica* script for $\mathbf{K}^e$ by analytical integration is shown in Figure E12.1.

```
ClearAll[EI,EIi,EIj,Le,ξ,ℓ]; Le=ℓ;
Be={{6*ξ,(3*ξ-1)*Le,-6*ξ,(3*ξ+1)*Le}};
EI=EI1*(1-ξ)/2+EI2*(1+ξ)/2;
Ke=1/(2*Le^3)*Integrate[EI*Transpose[Be].Be,{ξ,-1,1}];
Ke=Simplify[Ke];
Print["Ke for variable xsec beam:\n", Ke//MatrixForm];
ClearAll[EI];  Ke=Simplify[Ke/.{EI2->EI,EI2->EI}];
Print["Ke for EI1=EI2=EI is ", Ke//MatrixForm];
```

Ke for variable xsec beam:

$$\begin{pmatrix} \frac{6\,(\text{EI1}+\text{EI2})}{\ell^3} & \frac{2\,(2\,\text{EI1}+\text{EI2})}{\ell^2} & -\frac{6\,(\text{EI1}+\text{EI2})}{\ell^3} & \frac{2\,(\text{EI1}+2\,\text{EI2})}{\ell^2} \\ \frac{2\,(2\,\text{EI1}+\text{EI2})}{\ell^2} & \frac{3\,\text{EI1}+\text{EI2}}{\ell} & -\frac{2\,(2\,\text{EI1}+\text{EI2})}{\ell^2} & \frac{\text{EI1}+\text{EI2}}{\ell} \\ -\frac{6\,(\text{EI1}+\text{EI2})}{\ell^3} & -\frac{2\,(2\,\text{EI1}+\text{EI2})}{\ell^2} & \frac{6\,(\text{EI1}+\text{EI2})}{\ell^3} & -\frac{2\,(\text{EI1}+2\,\text{EI2})}{\ell^2} \\ \frac{2\,(\text{EI1}+2\,\text{EI2})}{\ell^2} & \frac{\text{EI1}+\text{EI2}}{\ell} & -\frac{2\,(\text{EI1}+2\,\text{EI2})}{\ell^2} & \frac{\text{EI1}+3\,\text{EI2}}{\ell} \end{pmatrix}$$

FIGURE E12.1. Script to solve Exercise 12.1

Transcribing the result:

$$\mathbf{K}^e = \frac{1}{\ell^3}\begin{bmatrix} 6(EI_1+EI_2) & 2(2EI_1+EI_2)\ell & -6(EI_1+EI_2) & 2(EI_1+EI_2)\ell \\ & (3EI_1+EI_2)\ell^2 & -2(2EI_1+EI_2)\ell & (EI_1+EI_2)\ell^2 \\ & & 6(EI_1+EI_2) & -2(2EI_1+2EI_2)\ell \\ symm & & & (EI_1+2EI_2)\ell^2 \end{bmatrix}. \qquad \text{(E12.10)}$$

The output of the check $EI_1 = EI_2 = EI$ is omitted to save space, but it does reproduce the matrix (12.20).

EXERCISE 12.2 A *Mathematica* script for $\mathbf{f}^e$ by analytical integration is shown in Figure E12.2.

```
ClearAll[q,q1,q2,ξ,ℓ]; Le=ℓ;
Ne={2*(1-ξ)^2*(2+ξ), (1-ξ)^2*(1+ξ)*Le,
    2*(1+ξ)^2*(2-ξ),-(1+ξ)^2*(1-ξ)*Le}/8;
q=q1*(1-ξ)/2+q2*(1+ξ)/2;
fe=Simplify[ (Le/2)*Integrate[q*Ne,{ξ,-1,1}] ];
Print["fe^T for lin varying load q:\n",fe];
ClearAll[q]; fe=Simplify[fe/.{q1->q,q2->q}];
Print["check for q1=q2=q: ",fe];
```

fe^T for lin varying load q:

$\{\frac{1}{20}(7\,q1+3\,q2)\,\ell \quad \frac{1}{60}(3\,q1+2\,q2)\,\ell^2 \quad \frac{1}{20}(3\,q1+7\,q2)\,\ell \quad -\frac{1}{60}(2\,q1+3\,q2)\,\ell^2\}$

FIGURE E12.2. Script to solve Exercise 12.2

Transcribing the result:

$$\mathbf{f}^e = \frac{\ell}{60}\,[\,3(7q_1+3q_2) \quad \ell(3q_1+2q_2) \quad 3(3q_1+7q_2) \quad \ell(2q_1+3q_2)\,]^T . \tag{E12.11}$$

Table 12.1. Results From Exercise 12.6

	Monomial degree						
	0	1	2	3	4	5	6
Exact	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{2}{7}$
One-point rule (E12.6)	2	0	0				
Two-point rule (E12.7)	2	0	$\frac{2}{3}$	0	$\frac{2}{9}$		
Three-point rule (E12.8)	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{6}{25}$

The output of the check $q_i = q_i = q$ is omitted to save space, but it does reproduce (12.21).