1. (1) 목적함수 5점, 구속조건 5점

(design variables) N =number of tubes, R =radius of each tube (cm)

(objective function) maximize surface area of tubes N(2πR)l (cm^2)

(constraints) R ≥ 0.5, N(πR^2) ≤ 2000, N ≥ 0 (integer)

$$\left.\begin{array}{l}\text{maximize } S = N(2\pi R)l \\ \text{subject to} \\ R \ge 0.5 \\ N(\pi R^2) \le 2000 \\ N \ge 0\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{minimize } f = -2\pi l N R \\ \text{subject to} \\ g_1 = 0.5 - R \le 0 \\ g_2 = \pi N R^2 - 2000 \le 0 \\ g_3 = -N \le 0\end{array}\right.$$

(2)

(design variables) x_1 = product A in kg, x_2 = product B in kg

(objective function) maximize the profit = $10x_1 + 8x_2$

(constraints) $0.4x_1 + 0.5x_2 \le 100$ (raw material C), $0.6x_1 + 0.5x_2 \le 80$ (raw material D),

$0 \le x_1 \le 70$, $0 \le x_2 \le 110$ (limits on products)

(3)

Design variables: x_{ij} = number of items produced at the i-th facility shipped to j-th distribution center where i = 1 to m, j = 1 to n

Objective function: cost = $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \quad i = 1,\ldots,m,\ j = 1,\ldots,n$

Constraints:

(capacity of manufacturing facility) $\sum_{j=1}^{n} x_{ij} \le b_i \quad i = 1,\ldots,m$

(demand) $\sum_{i=1}^{m} x_{ij} \ge a_j \quad j = 1,\ldots,n$

(side) $x_{ij} \ge 0$ for all i and j

2.

$$\nabla f = \begin{bmatrix} 3x_1^2 + 4x_1 \\ 3x_2^2 + 8x_2 \end{bmatrix} = 0 \rightarrow (0,0)(0,-\frac{8}{3})(-\frac{4}{3},0)(-\frac{4}{3},-\frac{8}{3}), \nabla^2 f = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

$\mathbf{x}$	Hessian	nature	$f(\mathbf{x})$
(0, 0)	Positive definite	Local minimum	6
(0, -8/3)	Indefinite	Saddle point	418/27
(-4/3, 0)	Indefinite	Saddle point	194/27
(-4/3, -8/3)	Negative definite	Local maximum	50/3

3.

$$\begin{cases} \text{unscaled problem}: \nabla f + u_u \nabla h = 0 \rightarrow \nabla f = -u_u \nabla h \\ \text{scaled problem}: k_1 \nabla f + u_s k_2 \nabla h = 0 \rightarrow (-u_u k_1 + u_s k_2)\nabla h = 0 \end{cases}$$

assuming that at a solution point $\nabla h \neq 0$, $-u_u k_1 + u_s k_2 = 0 \rightarrow u_s = \frac{k_1}{k_2} u_u$

4. $\boldsymbol{x}^*$ = (0, 0) is not an optimal point. (10 pts.)

$$L = (x_1 - 1)^2 + x_2^{\,2} + u(x_1 - x_2^{\,2} + s^2)$$
$$x = (0,0) \rightarrow s = 0$$
(necessary condition)

$$\left.\begin{aligned} \partial L/\partial x_1 &= 2(x_1 - 1) + u = 0 \\ \partial L/\partial x_2 &= 2x_2 - 2ux_2 = 0 \\ x_1 - x_2^{\,2} + s^2 &= 0 \\ us &= 0 \end{aligned}\right\} \rightarrow \begin{cases} u = 2 \geq 0 \\ \text{satisfy!} \end{cases}$$

(sufficient condition)

$$\nabla^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2-2u \end{bmatrix} \rightarrow \nabla^2 L(x^*, u^*) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}: \text{indefinite}$$

$$\nabla g(x^*)^T \boldsymbol{y} = 0 \rightarrow [1 \quad 0]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \rightarrow \boldsymbol{y} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$$

$$\boldsymbol{y}^T \nabla^2 L \boldsymbol{y} = [0 \quad y_2]\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ y_2 \end{bmatrix} = -2y_2^{\,2} < 0: \text{NOT satisfy!}$$

5.

(1) problem formulation

$$\begin{aligned} &\textit{Minimize} \quad f = 400(0.5\pi D^2 + \pi DH) \\ &\textit{subject to} \quad h_1 = \frac{\pi D^2 H}{4} - 250\pi = 0 \\ &\qquad\qquad\quad g_1 = H - 8D \leq 0 \end{aligned}$$

(2) check for convexity: nonlinear equality constraint → nonconvex

$$\nabla f = \begin{bmatrix} 400\pi D + 400\pi H \\ 400\pi D \end{bmatrix}, \quad \nabla^2 f = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix}$$

since Hessian of the cbjective function is NOT positive definite,
this is NOT a convex programming problem.

(3) KT necessary conditions

$$L = 400\left(0.5\pi D^2 + \pi DH\right) + v_1\left(\frac{\pi D^2 H}{4} - 250\pi\right) + u_1\left(H - 8D\right)$$

$$\frac{\partial L}{\partial D} = 400\pi D + 400\pi H + v_1\frac{\pi DH}{2} + u_1\left(-8\right) = 0$$

$$\frac{\partial L}{\partial H} = 400\pi D + v_1\frac{\pi D^2}{4} + u_1 = 0$$

$$h_1 = 0$$

$$g_1 \le 0,\ \ u_1 g_1 = 0,\ \ u_1 \ge 0$$

(4) solve the KT conditions

$$i)\ g_1 = 0 \to H = 8D \to D = 5, H = 40, v_1 = -226.7, u_1 = -1832.5 < 0(\times)$$

$$ii)\ u_1 = 0 \to D = H = 10, v_1 = -160, g_1 = H - 8D = -70 < 0 \to f^* = 60000\pi$$

sufficiency check:

$$\nabla^2 L = \nabla^2 f + v_1 \nabla^2 h_1 = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix} - 160\begin{bmatrix} 0.5\pi H & 0.5\pi D \\ 0.5\pi D & 0 \end{bmatrix} = \begin{bmatrix} 400\pi - 80\pi H & 400\pi - 80\pi D \\ 400\pi - 80\pi D & 0 \end{bmatrix}$$

$$\nabla^2 L\left(\boldsymbol{x}^*\right) = \begin{bmatrix} -400\pi & -400\pi \\ -400\pi & 0 \end{bmatrix} : \text{NOT positive definite!}$$

$$\nabla h_1\left(\boldsymbol{x}^*\right)^T \boldsymbol{d} = 25\pi\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \to \boldsymbol{d} = c\begin{bmatrix} 1 & -2 \end{bmatrix}^T$$

$$\boldsymbol{d}^T \nabla^2 L\left(\boldsymbol{x}^*\right)\boldsymbol{d} = -400\pi c^2\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1200\pi c^2 > 0 \to \text{isolated local minimum}$$

(5) post-optimality analysis

$$h_1 : 250\pi \to 255\pi$$

$$\Delta f = -v_1 \Delta b = -(-160)(255\pi - 250\pi) = 800\pi$$