1.3

$(a)\ \frac{dv}{dt} = g - \frac{c'}{m}v^2 \rightarrow \frac{m}{c'}\frac{dv}{dt} = \frac{m}{c'}g - v^2 \xrightarrow{a=\sqrt{mg/c'}} \frac{m}{c'}\frac{dv}{dt} = a^2 - v^2$

$\int \frac{dv}{a^2 - v^2} = \int \frac{c'}{m}dt \rightarrow \frac{1}{a}\tanh^{-1}\frac{v}{a} = \frac{c'}{m}t + C \xrightarrow{v(0)=0} C = 0$

$v = \sqrt{\frac{mg}{c'}}\tanh\left(\sqrt{\frac{gc'}{m}}t\right)$

(b) Using Euler's method, the first two steps can be computed as

$v(2) = 0 + \left[9.8 - \frac{0.225}{68.1}(0)^2\right]2 = 19.6$

$v(4) = 19.6 + \left[9.8 - \frac{0.225}{68.1}(19.6)^2\right]2 = 36.6615$

1.12 (a) The force balance can be written as:

$$m\frac{dv}{dt} = -mg(0)\frac{R^2}{(R+x)^2} + cv$$

Dividing by mass gives

$$\frac{dv}{dt} = -g(0)\frac{R^2}{(R+x)^2} + \frac{c}{m}v$$

(b) Recognizing that $dx/dt = v$, the chain rule is

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

Setting drag to zero and substituting this relationship into the force balance gives

$$\frac{dv}{dx} = -\frac{g(0)}{v}\frac{R^2}{(R+x)^2}$$

(c) Using separation of variables

$$v\,dv = -g(0)\frac{R^2}{(R+x)^2}dx$$

Integrating gives

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + C$$

Applying the initial condition yields

$$\frac{v_0^2}{2} = g(0)\frac{R^2}{R+0} + C$$

which can be solved for $C = v_0^2/2 - g(0)R$, which can be substituted back into the solution to give

$$\frac{v^2}{2} = g(0)\frac{R^2}{R+x} + \frac{v_0^2}{2} - g(0)R$$

or

$$v = \pm\sqrt{v_0^2 + 2g(0)\frac{R^2}{R+x} - 2g(0)R}$$

Note that the plus sign holds when the object is moving upwards and the minus sign holds when it is falling.

(d) Euler's method can be developed as

$$v(x_{i+1}) = v(x_i) + \left[-\frac{g(0)}{v(x_i)} \frac{R^2}{(R + x_i)^2} \right] (x_{i+1} - x_i)$$

The first step can be computed as

$$v(10{,}000) = 1{,}400 + \left[-\frac{9.8}{1{,}400} \frac{(6.37 \times 10^6)^2}{(6.37 \times 10^6 + 0)^2} \right] (10{,}000 - 0) = 1{,}400 + (-0.007)10{,}000 = 1{,}330$$

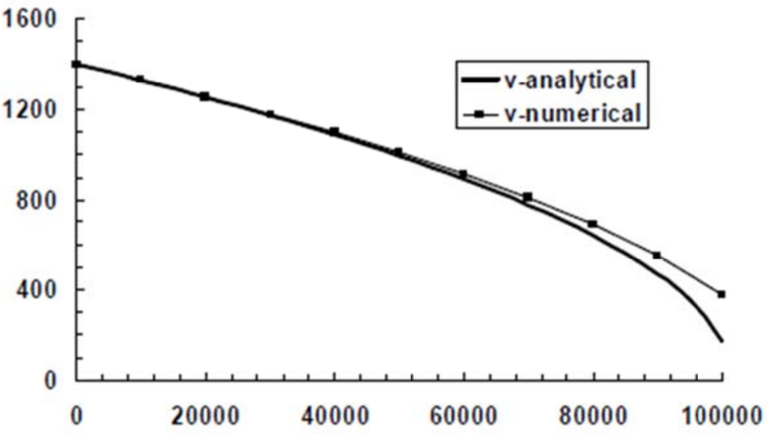
The remainder of the calculations can be implemented in a similar fashion as in the following table

x	*v*	*dv/dx*	*v*-analytical
0	1400.000	-0.00700	1400.000
10000	1330.000	-0.00735	1328.272
20000	1256.547	-0.00775	1252.688
30000	1179.042	-0.00823	1172.500
40000	1096.701	-0.00882	1086.688
50000	1008.454	-0.00957	993.796
60000	912.783	-0.01054	891.612
70000	807.413	-0.01188	776.473
80000	688.661	-0.01388	641.439
90000	549.864	-0.01733	469.650
100000	376.568	-0.02523	174.033

For the analytical solution, the value at 10,000 m can be computed as

$$v = \sqrt{1{,}400^2 + 2(9.8)\frac{(6.37 \times 10^6)^2}{(6.37 \times 10^6 + 10{,}000)^2} - 2(9.8)(6.37 \times 10^6)} = 1{,}328.272$$

The remainder of the analytical values can be implemented in a similar fashion as in the last column of the above table. The numerical and analytical solutions can be displayed graphically.



3.6 For the first series, after 20 terms are summed, the result is

	A	B	C	D	E	F	G	H
1	x	5						
2	n	n!	x^n/n!	Sign	Series	True Value	et (%)	ea(%)
3	0	1	1	1	1.000000E+00	6.737947E-03	14741.32	
4	1	1	5	-1	-4.000000E+00	6.737947E-03	59465.26	125.0000000
5	2	2	12.5	1	8.500000E+00	6.737947E-03	126051.2	147.0588235
6	3	6	20.83333	-1	-1.233333E+01	6.737947E-03	183142.9	168.9189189
7	4	24	26.04167	1	1.370833E+01	6.737947E-03	203349.7	189.9696049
8	5	120	26.04167	-1	-1.233333E+01	6.737947E-03	183142.9	211.1486486
9	6	720	21.70139	1	9.368056E+00	6.737947E-03	138934.3	231.6530764
10	7	5040	15.50099	-1	-6.132937E+00	6.737947E-03	91120.85	252.7499191
11	8	40320	9.68812	1	3.555184E+00	6.737947E-03	52663.6	272.5068890
12	9	362880	5.382289	-1	-1.827105E+00	6.737947E-03	27216.65	294.5801032
13	10	3628800	2.691144	1	8.640391E-01	6.737947E-03	12723.48	311.4609662
14	11	39916800	1.223247	-1	-3.592084E-01	6.737947E-03	5431.125	340.5397724
15	12	4.79E+08	0.509686	1	1.504780E-01	6.737947E-03	2133.292	338.7115011
16	13	6.23E+09	0.196033	-1	-4.555520E-02	6.737947E-03	776.0992	430.3202153
17	14	8.72E+10	0.070012	1	2.445667E-02	6.737947E-03	262.9692	286.2690254
18	15	1.31E+12	0.023337	-1	1.119380E-03	6.737947E-03	83.38693	2084.8412684
19	16	2.09E+13	0.007293	1	8.412283E-03	6.737947E-03	24.84936	86.6935085
20	17	3.56E+14	0.002145	-1	6.267312E-03	6.737947E-03	6.984846	34.2247480
21	18	6.4E+15	0.000596	1	6.863137E-03	6.737947E-03	1.857988	8.6815321
22	19	1.22E+17	0.000157	-1	6.706341E-03	6.737947E-03	0.469074	2.3380286
23	20	2.43E+18	3.92E-05	1	6.745540E-03	6.737947E-03	0.112692	0.5811105

The result oscillates at first. By $n = 20$ (21 terms), it is starting to converge on the true value. However, the relative error is still a substantial 0.11%. If carried out further to $n = 27$, the series eventually converges to within 7 significant digits.

In contrast the second series converges much faster. It attains 6 significant digits by $n = 20$ with a percent relative error of 8.1×10^{-6}%.

	A	B	C	D	E	F	G	H
1	x	5						
2	n	n!	x^n/n!	Series	1/Series	True Value	et (%)	ea(%)
3	0	1	1	1.000000E+00	1.000000E+00	6.737947E-03	1.47E+04	
4	1	1	5	6.000000E+00	1.666667E-01	6.737947E-03	2.37E+03	5.00E+02
5	2	2	12.5	1.850000E+01	5.405405E-02	6.737947E-03	7.02E+02	2.08E+02
6	3	6	20.83333	3.933333E+01	2.542373E-02	6.737947E-03	2.77E+02	1.13E+02
7	4	24	26.04167	6.537500E+01	1.529637E-02	6.737947E-03	1.27E+02	6.62E+01
8	5	120	26.04167	9.141667E+01	1.093892E-02	6.737947E-03	6.23E+01	3.98E+01
9	6	720	21.70139	1.131181E+02	8.840322E-03	6.737947E-03	3.12E+01	2.37E+01
10	7	5040	15.50099	1.286190E+02	7.774898E-03	6.737947E-03	1.54E+01	1.37E+01
11	8	40320	9.68812	1.383072E+02	7.230283E-03	6.737947E-03	7.31E+00	7.53E+00
12	9	362880	5.382289	1.436895E+02	6.959453E-03	6.737947E-03	3.29E+00	3.89E+00
13	10	3628800	2.691144	1.463806E+02	6.831506E-03	6.737947E-03	1.39E+00	1.87E+00
14	11	39916800	1.223247	1.476038E+02	6.774891E-03	6.737947E-03	5.48E-01	8.36E-01
15	12	4.79E+08	0.509686	1.481135E+02	6.751577E-03	6.737947E-03	2.02E-01	3.45E-01
16	13	6.23E+09	0.196033	1.483096E+02	6.742653E-03	6.737947E-03	6.98E-02	1.32E-01
17	14	8.72E+10	0.070012	1.483796E+02	6.739472E-03	6.737947E-03	2.26E-02	4.72E-02
18	15	1.31E+12	0.023337	1.484029E+02	6.738412E-03	6.737947E-03	6.90E-03	1.57E-02
19	16	2.09E+13	0.007293	1.484102E+02	6.738081E-03	6.737947E-03	1.99E-03	4.91E-03
20	17	3.56E+14	0.002145	1.484124E+02	6.737983E-03	6.737947E-03	5.42E-04	1.45E-03
21	18	6.4E+15	0.000596	1.484130E+02	6.737956E-03	6.737947E-03	1.40E-04	4.01E-04
22	19	1.22E+17	0.000157	1.484131E+02	6.737949E-03	6.737947E-03	3.45E-05	1.06E-04
23	20	2.43E+18	3.92E-05	1.484131E+02	6.737948E-03	6.737947E-03	8.11E-06	2.64E-05

3.7 The true value can be computed as

$$f'(0.577) = \frac{6(0.577)}{(1 - 3 \times 0.577^2)^2} = 2{,}352{,}911$$

Using 3-digits with chopping

$$\begin{aligned}
6x &= 6(0.577) = 3.462 \xrightarrow{\text{chopping}} 3.46 \\
x &= 0.577 \\
x^2 &= 0.332929 \xrightarrow{\text{chopping}} 0.332 \\
3x^2 &= 0.996 \\
1 - 3x^2 &= 0.004
\end{aligned}$$

$$f'(0.577) = \frac{3.46}{(1 - 0.996)^2} = \frac{3.46}{0.004^2} = 216{,}250$$

This represents a percent relative error of

$$\varepsilon_t = \left| \frac{2{,}352{,}911 - 216{,}250}{2{,}352{,}911} \right| = 90.8\%$$

Using 4-digits with chopping

$$\begin{aligned}
6x &= 6(0.577) = 3.462 \xrightarrow{\text{chopping}} 3.462 \\
x &= 0.577 \\
x^2 &= 0.332929 \xrightarrow{\text{chopping}} 0.3329 \\
3x^2 &= 0.9987 \\
1 - 3x^2 &= 0.0013
\end{aligned}$$

$$f'(0.577) = \frac{3.462}{(1 - 0.9987)^2} = \frac{3.462}{0.0013^2} = 2{,}048{,}521$$

This represents a percent relative error of

$$\varepsilon_t = \left| \frac{2{,}352{,}911 - 2{,}048{,}521}{2{,}352{,}911} \right| = 12.9\%$$

Although using more significant digits improves the estimate, the error is still considerable. The problem stems primarily from the fact that we are subtracting two nearly equal numbers in the denominator. Such subtractive cancellation is worsened by the fact that the denominator is squared.

3.8 First, the correct result can be calculated as

$$y = 1.37^3 - 7(1.37)^2 + 8(1.37) - 0.35 = 0.043053$$

(*a*) Using 3-digits with chopping

1.37^3	$\rightarrow$	2.571353	$\rightarrow$	2.57
$-7(1.37)^2$	$\rightarrow$	$-7(1.87) \rightarrow$	-13.0	
$8(1.37)$	$\rightarrow$	10.96	$\rightarrow$	10.9
				$-\underline{\ 0.35}$
				0.12

This represents an error of

$$\varepsilon_t = \left| \frac{0.043053 - 0.12}{0.043053} \right| = 178.7\%$$

(*b*) Using 3-digits with chopping

$$y = ((1.37 - 7)1.37 + 8)1.37 - 0.35$$
$$y = (-5.63 \times 1.37 + 8)1.37 - 0.35$$
$$y = (-7.71 + 8)1.37 - 0.35$$
$$y = 0.29 \times 1.37 - 0.35$$
$$y = 0.397 - 0.35$$
$$y = 0.047$$

This represents an error of

$$\varepsilon_t = \left| \frac{0.043053 - 0.047}{0.043053} \right| = 9.2\%$$

Hence, the second form is superior because it tends to minimize round-off error.

4.5 True value: $f(3) = 554$.

<u>zero order:</u>

$$f(3) = f(1) = -62 \qquad \varepsilon_t = \left|\frac{554-(-62)}{554}\right| \times 100\% = 111.191\%$$

<u>first order:</u>

$$f(3) = -62 + f'(1)(3-1) = -62 + 70(2) = 78 \qquad \varepsilon_t = 85.921\%$$

<u>second order:</u>

$$f(3) = 78 + \frac{f''(1)}{2}(3-1)^2 = 78 + \frac{138}{2}4 = 354 \qquad \varepsilon_t = 36.101\%$$

<u>third order:</u>

$$f(3) = 354 + \frac{f^{(3)}(1)}{6}(3-1)^3 = 354 + \frac{150}{6}8 = 554 \qquad \varepsilon_t = 0\%$$

Thus, the third-order result is perfect because the original function is a third-order polynomial.

4.6 True value:

$$f'(x) = 75x^2 - 12x + 7$$

$$f'(2) = 75(2)^2 - 12(2) + 7 = 283$$

function values:

$x_{i-1} = 1.8$ $\quad f(x_{i-1}) = 50.96$
$x_i = 2$ $\quad f(x_i) = 102$
$x_{i+1} = 2.2$ $\quad f(x_{i+1}) = 164.56$

forward:

$$f'(2) = \frac{164.56 - 102}{0.2} = 312.8 \qquad \varepsilon_t = \left|\frac{283 - 312.8}{283}\right| \times 100\% = 10.53\%$$

backward:

$$f'(2) = \frac{102 - 50.96}{0.2} = 255.2 \qquad \varepsilon_t = \left|\frac{283 - 255.2}{283}\right| \times 100\% = 9.823\%$$

centered:

$$f'(2) = \frac{164.56 - 50.96}{2(0.2)} = 284 \qquad \varepsilon_t = \left|\frac{283 - 284}{283}\right| \times 100\% = 0.353\%$$

Both the forward and backward have errors that can be approximated by (recall Eq. 4.15),

$$|E_t| \approx \frac{f''(x_i)}{2}h$$

$$f''(2) = 150x - 12 = 150(2) - 12 = 288$$

$$|E_t| \approx \frac{288}{2}0.2 = 28.8$$

This is very close to the actual error that occurred in the approximations

forward: $\quad |E_t| \approx |283 - 312.8| = 29.8$
backward: $\quad |E_t| \approx |283 - 255.2| = 27.8$

The centered approximation has an error that can be approximated by,

$$E_t \approx -\frac{f^{(3)}(x_i)}{6}h^2 = -\frac{150}{6}0.2^2 = -1$$

which is exact: $E_t = 283 - 284 = -1$. This result occurs because the original function is a cubic equation which has zero fourth and higher derivatives.

4.7 True value:

$$f''(x) = 150x - 12$$
$$f''(2) = 150(2) - 12 = 288$$

$\underline{h = 0.25:}$

$$f''(2) = \frac{f(2.25) - 2f(2) + f(1.75)}{0.25^2} = \frac{182.1406 - 2(102) + 39.85938}{0.25^2} = 288$$

$\underline{h = 0.125:}$

$$f''(2) = \frac{f(2.125) - 2f(2) + f(1.875)}{0.125^2} = \frac{139.6738 - 2(102) + 68.82617}{0.125^2} = 288$$

Both results are exact because the errors are a function of 4^{th} and higher derivatives which are zero for a 3^{rd}-order polynomial.