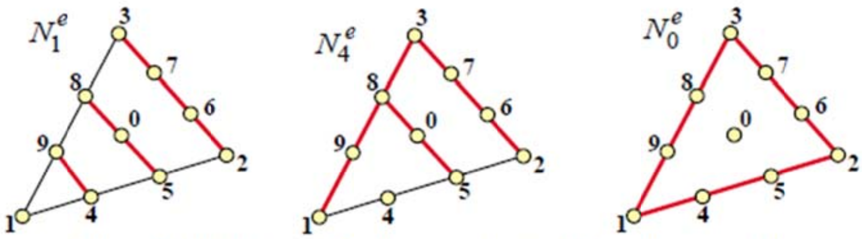

FIGURE E18.6. Lines used for direct construction of shape functions of the 10-node cubic triangle.

EXERCISE 18.1

(a) The shape functions for nodes 1, 4 and 0 of the 10-node cubic triangle can be obtained using the lines marked in red in Figure E18.6. The process is as follows.

$$\begin{aligned} N_1^e(\zeta_1,\zeta_2,\zeta_3) &= c_1\,\zeta_1(\zeta_1-\tfrac{1}{3})(\zeta_1-\tfrac{2}{3})\\ N_1^e(1,0,0) &= c_1\times 1\times \tfrac{2}{3}\times\tfrac{1}{3}=\tfrac{2}{9}c_1=1, \qquad \text{hence} \quad c_1=\tfrac{9}{2},\\ N_1^e &= \tfrac{9}{2}\zeta_1(\zeta_1-\tfrac{1}{3})(\zeta_1-\tfrac{2}{3})=\tfrac{1}{2}\zeta_1(3\zeta_1-1)(3\zeta_1-2) \end{aligned} \tag{E18.4}$$

$$\begin{aligned} N_4^e(\zeta_1,\zeta_2,\zeta_3) &= c_4\,\zeta_1\zeta_2(\zeta_1-\tfrac{1}{3})\\ N_4^e(\tfrac{2}{3},\tfrac{1}{3},0) &= c_4\times\tfrac{2}{3}\times\tfrac{1}{3}\times\tfrac{1}{3}=\tfrac{2}{27}c_4=1, \qquad \text{hence} \quad c_4=\tfrac{27}{2},\\ N_4^e &= \tfrac{27}{2}\zeta_1\zeta_2(\zeta_1-\tfrac{1}{3})=\tfrac{9}{2}\zeta_1\zeta_2(3\zeta_1-1) \end{aligned} \tag{E18.5}$$

$$\begin{aligned} N_0^e(\zeta_1,\zeta_2,\zeta_3) &= c_0\,\zeta_1\zeta_2\zeta_3\\ N_0^e(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) &= c_{10}\times\tfrac{1}{3}\times\tfrac{1}{3}\times\tfrac{1}{3}=\tfrac{1}{27}c_{10}=1, \qquad \text{hence} \quad c_{10}=27,\\ N_0^e &= 27\zeta_1\zeta_2\zeta_3. \end{aligned} \tag{E18.6}$$

The remaining shape functions can be obtained by appropriate index substitutions:

$$\begin{aligned} N_2^e &= \tfrac{9}{2}\zeta_2(\zeta_2-\tfrac{1}{3})(\zeta_2-\tfrac{2}{3}), & N_3^e &= \tfrac{9}{2}\zeta_3(\zeta_3-\tfrac{1}{3})(\zeta_3-\tfrac{2}{3}),\\ N_4^e &= \tfrac{27}{2}\zeta_1\zeta_2(\zeta_1-\tfrac{1}{3}), & N_5^e &= \tfrac{27}{2}\zeta_1\zeta_2(\zeta_2-\tfrac{1}{3}),\\ N_6^e &= \tfrac{27}{2}\zeta_2\zeta_3(\zeta_2-\tfrac{1}{3}), & N_7^e &= \tfrac{27}{2}\zeta_2\zeta_3(\zeta_3-\tfrac{1}{3}),\\ N_8^e &= \tfrac{27}{2}\zeta_3\zeta_1(\zeta_3-\tfrac{1}{3}), & N_9^e &= \tfrac{27}{2}\zeta_3\zeta_1(\zeta_1-\tfrac{1}{3}). \end{aligned} \tag{E18.7}$$

Here factors 9/2 and 27/2 may be partially absorbed to in the factors to make $\zeta_1-\frac{1}{3}\to 3\zeta_1-1$, etc.

(b) The verification that the sum is unity is done with the *Mathematica* script shown in Figure E18.7. Notice the substitution enforcing $\zeta_1+\zeta_2+\zeta_3=1$.

```
Nf={ζ1*(3*ζ1-1)*(3*ζ1-2)/2, ζ2*(3*ζ2-1)*(3*ζ2-2)/2,
    ζ3*(3*ζ3-1)*(3*ζ3-2)/2, 9*ζ1*ζ2*(3*ζ1-1)/2, 9*ζ1*ζ2*(3*ζ2-1)/2,
    9*ζ2*ζ3*(3*ζ2-1)/2, 9*ζ2*ζ3*(3*ζ3-1)/2, 9*ζ3*ζ1*(3*ζ3-1)/2,
    9*ζ3*ζ1*(3*ζ1-1)/2, 27*ζ1*ζ2*ζ3};
S=Simplify[Sum[Nf[[i]],{i,10}]/.{ζ3->1-ζ1-ζ2}]; Print[S];
```

```
1
```

FIGURE E18.7. Unit sum verification for 10-node cubic triangle.

EXERCISE 18.6 The shape function for the internal node 5 is the bubble function $N_5 = (1-\xi^2)(1-\eta^2)$, which also appears in Eq. (15.17) as N_9 of the 9-node quad. Take $N_i = \bar{N}_i + c_5 N_5$, where $\bar{N}_i$, $i = 1, 2, 3, 4$ are the shape functions (16.15) of the four-node bilinear quadrilateral. These functions are not zero at node 5, hence the correction by a multiple of the bubble function. Evaluation at node 5 ($\xi = \eta = 0$) shows that $c_5 = -1/4$. The corrected corner shape functions are $N_i = \bar{N}_i - N_5/4$, $i = 1, 2, 3, 4$. For example,

$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2). \qquad \text{(E18.10)}$$

These four corner shape functions vary linearly over all sides because the bubble function N_5 vanishes over the 4 sides. Hence interelement continuity is maintained.

Unit sum check: $N_1 + N_2 + N_3 + N_4 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 - 4 \times \tfrac{1}{4} N_5 + N_5 = \bar{N}_1 + \bar{N}_2 + \bar{N}_3 + \bar{N}_4 = 1$, since it is known that the shape functions $\bar{N}_i$ of the 4-node quad add up to one.