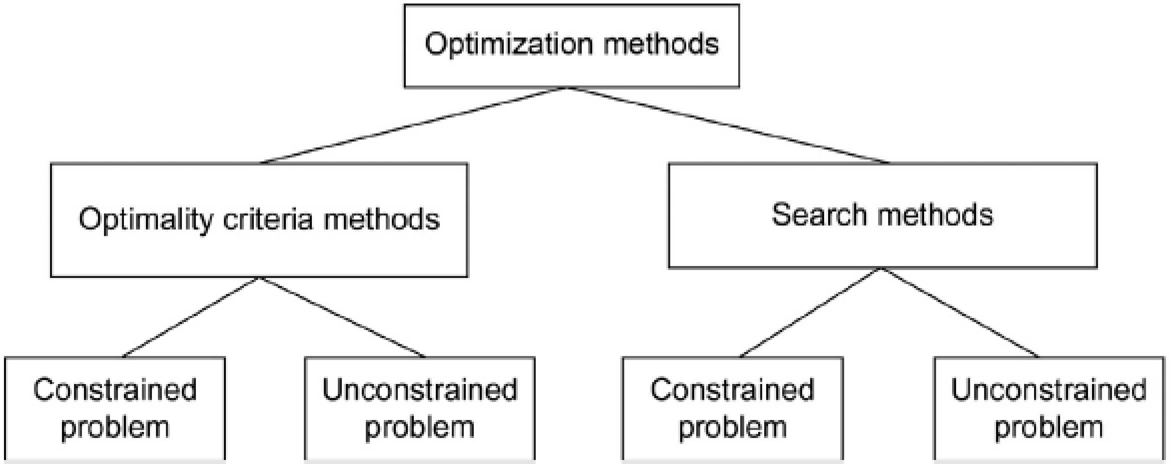
Contents

- Definitions of global and local minima
- Review of some basic calculus concepts
- Concepts of necessary and sufficient conditions
- Optimality conditions: unconstrained problem
- Necessary conditions: equality-constrained problem
- Necessary conditions for a general constrained problem
- Post-optimality analysis: the physical meaning of Lagrange multipliers
- Global optimality
- Second-order conditions for constrained optimization
- ~~Duality in nonlinear programming~~

Classification of Optimization Methods

- Optimality criteria methods (Ch.4~5)
  - Conditions a function must satisfy at its minimum point
  - Seeking solutions to the optimality conditions
- Search methods (Ch.6~13)
  - Numerically searching the design space: direct approach
  - Start with an estimate of the optimum design
  - Search the design space for optimum points



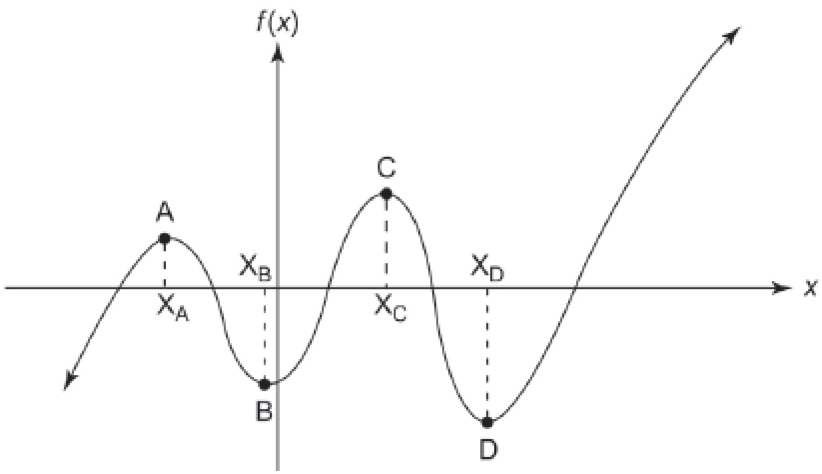
Minimum

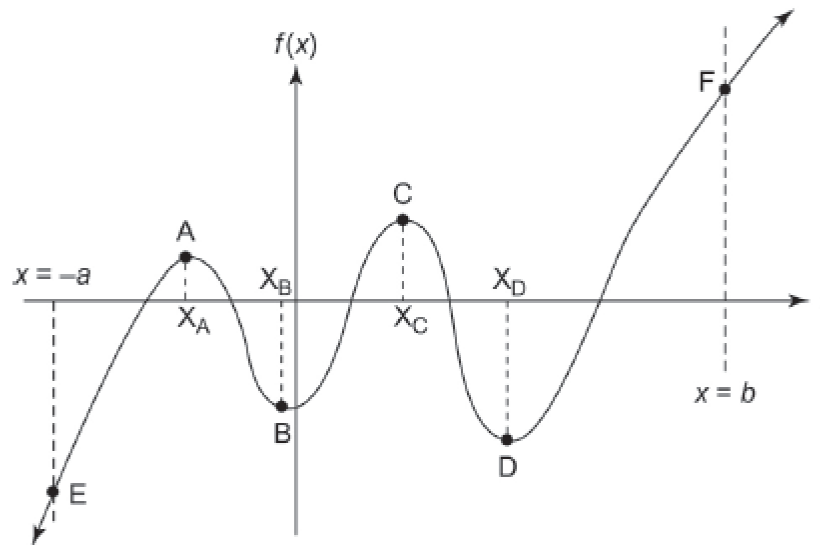
- Global (absolute) minimum

$$f(\mathbf{x}^*) \le f(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in the feasible region (constraint set } S)$$

- Local (relative) minimum

$$f(\mathbf{x}^*) \le f(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in a small } \textit{neighborhood } N \text{ of } \mathbf{x}^*$$
$$\text{in the feasible region (constraint set } S)$$
$$N = \{\mathbf{x} | \mathbf{x} \in S \quad \text{with} \quad \|\mathbf{x} - \mathbf{x}^*\| < \delta\} \text{ for some small } \delta > 0$$





Example 4.1+2+3

- Find the local and global minima for the function $f(x, y)$ using the graphical method

Minimize

$$f(x,y) = (x-4)^2 + (y-6)^2 \quad \text{(a)}$$

subject to

$$g_1 = x + y - 12 \le 0 \quad \text{(b)}$$

$$g_2 = x - 8 \le 0 \quad \text{(c)}$$

$$g_3 = -x \le 0\ (x \ge 0) \quad \text{(d)}$$

$$g_4 = -y \le 0\ (y \ge 0) \quad \text{(e)}$$

Minimize

$$f(x,y) = (x-10)^2 + (y-8)^2 \quad \text{(a)}$$

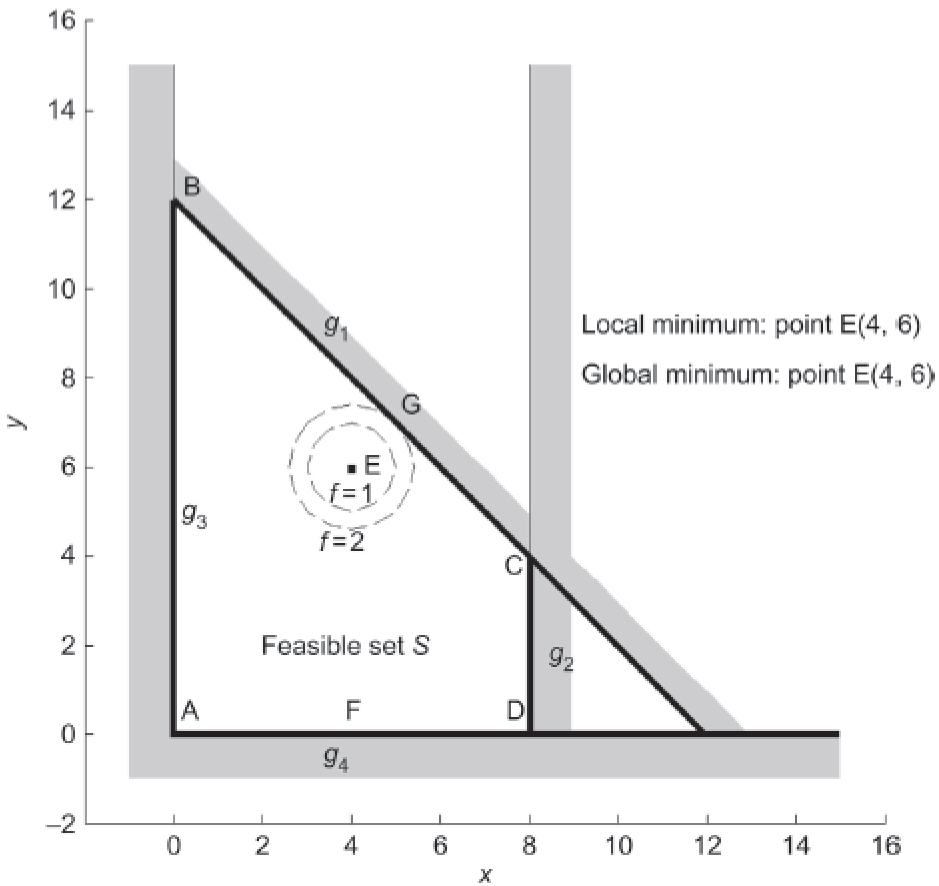
subject to the same constraints as in Example 4.1, Eqs. (b)–(e).

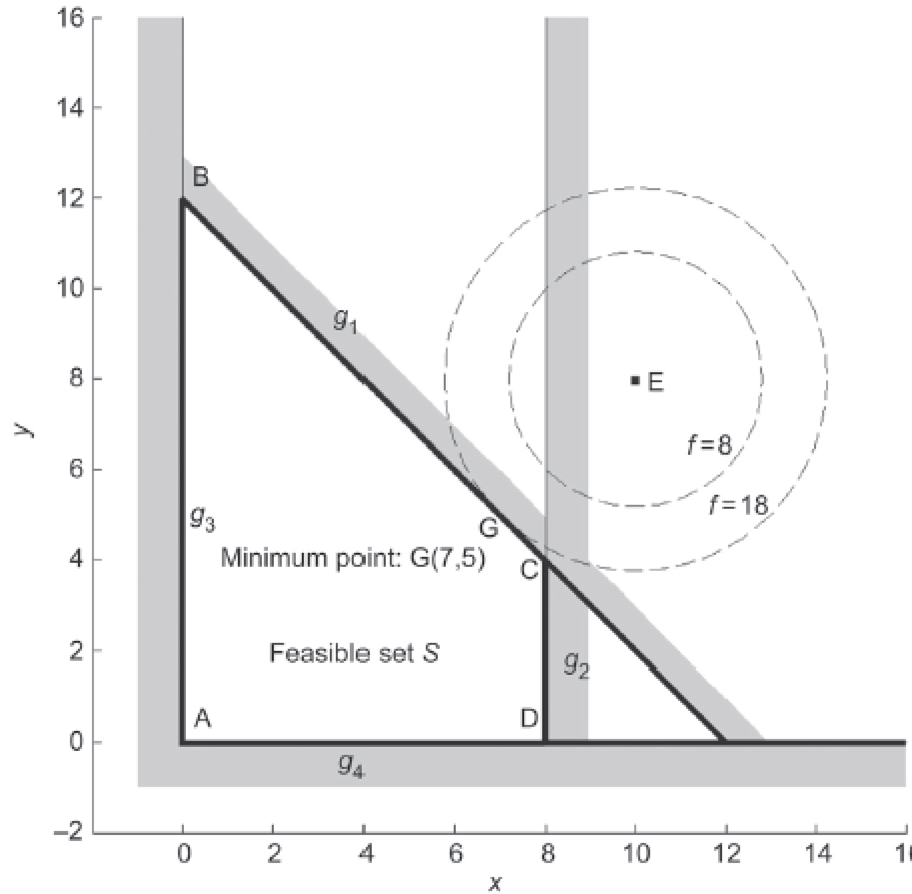
Maximize

$$f(x,y) = (x-4)^2 + (y-6)^2 \quad \text{(a)}$$

subject to the same constraints as in Example 4.1, Eqs. (b)–(e).

Example 4.1+2+3





Weierstrass Theorem

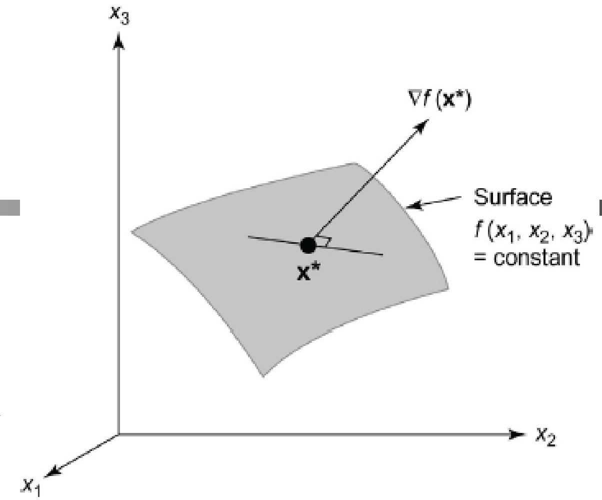
- Existence of global minimum
  - If $f(x)$ is continuous on a nonempty feasible set S which is closed and bounded, then $f(x)$ has a global minimum in S.
  - A set S is *closed* if it includes all its boundary points and every sequence of points has a subsequence that converges to a point in the set.
  - A set S is *bounded* if for any point $\mathbf{x} \in S$, $\mathbf{x}^T\mathbf{x} < c$, where c is a finite number.
  - The theorem does not rule out the possibility of a global minimum if its conditions are not met. (not an "if and only if" theorem)

$$e.g., \quad f(x) = -1/x$$

$$\begin{cases} S = \{x \mid 0 < x \le 1\} & : \text{ not closed} \\ S = \{x \mid 0 \le x \le 1\} & : \text{ closed and bounded, not continuous} \end{cases}$$

$$e.g., \quad f(x) = x^2 \text{ subject to } -1 < x < 1$$

Fundamentals



- Gradient vector
  - Normal to the tangent plane at the point **x***

$$\nabla f\left(\mathbf{x}^*\right)=\frac{\partial f\left(\mathbf{x}^*\right)}{\partial \mathbf{x}}=\text{grad } f\left(\mathbf{x}^*\right)=\left[\frac{\partial f\left(\mathbf{x}^*\right)}{\partial x_1} \quad \cdots \quad \frac{\partial f\left(\mathbf{x}^*\right)}{\partial x_n}\right]^T$$

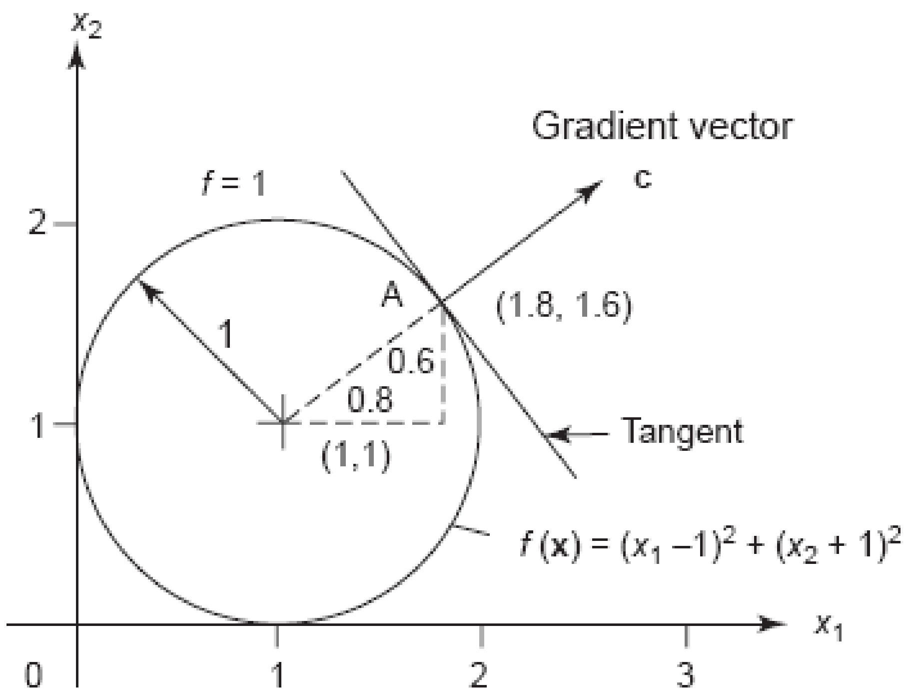
- Hessian matrix
  - Always a symmetric matrix

$$\mathbf{H}=\nabla^2 f\left(\mathbf{x}^*\right)=\frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial \mathbf{x} \partial \mathbf{x}}=\begin{bmatrix} \frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial x_1{}^2} & \cdots & \frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial x_n{}^2} \end{bmatrix}^T=\left[\frac{\partial^2 f\left(\mathbf{x}^*\right)}{\partial x_i \partial x_j}\right]^T$$

Example

– 점 x* = (1.8, 1.6)에서 다음 함수의 경사도벡터(Gradient)를 구하라.

$$f(\boldsymbol{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$$



Taylor Series Expansion (1)

– Polynomial approximation in a neighborhood of any point in terms of its value and derivatives

- Single variable

$$f(x) = f(x^*) + \frac{df(x^*)}{dx}(x - x^*) + \frac{1}{2}\frac{d^2 f(x^*)}{dx^2}(x - x^*)^2 + R$$

$$x - x^* = d \;:\; \text{small change in the point } x^*$$

$$f(x^* + d) = f(x^*) + \frac{df(x^*)}{dx}d + \frac{1}{2}\frac{d^2 f(x^*)}{dx^2}d^2 + R$$

- Two variables

$$f(x_1, x_2) = f(x_1^*, x_2^*) + \frac{\partial f}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2^*)$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial f}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*)^2\right] + R$$

$$f(x_1, x_2) = f(x_1^*, x_2^*) + \sum_{i=1}^{2}\frac{\partial f}{\partial x_1}(x_i - x_i^*) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_i - x_i^*)(x_j - x_j^*) + R$$

Taylor Series Expansion (2)

– Matrix notation

$$f(\boldsymbol{x}) = f(\boldsymbol{x}^*) + \nabla f^T(\boldsymbol{x}-\boldsymbol{x}^*) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}^*)^T \boldsymbol{H}(\boldsymbol{x}-\boldsymbol{x}^*) + R$$

$$\boldsymbol{x}-\boldsymbol{x}^* = \boldsymbol{d}$$

$$f(\boldsymbol{x}^* + \boldsymbol{d}) = f(\boldsymbol{x}^*) + \nabla f^T \boldsymbol{d} + \frac{1}{2}\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} + R$$

$$\Delta f = f(\boldsymbol{x}^* + \boldsymbol{d}) - f(\boldsymbol{x}^*) = \nabla f^T \boldsymbol{d} + \frac{1}{2}\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} + R$$

first order change in $f(\boldsymbol{x})$ at $\boldsymbol{x}^*$

$$\delta f = \nabla f^T \delta\mathbf{x} \quad \text{where} \quad \delta\mathbf{x} = \boldsymbol{x}-\boldsymbol{x}^*$$

– examples

$$f(x) = \cos x \;\; @ \; x^* = 0$$

$$f(\mathbf{x}) = 3x_1^{\;3} x_2 \;\; @ \; x^* = (1,1)$$

$$f(\mathbf{x}) = x_1^{\;2} + x_2^{\;2} - 4x_1 - 2x_2 + 4 \;\; @ \; x^* = (1,2)$$

Quadratic Form

– Special nonlinear function having only second-order terms

$$F(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_i x_j = \frac{1}{2}\sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} p_{ij}x_j\right) \xrightarrow{y_i=\sum_{j=1}^{n} p_{ij}x_j \to \boldsymbol{y}=\boldsymbol{Px}} F(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i y_i$$

$$F(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{y} = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Px}$$

$$= \frac{1}{2}\left\{\begin{matrix}\left[p_{11}x_1{}^2 + \cdots + p_{nn}x_n{}^2\right] + \left[(p_{12}+p_{21})x_1x_2 + \cdots + (p_{1n}+p_{n1})x_1x_n\right] \\ + \cdots + \left[(p_{n-1,n}+p_{n,n-1})x_{n-1}x_n\right]\end{matrix}\right\}$$

$$\xrightarrow{a_{ij}=\frac{1}{2}(p_{ij}+p_{ji}) \to a_{ij}+a_{ji}=p_{ij}+p_{ji}} F(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Ax}$$

$$\nabla F(\boldsymbol{x}) = \boldsymbol{Ax}, \quad \frac{\partial^2 F(\boldsymbol{x})}{\partial x_j \partial x_i} = a_{ij}$$

Example

– 다음 이차형식에서 경사도벡터와 헷시안행렬을 계산하라.

$$F(x_1, x_2, x_3) = \frac{1}{2}\left(2x_1^{\,2} + 2x_1x_2 + 4x_1x_3 - 6x_2^{\,2} - 4x_2x_3 + 5x_3^{\,2}\right)$$

$$F(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & -6 & -4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & -6 & -2 \\ 2 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Form of a Matrix

- Check the form of a matrix
  - Eigenvalue
  - Principal minors
    - If no two consecutive principal minors are zero

form	definition	eigenvalue	principal minors
positive definite	$x^T A x > 0$	$\lambda_i > 0$	$M_k > 0 \left(k = 1, \ldots, n\right)$
positive semidefinite	$x^T A x \geq 0$	$\lambda_i \geq 0$	$M_k > 0 \left(k = 1, \ldots, r\right)$
negative definite	$x^T A x < 0$	$\lambda_i < 0$	$\left.\begin{matrix} M_k < 0 \left(\text{odd } k\right) \\ M_k > 0 \left(\text{even } k\right) \end{matrix}\right\} k = 1, \ldots, n$
negative semidefinite	$x^T A x \leq 0$	$\lambda_i \leq 0$	$\left.\begin{matrix} M_k < 0 \left(\text{odd } k\right) \\ M_k > 0 \left(\text{even } k\right) \end{matrix}\right\} k = 1, \ldots, r$
indefinite	?	some $\lambda_i < 0$ other $\lambda_i > 0$	

Example

– 다음 행렬의 형태를 결정하라.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{positive definite}$$

$$B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \text{negative semidefinite}$$

필요조건과 충분조건의 개념

- 최적점은 필요조건을 만족하여야 한다. 필요조건을 만족하는 점을 후보최적점(candidate optimum point)이라 한다. 필요조건을 만족하지 않는 점은 최적점이 될 수 없다.
- 필요조건을 만족한다고 해서 최적점인 것은 아니다. 즉, 비최적점도 필요조건을 만족시킬 수 있다.
- 충분조건을 만족하는 후보최적점은 실제로 최적점이다.
- 충분조건을 사용할 수 없거나 충분조건을 만족하지 않으면 후보점의 최적성 여부에 대하여 어떤 결론도 내릴 수 없게 된다.

비제약최적설계 문제

- 공학의 실제응용에서 자주 나타나는 문제는 아니지만, 제약문제의 최적성조건들은 비제약문제를 논리적으로 확장한 것이므로 개념이해가 중요
- 최적성조건(Optimality Conditions)을 이용하는 방법
  - 어떤 설계점이 주어지면 그 점의 후보최적점 여부 판정
  - 후보최적점을 계산
- 국부적 최적성조건
  - ***x**** 를 ***f(x)*** 의 국부적 최소점이라 하고 ***x*** 를 ***x**** 에 매우 가까운 점이라 하면, ***f(x)***는 ***x**** 에서 국부적 최소이므로 그 점으로부터 매우 작은 거리를 움직였을 때 함수값이 감소될 수 없다.

$$\mathbf{d} = \mathbf{x} - \mathbf{x}^*$$

$$\Delta f = f(\mathbf{x}) - f(\mathbf{x}^*) \geq 0$$

일변수함수의 최적성조건

- 일차 필요조건 (First-order necessary conditions)

$$f(x) = f(x^*) + f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

$$\Delta f = f(x) - f(x^*) \cong f'(x^*)d \geq 0$$

$$\rightarrow f'(x^*) = 0$$

  – 국부적 최소 또는 최대점, 변곡점 : stationary point

- 충분조건 (Sufficient conditions)

$$\Delta f = \frac{1}{2}f''(x^*)d^2 + R \rightarrow f''(x^*) > 0$$ （최소점에서 함수의 곡률이 양수）

- 이차 필요조건 (Second-order necessary conditions)

$$f''(x^*) \geq 0$$

$$\text{if } f''(x^*) = 0,\ f'''(x^*) = 0 \text{ and } f^{(IV)}(x^*) > 0$$

Examples

- Local minimum points using first-order necessary conditions

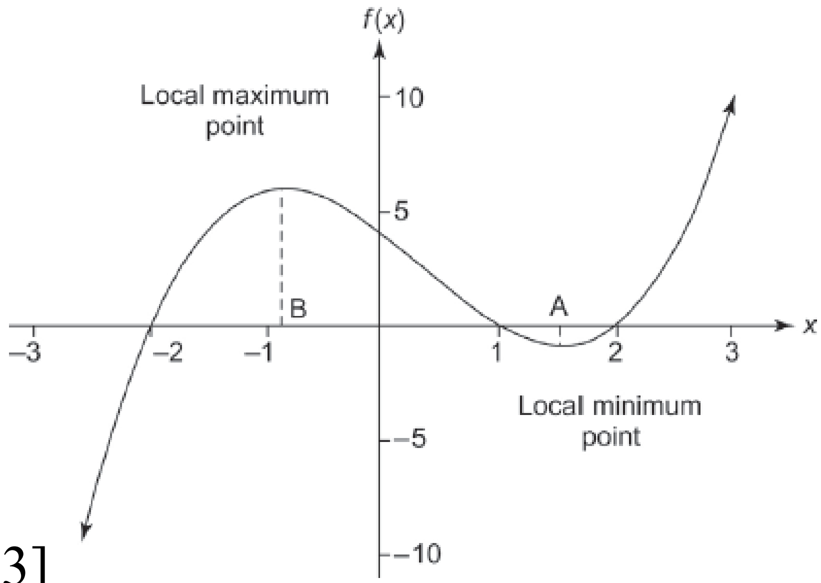
$$1.\ f(x) = \sin x$$

$$2.\ f(x) = x^2 - 4x + 4$$

$$3.\ f(x) = x^3 - x^2 - 4x + 4$$

$$4.\ f(x) = x^4$$

$$5.\ f(x) = ax + \frac{b}{x} \quad (a, b > 0)\text{[section 2.3]}$$



다변수함수의 최적성조건

- 일차 필요조건

$$f(\boldsymbol{x}) = f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} + \frac{1}{2}\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} + R$$

$$\Delta f = f(\boldsymbol{x}) - f(\boldsymbol{x}^*) = \nabla f(\boldsymbol{x}^*)^T \boldsymbol{d} + \frac{1}{2}\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} + R \geq \boldsymbol{0}$$

$$\rightarrow \nabla f(\boldsymbol{x}^*) = \boldsymbol{0}, \qquad \frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} = 0;\; i = 1, \ldots, n$$

- (이차)충분조건

$$\Delta f = \frac{1}{2}\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} + R \rightarrow \boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} > 0$$ (헷시안행렬 ***H*** 가 positive definite)

- 이차 필요조건

$$\boldsymbol{d}^T \boldsymbol{H}\boldsymbol{d} \geq 0$$

Examples

1. $f(x) = x^2 - 2x + 2$ (effects of scaling or adding constants to a function)

$\rightarrow [f(x)+1], [2f(x)], [-f(x)]$

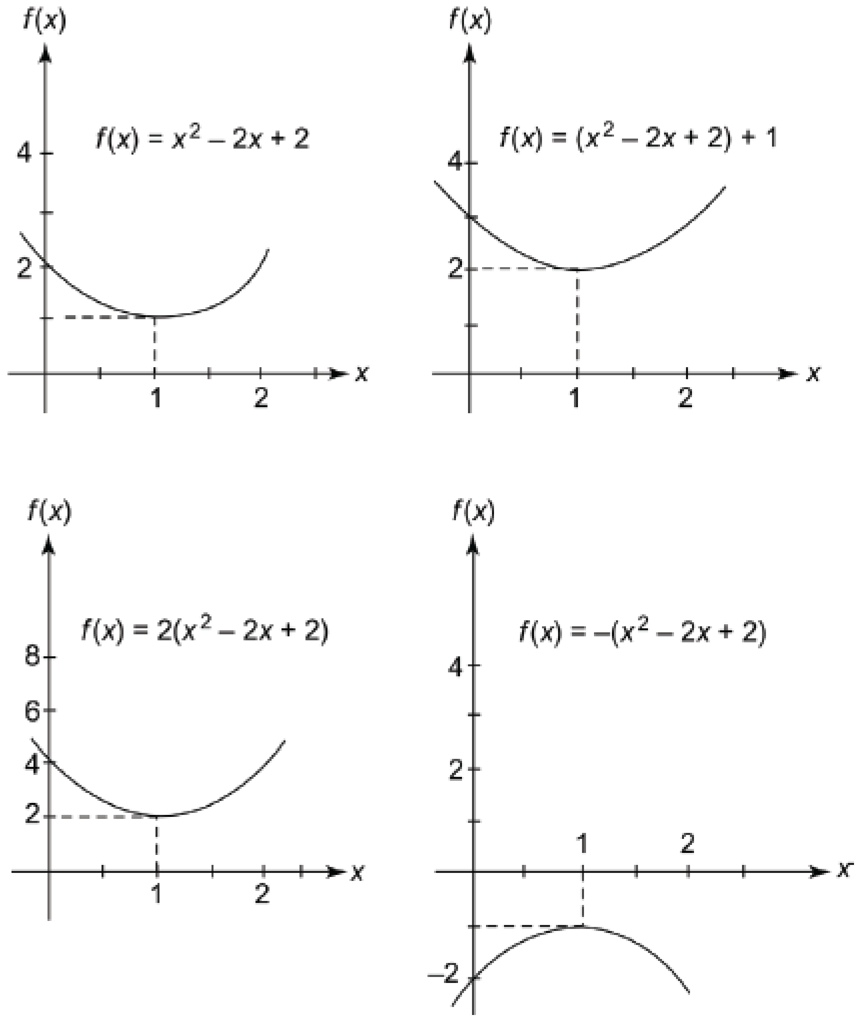
2. $f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 + x_2 + 8$

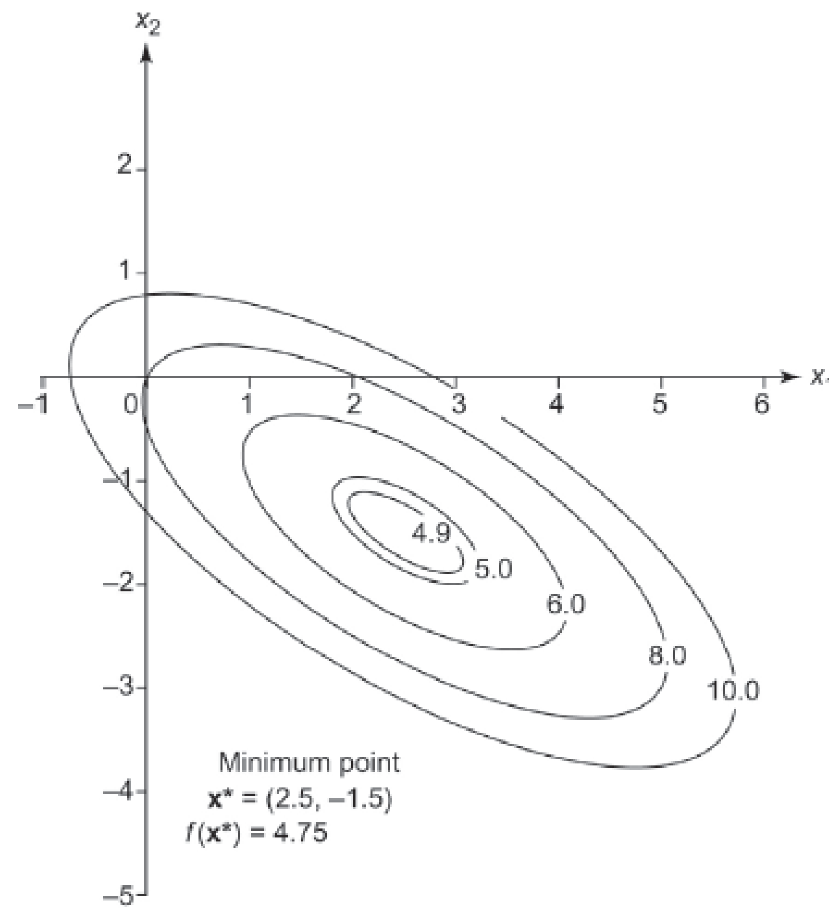
3. [section 2.8 cylindrical tank design] $\left.\begin{aligned} \bar{f} &= R^2 + RH \\ h &= \pi R^2 H - V = 0 \end{aligned}\right\} \rightarrow \bar{\bar{f}} = R^2 + \dfrac{V}{\pi R}$

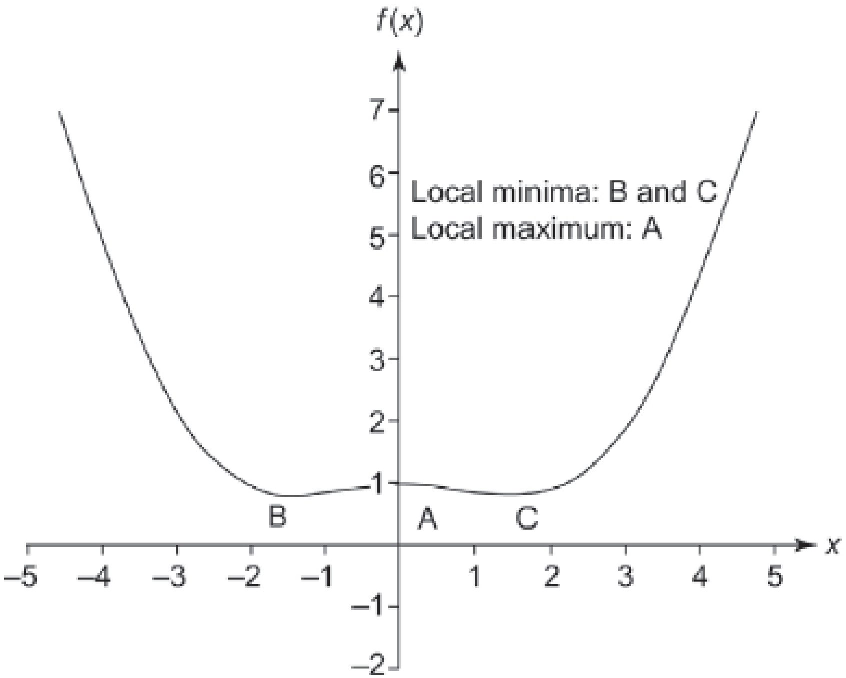
$$R^* = \left(\frac{V}{2\pi}\right)^{1/3},\ H^* = \left(\frac{4V}{\pi}\right)^{1/3},\ \bar{f}^* = 3\left(\frac{V}{2\pi}\right)^{2/3}$$

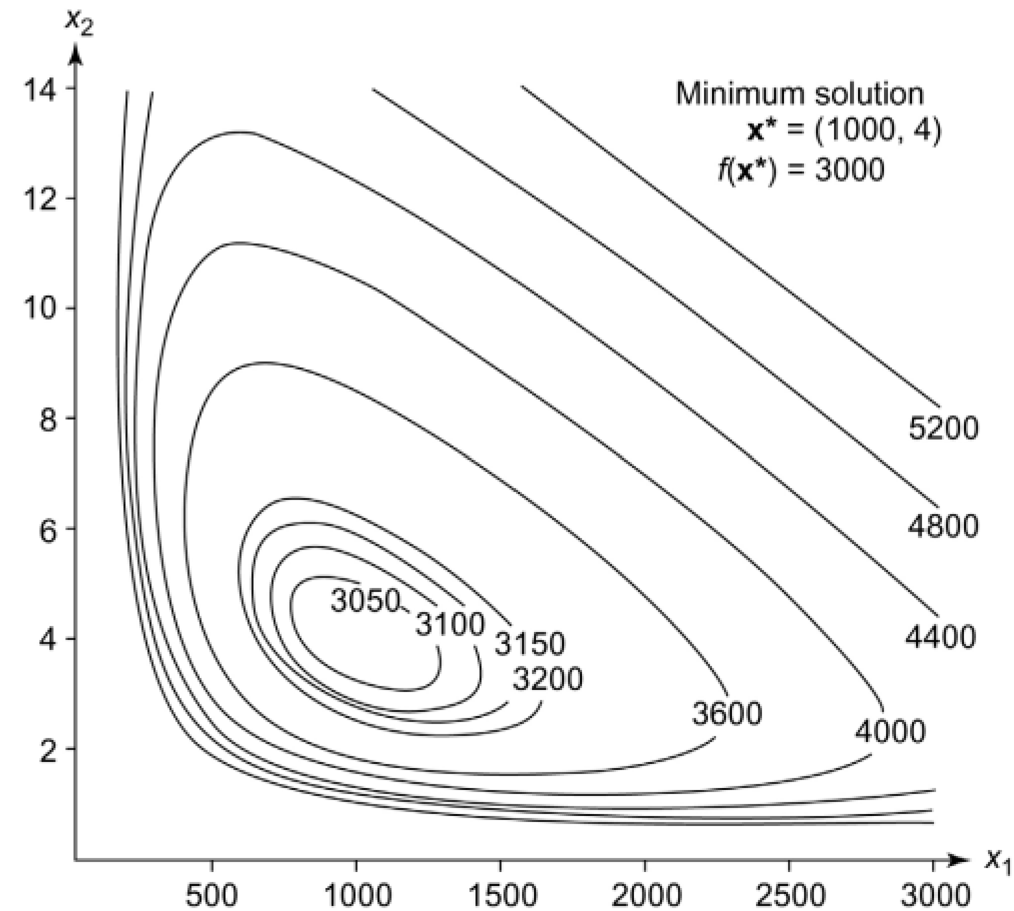
4. $f(x) = \dfrac{1}{3}x^2 + \cos x \rightarrow \begin{cases} x^* = 0 \\ x^* = 1.496 \\ x^* = -1.496 \end{cases}$

5. $f(\mathbf{x}) = x_1 + \dfrac{4.0E+06}{x_1x_2} + 250x_2 \rightarrow x^* = (1000, 4),\ f^* = 3000$


f(x)
f(x) = x^2 – 2x + 2
f(x) = (x^2 – 2x + 2) + 1
f(x) = 2(x^2 – 2x + 2)
f(x) = –(x^2 – 2x + 2)
x


x2
x1
4.9
5.0
6.0
8.0
10.0
Minimum point
x* = (2.5, –1.5)
f(x*) = 4.75


f(x)
Local minima: B and C
Local maximum: A
B
A
C
x


x2
x1
Minimum solution
x* = (1000, 4)
f(x*) = 3000
3050
3100
3150
3200
3600
4000
4400
4800
5200

Optimality Conditions for Unconstrained Function

TABLE 4.1(a) Optimality Conditions for Unconstrained One Variable Problems

Problem: Find x to minimize $f(x)$

First-order necessary condition: $f' = 0$. Any point satisfying this condition is called a stationary point; it can be a local maximum, local minimum, or neither of the two (inflection point)

Second-order necessary condition for a local minimum: $f'' \geq 0$

Second-order necessary condition for a local maximum: $f'' \leq 0$

Second-order sufficient condition for a local minimum: $f'' > 0$

Second-order sufficient condition for a local maximum: $f'' < 0$

Higher-order necessary conditions for a local minimum or local maximum: calculate a higher-ordered derivative that is not 0; all odd-ordered derivatives below this one must be 0

Higher-order sufficient condition for a local minimum: highest nonzero derivative must be even-ordered and positive

TABLE 4.1(b) Optimality Conditions for Unconstrained Function of Several Variables

Problem: Find $\mathbf{x}$ to minimize $f(\mathbf{x})$

First-order necessary condition: $\nabla f = \mathbf{0}$. Any point satisfying this condition is called a stationary point; it can be a local minimum, local maximum, or neither of the two (inflection point)

Second-order necessary condition for a local minimum: $\mathbf{H}$ must be at least positive semidefinite

Second-order necessary condition for a local maximum: $\mathbf{H}$ must be at least negative semidefinite

Second-order sufficient condition for a local minimum: $\mathbf{H}$ must be positive definite

Second-order sufficient condition for a local maximum: $\mathbf{H}$ must be negative definite

제약최적설계 문제

- 등호제약조건과 부등호제약조건을 만족하면서 목적함수를 최소화하는 설계변수벡터를 찾는 것

$$f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$$
$$h_j(\boldsymbol{x}) = 0; \quad j = 1, \ldots, p$$
$$g_i(\boldsymbol{x}) \le 0; \quad i = 1, \ldots, m$$

- 제약함수들이 최적해를 구하는데 결정적인 역할
  - 해가 존재하지 않을 수도 있음: 과제약 (overconstrained)
- Equality constraints are always active for any feasible design, whereas an inequality constraint may not be active at a feasible point

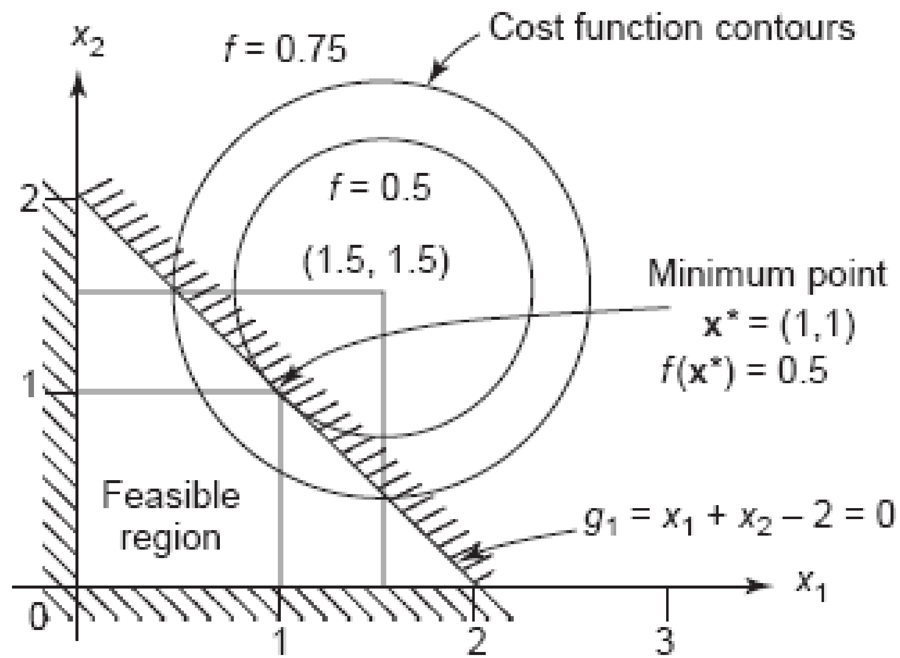
Example 4.24

- 다음 함수를 최소화하라.

  $$f(\boldsymbol{x}) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

  여기서 제약조건은 다음과 같다.

  $$g_1(\boldsymbol{x}) = x_1 + x_2 - 2 \le 0$$
  $$g_2(\boldsymbol{x}) = -x_1 \le 0$$
  $$g_3(\boldsymbol{x}) = -x_2 \le 0$$



필요조건: 등호제약조건

- 정칙점 (regular point)
  - 등호제약조건을 만족하고 모든 제약함수의 경사도벡터들이 일차독립
  - 일차독립: 두개의 경사도벡터가 서로 평행하지 않고 어떤 경사도벡터도 다른 경사도벡터들의 선형결합으로 표현할 수 없다는 것을 의미
- 라그란지승수 (Lagrange multiplier)
  - 각각의 제약조건에 대응하는 승수 (scalar multiplier)
  - 목적함수나 제약함수의 형태에 따라 좌우

Example 4.27

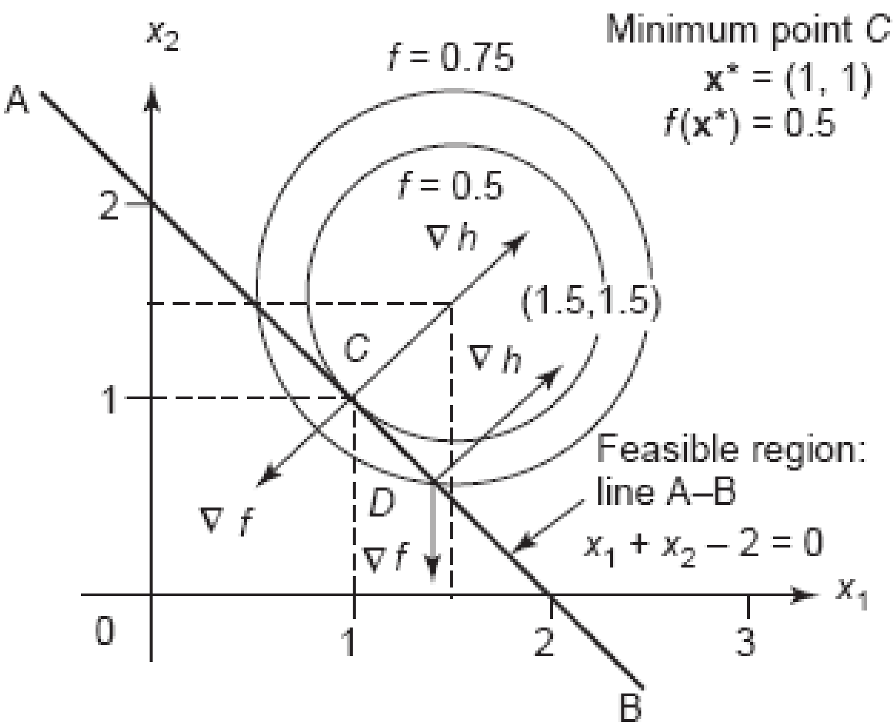
- 다음 함수를 최소화하는 x_1 과 x_2 를 구하라.

$$f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

$$h(x_1, x_2) = x_1 + x_2 - 2 = 0$$

$$\text{Minimize} \quad f(x_1, x_2)$$

$$\text{subject to} \quad h(x_1, x_2) = 0 \rightarrow x_2 = \phi(x_1)$$



등호제약조건이 양함수가 아닌 경우

$$\frac{df(x_1, x_2)}{dx_1} = \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{dx_2}{dx_1} = 0$$

$$\rightarrow \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}\frac{d\phi}{dx_1} = 0 \quad @\,\text{optimum}$$

$$\frac{dh(x_1^*, x_2^*)}{dx_1} = \frac{\partial h(x_1^*, x_2^*)}{\partial x_1} + \frac{\partial h(x_1^*, x_2^*)}{\partial x_2}\frac{d\phi}{dx_1} = 0 \rightarrow \frac{d\phi}{dx_1} = -\frac{\partial h(x_1^*, x_2^*)/\partial x_1}{\partial h(x_1^*, x_2^*)/\partial x_2}$$

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} - \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}\left[\frac{\partial h(x_1^*, x_2^*)/\partial x_1}{\partial h(x_1^*, x_2^*)/\partial x_2}\right] = 0$$

$$v = -\frac{\partial f(x_1^*, x_2^*)/\partial x_2}{\partial h(x_1^*, x_2^*)/\partial x_2}$$

$$\begin{cases} \dfrac{\partial f(x_1^*, x_2^*)}{\partial x_1} + v\dfrac{\partial h(x_1^*, x_2^*)}{\partial x_1} = 0 \\ \dfrac{\partial f(x_1^*, x_2^*)}{\partial x_2} + v\dfrac{\partial h(x_1^*, x_2^*)}{\partial x_2} = 0 \end{cases}$$

$$h(x_1, x_2) = 0$$

라그란지승수의 기하학적 의미

$$L(x_1, x_2, v) = f(x_1, x_2) + vh(x_1, x_2)$$

$$\left.\begin{aligned} \frac{\partial L(x_1^*, x_2^*)}{\partial x_1} = 0 \\ \frac{\partial L(x_1^*, x_2^*)}{\partial x_2} = 0 \end{aligned}\right\} \rightarrow \nabla L(x_1^*, x_2^*) = \nabla f(\mathbf{x}^*) + v\nabla h(\mathbf{x}^*) = 0$$

$$\boxed{\nabla f(\mathbf{x}^*) = -v\nabla h(\mathbf{x}^*)}$$

- 후보최적점에서 목적함수 및 제약함수들의 경사도벡터들은 동일 작용선상에 있고 서로 비례함
- 라그란지승수는 비례상수 (제약을 가하기 위해 필요한 힘으로 해석 가능)
- 등호제약조건에 대한 라그란지승수의 부호는 제약이 없음

라그란지승수정리

– 등호제약조건으로 $h_j(\boldsymbol{x}) = 0;\ j = 1, \ldots, p$ 가 있는 $f(x)$ 의 최소화문제를 고려해 보자. x* 를 이 문제의 국부적 최소인 정칙점이라고 하면 다음을 만족하는 라그란지승수 $v_j^*,\ j = 1, \ldots, p$ 가 존재한다.

$$\frac{df(x^*)}{dx_i} + \sum_{j=1}^{p} v_j^* \frac{dh_j(x^*)}{dx_i} = 0 \rightarrow \frac{df(x^*)}{dx_i} = -\sum_{j=1}^{p} v_j^* \frac{dh_j(x^*)}{dx_i}; \quad i = 1, \ldots, n$$

$$h_j(x^*) = 0; \quad j = 1, \ldots, p$$

– 후보최적점에서 목적함수의 경사도벡터는 제약함수의 경사도벡터들의 선형결합

$$L(x, v) = f(x) + \sum_{j=1}^{p} v_j h_j(x) = f(x) + v^T h(x)$$

$$\rightarrow \begin{cases} \nabla L(x^*, v^*) = 0 \quad \text{or} \quad \dfrac{\partial L(x^*, v^*)}{\partial x_i} = 0; \quad i = 1, \ldots, n \\ \dfrac{\partial L(x^*, v^*)}{\partial v_j} = h_j(x^*) = 0; \quad j = 1, \ldots, p \end{cases}$$

Example 4.25 Cylindrical Tank Design

- Section 2.8 → Example 4.21

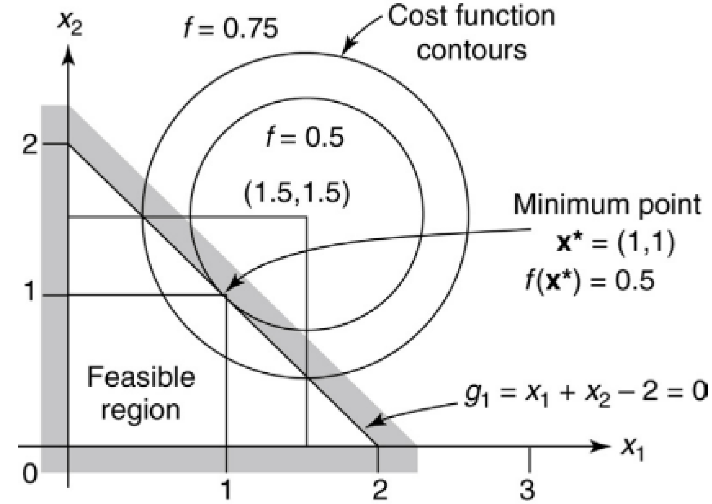
$$\left.\begin{aligned}&\underset{R,l}{\text{minimize}}\ \ \bar{f} = R^2 + RH\\&\text{subject to}\ \ h = \pi R^2 H - V = 0\end{aligned}\right\} \rightarrow R^* = \left(\frac{V}{2\pi}\right)^{1/3},\ V^* = \left(\frac{4V}{\pi}\right)^{1/3},\ \bar{f}^* = 3\left(\frac{V}{2\pi}\right)^{1/3}$$

Example 4.26~28 Role of Inequalities

– status of the inequality constraint (active or inactive) must be determined as a part of the solution for the optimization problem

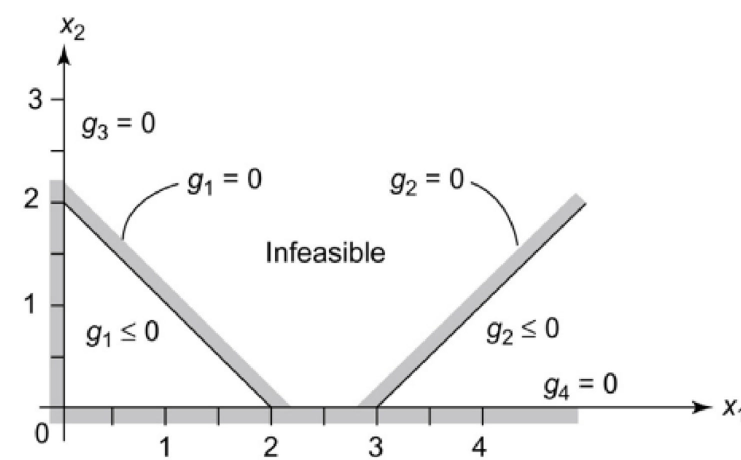
$$\begin{cases} \underset{\mathbf{x}}{\text{Minimize}} \; f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 \\ \underset{\mathbf{x}}{\text{Minimize}} \; f(x_1, x_2) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \end{cases}$$

$$\text{subject to} \; g_1(\mathbf{x}) = x_1 + x_2 - 2 \le 0$$
$$g_2(\mathbf{x}) = -x_1 \le 0$$
$$g_3(\mathbf{x}) = -x_2 \le 0$$



$$\underset{\mathbf{x}}{\text{Minimize}} \; f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2$$

$$\text{subject to} \; g_1(\mathbf{x}) = x_1 + x_2 - 2 \le 0$$
$$g_2(\mathbf{x}) = -x_1 + x_2 + 3 \le 0$$
$$g_3(\mathbf{x}) = -x_1 \le 0$$
$$g_4(\mathbf{x}) = -x_2 \le 0$$



필요조건: 부등호제약조건

- 완화변수(slack variable)를 더하여 등호제약조건으로 변환

$$g_i(\boldsymbol{x}) \le 0 \rightarrow g_i(\boldsymbol{x}) + s_i^2 = 0 \quad i = 1, \ldots, m$$

$$\text{slack variable}: s_i^2 \ \text{(why? avoid additional constraints } s_i \ge 0)$$

$$\begin{cases} s_i^2 = 0 : \text{equality} \rightarrow \text{active (tight) constraint} \\ s_i^2 > 0 : \text{inequality} \rightarrow \text{inactive constraint} \end{cases}$$

- “≤ type” 제약조건의 라그란지승수에 대한 추가적인 필요조건
  – u_j^* 는 j번째 부등호제약조건에 대한 라그란지승수: $u_j^* \ge 0 \ (j = 1, \ldots, m)$

	minimize	maximize
$g_i(\boldsymbol{x}) \le 0$	$u_i^* \ge 0$	$u_i^* \le 0$
$g_i(\boldsymbol{x}) \ge 0$	$u_i^* \le 0$	$u_i^* \ge 0$

Example 4.29

$$\textit{Minimize } f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

$$\textit{subject to } g(\mathbf{x}) = x_1 + x_2 - 2 \le 0$$

Karush-Kuhn-Tucker necessary conditions (1)

- x^* 가 제약집합내의 정칙점이고, 다음의 제약조건 하에서 함수 $f(x)$의 국부적 최소점이라 하자.

$$h_i(\boldsymbol{x}) = 0; \quad i = 1, \ldots, p \quad \text{and} \quad g_j(\boldsymbol{x}) \le 0; \quad j = 1, \ldots, m$$

- 이 문제의 라그란지함수를 다음과 같이 정의한다.

$$L(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{s}) = f(\boldsymbol{x}) + \sum_{i=1}^{p} v_i h_i(\boldsymbol{x}) + \sum_{i=1}^{m} u_i \left[g_i(\boldsymbol{x}) + s_i^{\,2} \right] = f(\boldsymbol{x}) + \boldsymbol{v}^T \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{u}^T \left[\boldsymbol{g}(\boldsymbol{x}) + \mathrm{s}^2 \right]$$

- 그러면 다음 조건을 만족하는 라그란지승수 v* 와 u* 가 존재한다.

$$\begin{cases} \dfrac{\partial L}{\partial x_j} \equiv \dfrac{\partial f}{\partial x_j} + \displaystyle\sum_{i=1}^{p} v_i^{\,*} \dfrac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^{\,*} \dfrac{\partial g_i}{\partial x_j} = 0; \quad j = 1, \ldots, n \\ h_i(\mathbf{x}^*) = 0; \quad i = 1, \ldots, p \\ g_i(\mathbf{x}^*) + s_i^{\,2} = 0; \quad i = 1, \ldots, m \\ u_i^{\,*} s_i = 0; \quad i = 1, \ldots, m \ \ \text{(switching conditions)} \\ u_i^{\,*} \ge 0; \quad i = 1, \ldots, m \end{cases}$$

Karush-Kuhn-Tucker necessary conditions (2)

- 1차 필요조건
  - 어떤 주어진 점에 대한 최적성을 점검 / 후보최적점을 결정
- 기하학적 의미
  - 목적함수의 음의 경사도벡터방향이 제약함수의 경사도벡터들의 선형결합이며, 라그란지승수가 선형결합의 상수로서 사용

$$-\frac{\partial f}{\partial x_j} = \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i^* \frac{\partial g_i}{\partial x_j}; \quad j = 1, \ldots, n$$

- 미지수의 개수: x, u, s, v (n+2m+p) = # of eqns
- 전환조건(switching condition) 또는 보충완화조건 (complementary slackness condition)

$$\begin{cases} g_i(\mathbf{x}^*) < 0 \text{ (inactive, } s_i^2 > 0) \rightarrow u_i^* = 0 \\ g_i(\mathbf{x}^*) = 0 \text{ (active, } s_i^2 = 0) \rightarrow u_i^* \geq 0 \end{cases}$$

Karush-Kuhn-Tucker necessary conditions (3)

- K-T conditions are *not applicable* at the points that are not *regular*.
- Any point that *does not satisfy* K-T conditions *cannot be a local minimum* unless it is an irregular points.
- The points satisfying K-T conditions can be *constrained or unconstrained*.
- If there are equality constraints and no inequalities are active, then the points satisfying K-T conditions are *only stationary*.
- If some inequality constraints are active and their multipliers are positive, then the points satisfying K-T conditions cannot be local maxima for the cost function.
- The value of the *Lagrange multiplier* for each constraint depends on the functional form for the constraint.
  - Optimum solution ? / Lagrange multiplier ?

$$\text{(i)}\ \frac{x}{y}-10\leq 0\left(y>0\right)\quad \text{(ii)}\ x-10y\leq 0\quad \text{(iii)}\ \frac{0.1x}{y}-1\leq 0$$

Example 5.3

- Necessary conditions are applicable only if the assumption for regularity of x* is satisfied.
  - Gradients of all the active constraints @ x* is linearly independent

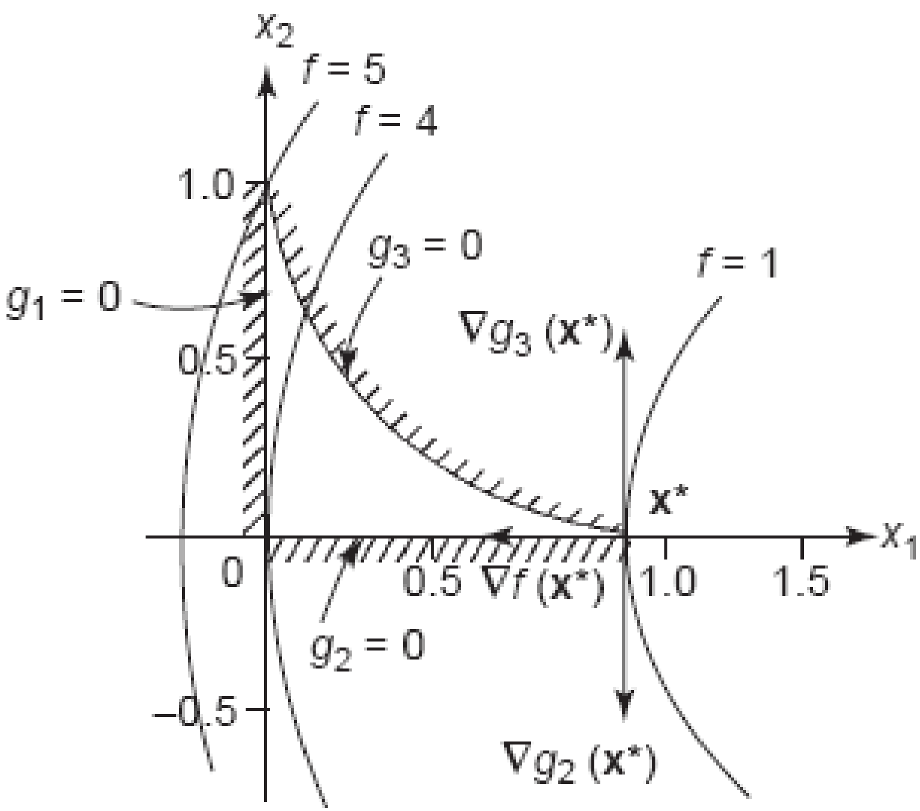
$$\text{Minimize } f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 + 4$$

$$\text{subject to } g_1 = -x_1 \le 0$$

$$g_2 = -x_2 \le 0$$

$$g_3 = x_2 - (1 - x_1)^3 \le 0$$

Check if the minimum point (1,0) satisfies K-T conditions.



Kuhn-Tucker 필요조건의 변형

- Without slack variables

$$\begin{cases} g_i(\mathbf{x}^*) + s_i^2 = 0 \rightarrow s_i^2 = -g_i(\mathbf{x}^*) \geq 0 \rightarrow g_i(\mathbf{x}^*) \leq 0 \\ u_i^* s_i = 0 \rightarrow u_i^* s_i^2 = 0 \rightarrow u_i^* g_i(\mathbf{x}^*) = 0 \end{cases}$$

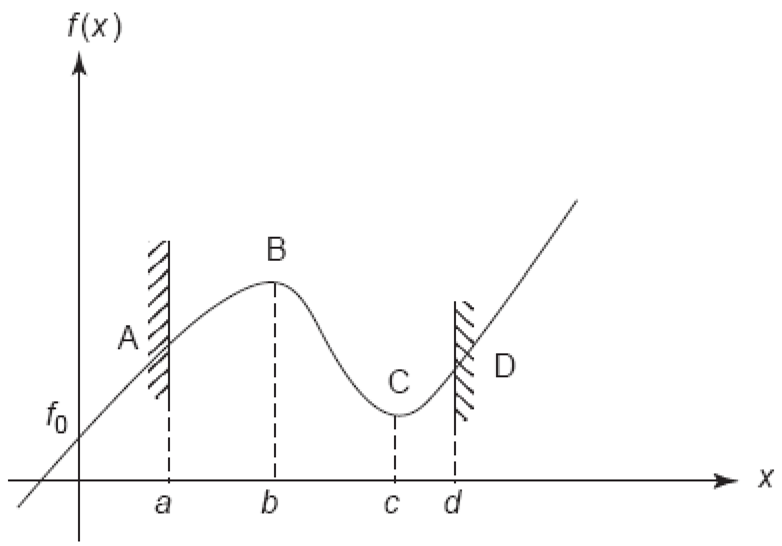
$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{i=1}^{m} u_i g_i(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T \mathbf{g}(\mathbf{x})$$

$$\begin{cases} \dfrac{\partial L}{\partial x_j} \equiv \dfrac{\partial f}{\partial x_j} + \sum_{i=1}^{p} v_i \dfrac{\partial h_i}{\partial x_j} + \sum_{i=1}^{m} u_i \dfrac{\partial g_i}{\partial x_j} = 0; \quad j = 1, \ldots, n \\ h_i(\mathbf{x}^*) = 0; \quad i = 1, \ldots, p \\ g_i(\mathbf{x}^*) \leq 0; \quad i = 1, \ldots, m \\ u_i^* g_i(\mathbf{x}^*) = 0; \quad i = 1, \ldots, m \ \text{(switching conditions)} \\ u_i^* \geq 0; \quad i = 1, \ldots, m \end{cases}$$

Example 4.30

Minimize $f(\boldsymbol{x}) = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0$

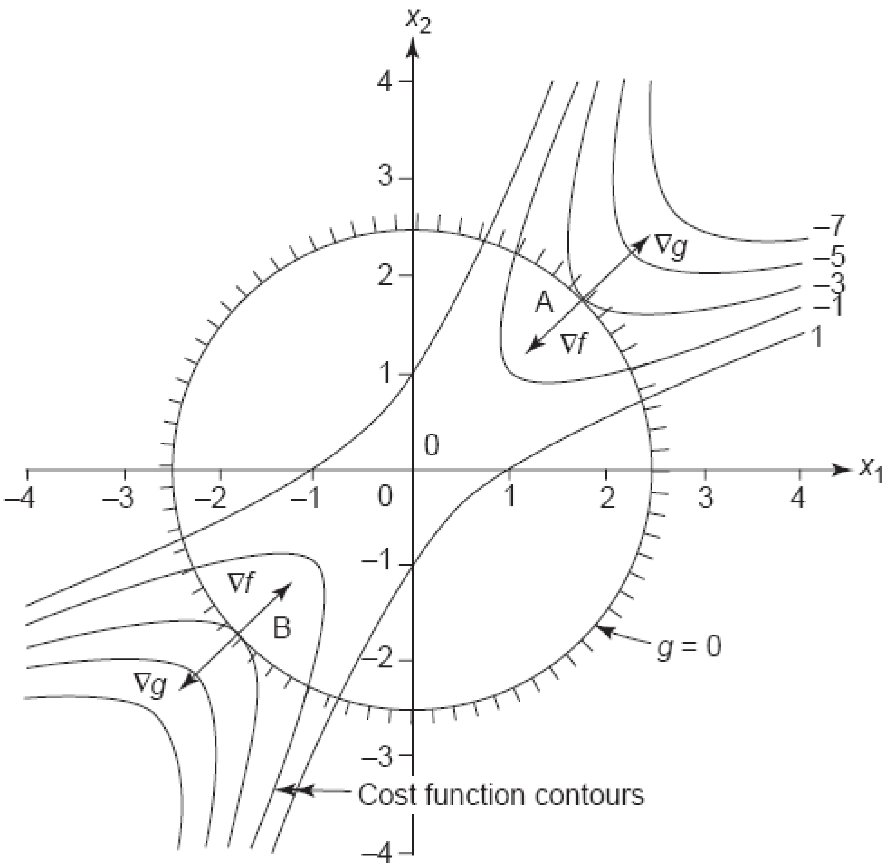
subject to $a \le x \le d$ $\left(0 < a < b < c < d \text{ and } f_0 \text{ are constants}\right)$



Example 4.31

$$Minimize \;\; f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2$$

$$subject\ to \;\; g = x_1^2 + x_2^2 - 6 \le 0$$



Perplexity

Based on the provided equations and inequality, let's analyze the solutions:

1. The system of equations:
   - 2x1 - 3x2 + 2ux1 = 0
   - 2x2 - 3x1 + 2ux2 = 0
   - x1^2 + x2^2 - 6 + s^2 = 0
   - u*s = 0
   - u >= 0
2. The solutions obtained:
   (-√3, √3, -5/2, 0)
   (√3, -√3, -5/2, 0)
   (-√3, -√3, 1/2, 0)
   (√3, √3, 1/2, 0)
   (0, 0, 0, -√6)
   (0, 0, 0, √6)
3. Verification of solutions:
   The valid solutions that satisfy all equations are:
   (-√3, √3, -2.5, 0)
   (√3, -√3, -2.5, 0)
   (-√3, -√3, 0.5, 0)
   (√3, √3, 0.5, 0)
   (0, 0, 0, -√6)
   (0, 0, 0, √6)
4. Considering the inequality u >= 0:
   We can eliminate solutions where u is negative.

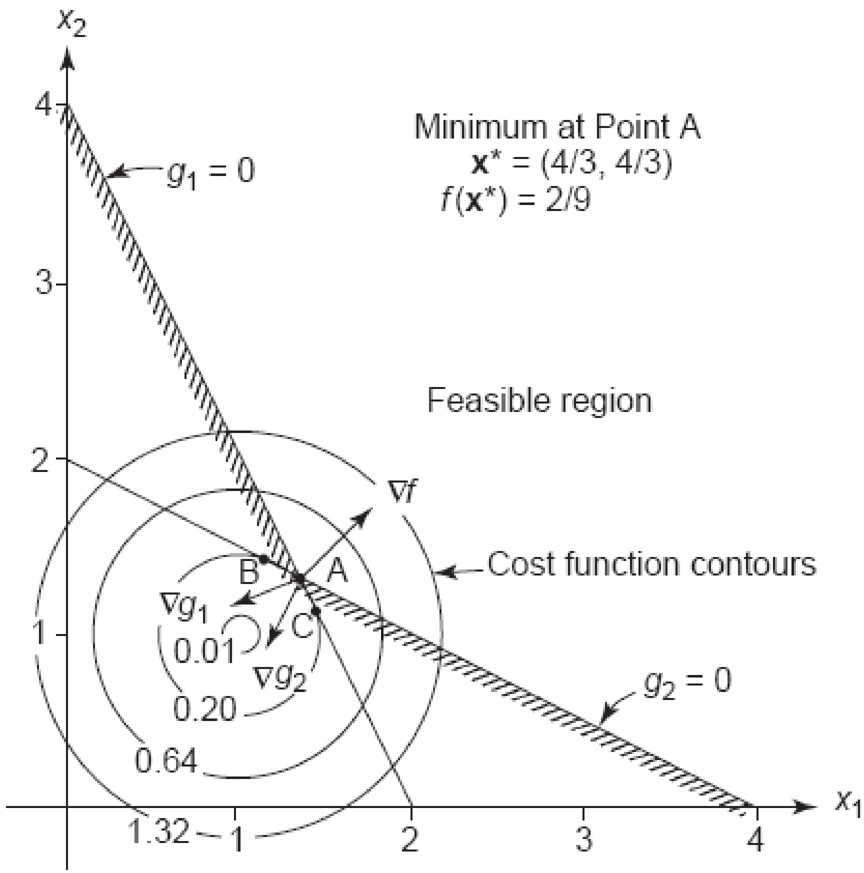
Therefore, the final solutions that satisfy all equations and the inequality are:

1. x1 = -√3, x2 = -√3, u = 0.5, s = 0
2. x1 = √3, x2 = √3, u = 0.5, s = 0
3. x1 = 0, x2 = 0, u = 0, s = -√6
4. x1 = 0, x2 = 0, u = 0, s = √6

These solutions satisfy all the given equations and the inequality u >= 0. The first two solutions have non-zero u and zero s, while the last two have zero u and non-zero s, which is consistent with the condition u*s = 0.

Example 4.32

$$\text{Minimize } f(\mathbf{x}) = x_1^{\,2} + x_2^{\,2} - 2x_1 - 2x_2 + 2$$
$$\text{subject to } g_1 = -2x_1 - x_2 + 4 \le 0$$
$$g_2 = -x_1 - 2x_2 + 4 \le 0$$



Summary of the KKT Solution Approach

- *to check* whether a given point is a candidate minimum
  - the point must be feasible
  - the gradient of the Lagrangian with respect to the design variables must be zero
  - the Lagrange multipliers for the inequality constraints must be nonnegative
- *to find* candidate minimum points
  - Several cases defined by the switching conditions must be considered and solved
    - Check all inequality constraints for feasibility
    - Calculate all of the Lagrange multipliers for each solution point
    - Ensure that the Lagrange multipliers for all of the inequality constraints are nonnegative

Limitation of the KKT Solution Approach

- addition of an inequality to the problem formulation doubles the number of KKT solution cases

No.	Cases	Active constraints
1	$u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0$	No inequality active
2	$s_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0$	*One* active inequality: $g_1 = 0$
3	$u_1 = 0, s_2 = 0, u_3 = 0, u_4 = 0$	*One* active inequality: $g_2 = 0$
4	$u_1 = 0, u_2 = 0, s_3 = 0, u_4 = 0$	*One* active inequality: $g_3 = 0$
5	$u_1 = 0, u_2 = 0, u_3 = 0, s_4 = 0$	*One* active inequality: $g_4 = 0$
6	$s_1 = 0, s_2 = 0, u_3 = 0, u_4 = 0$	*Two* active inequalities: $g_1 = 0, g_2 = 0$
7	$u_1 = 0, s_2 = 0, s_3 = 0, u_4 = 0$	*Two* active inequalities: $g_2 = 0, g_3 = 0$
8	$u_1 = 0, u_2 = 0, s_3 = 0, s_4 = 0$	*Two* active inequalities: $g_3 = 0, g_4 = 0$
9	$s_1 = 0, u_2 = 0, u_3 = 0, s_4 = 0$	*Two* active inequalities: $g_1 = 0, g_4 = 0$
10	$s_1 = 0, u_2 = 0, s_3 = 0, u_4 = 0$	*Two* active inequalities: $g_1 = 0, g_3 = 0$
11	$u_1 = 0, s_2 = 0, u_3 = 0, s_4 = 0$	*Two* active inequalities: $g_2 = 0, g_4 = 0$
12	$s_1 = 0, s_2 = 0, s_3 = 0, u_4 = 0$	*Three* active inequalities: $g_1 = 0, g_2 = 0, g_3 = 0$
13	$u_1 = 0, s_2 = 0, s_3 = 0, s_4 = 0$	*Three* active inequalities: $g_2 = 0, g_3 = 0, g_4 = 0$
14	$s_1 = 0, u_2 = 0, s_3 = 0, s_4 = 0$	*Three* active inequalities: $g_1 = 0, g_3 = 0, g_4 = 0$
15	$s_1 = 0, s_2 = 0, u_3 = 0, s_4 = 0$	*Three* active inequalities: $g_1 = 0, g_2 = 0, g_4 = 0$
16	$s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 0$	All *four* inequalities active

라그란지승수의 물리적 의미

- 후최적성해석(Post-optimality analysis) 또는 민감도해석(Sensitivity analysis)
  – 최적설계문제의 매개변수를 변화시켰을 때 최적해의 변화에 대한 고찰
- 제약한계값을 0으로부터 변화시켰을 때 목적함수의 최적해에는 어떤 변화?
  – 라그란지승수 (v*, u*)가 이러한 민감도문제에 대한 해답을 제공
- 왜 “≤ type” 제약조건에 대한 라그란지승수가 음수가 아니어야 하는가?
- 제약조건을 완화(relaxation)함에 따라 얻어지는 이점이나 속박(tightening)에 따른 불리한 점
- 목적함수와 제약함수를 축적화(scaling)했을때 라그란지승수의 변화?

제약한계변화의 영향

- b_i 와 e_j 는 0근처에서 매우 작은 변화량

$$\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{b},\boldsymbol{e}), f = f(\boldsymbol{b},\boldsymbol{e})$$

$$h_i(\boldsymbol{x}) = b_i;\quad i = 1,\ldots,p \quad \text{and} \quad g_j(\boldsymbol{x}) \le e_j;\quad j = 1,\ldots,m$$

- 제약함수의 민감도 정리
  - v_i*, u_j*: satisfying both necessary and sufficient conditions

$$\begin{cases} \dfrac{\partial L}{\partial b_i} \equiv \dfrac{\partial f}{\partial b_i} + \displaystyle\sum_{i=1}^{p} v_i \dfrac{\partial h_i}{\partial b_i} + \sum_{j=1}^{m} u_j \dfrac{\partial g_j}{\partial b_i} = 0 \rightarrow \dfrac{\partial f(\boldsymbol{x}^*(0,0))}{\partial b_i} = -v_i^*; \quad i = 1,\ldots,p \\ \dfrac{\partial L}{\partial e_j} \equiv \dfrac{\partial f}{\partial e_j} + \displaystyle\sum_{i=1}^{p} v_i \dfrac{\partial h_i}{\partial e_j} + \sum_{j=1}^{m} u_j \dfrac{\partial g_j}{\partial e_j} = 0 \rightarrow \dfrac{\partial f(\boldsymbol{x}^*(0,0))}{\partial e_j} = -u_j^*; \quad j = 1,\ldots,m \end{cases}$$

$$f(b_i, e_j) = f(0,0) + \frac{\partial f(0,0)}{\partial b_i} b_i + \frac{\partial f(0,0)}{\partial e_j} e_j = f(0,0) - v_i^* b_i - u_j^* e_j$$

$$\Delta f = f(b_i, e_j) - f(0,0) = -v_i^* b_i - u_j^* e_j$$

$$\Delta f = -\sum_i v_i^* b_i - \sum_j u_j^* e_j$$

Example 4.33 (←4.31)

- Nonnegativity of Lagrange multipliers

relax an inequality constraint $\left(g_j \le 0\right): e_j > 0$

$\rightarrow$ feasible set for the design problem expands

$\rightarrow$ expect the optimum cost function to reduce further or at the most remain unchanged

if $u_j^* < 0,$ then $\Delta f = -u_j^* e_j > 0$ (contradiction!)

$\therefore$ Lagrange multiplier corresponding to a "$\le$ *type*" constraint must be nonnegative.

$$\begin{aligned} &\textit{Minimize} \quad f\left(x_1, x_2\right) = x_1^2 + x_2^2 - 3x_1x_2 \\ &\textit{subject to} \quad g\left(x_1, x_2\right) = x_1^2 + x_2^2 - 6 \le 0 \end{aligned}$$

$$\Rightarrow x_1^* = x_2^* = \sqrt{3}, \quad u^* = \frac{1}{2}, \quad f\left(\mathbf{x}^*\right) = -3$$

$$\begin{cases} e = 1 \ \text{ (i.e., radius of circle } = \sqrt{6} \rightarrow \sqrt{7}) \ ? \\ f\left(0,1\right) = f\left(0,0\right) - u^* e = -3 - \left(0.5\right)\left(1\right) = -3.5 \\ e = -1 \ \text{ (smaller feasible region) } f\left(0,-1\right) = -2.5 \end{cases}$$

Effect of scaling a cost function

- No change on the optimum point
- Change in the optimum value for the cost function

$$\bar{f}(\mathbf{x}) = Kf(\mathbf{x}) \quad \text{where} \quad K > 0$$

$$\begin{cases} \bar{v}_i^* = Kv_i^*; & i = 1, \ldots, p \\ \bar{u}_j^* = Ku_j^*; & j = 1, \ldots, m \end{cases}$$

$$L = K\left(x_1^2 + x_2^2 - 3x_1x_2\right) + \bar{u}\left(x_1^2 + x_2^2 - 6 + \bar{s}^2\right)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x_1} = 2Kx_1 - 3Kx_2 + 2\bar{u}x_1 = 0 \\ \dfrac{\partial L}{\partial x_2} = 2Kx_2 - 3Kx_1 + 2\bar{u}x_2 = 0 \\ x_1^2 + x_2^2 - 6 + \bar{s}^2 = 0 \\ \bar{u}\bar{s} = 0 \\ \bar{u} \geq 0 \end{array}\right\} \rightarrow \begin{cases} x_1^* = x_2^* = \sqrt{3}, \quad \bar{u}^* = \dfrac{K}{2}, \quad \bar{f}(\mathbf{x}^*) = -3K \\ x_1^* = x_2^* = -\sqrt{3}, \quad \bar{u}^* = \dfrac{K}{2}, \quad \bar{f}(\mathbf{x}^*) = -3K \\ \bar{u}^* = Ku^* \end{cases}$$

Effect of scaling a constraint

- No change on the constraint boundary (no effect on the optimum solution)
- Change in the Lagrange multiplier

$$\begin{cases} \bar{v}_i^* = v_i^* / P_i; & i = 1, \ldots, p \\ \bar{u}_j^* = u_j^* / M_j; & j = 1, \ldots, m \quad \text{where} \quad M_j > 0 \end{cases}$$

$$L = \left(x_1^2 + x_2^2 - 3x_1x_2\right) + \bar{u}\left[M\left(x_1^2 + x_2^2 - 6\right) + \bar{s}^2\right]$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x_1} = 2x_1 - 3x_2 + 2\bar{u}Mx_1 = 0 \\ \dfrac{\partial L}{\partial x_2} = 2x_2 - 3x_1 + 2\bar{u}Mx_2 = 0 \\ M\left(x_1^2 + x_2^2 - 6\right) + \bar{s}^2 = 0 \\ \bar{u}\bar{s} = 0 \\ \bar{u} \geq 0 \end{array}\right\} \rightarrow \begin{cases} x_1^* = x_2^* = \sqrt{3}, \quad \bar{u}^* = \dfrac{1}{2M}, \quad \bar{f}\left(\mathbf{x}^*\right) = -3 \\ x_1^* = x_2^* = -\sqrt{3}, \quad \bar{u}^* = \dfrac{1}{2M}, \quad \bar{f}\left(\mathbf{x}^*\right) = -3 \\ \bar{u}^* = \dfrac{u^*}{M} \end{cases}$$

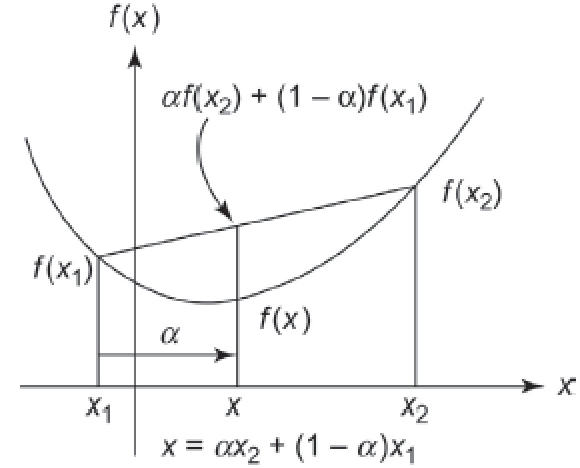
Global Optimality

- Question of *global optimum*
  - Weierstrass theorem (→ exhaustive search)
    - Cost function is continuous on a closed and bounded feasible region
  - Showing the optimization problem is convex
- Convex set *S*
  - If P_1 and P_2 are any points in *S*, then the entire line segment P_1 - P_2 is also in *S* $\quad \left[\mathbf{x} = \alpha \mathbf{x}^{(2)} + (1-\alpha)\mathbf{x}^{(1)}; \quad 0 \le \alpha \le 1\right]$
- Convex functions

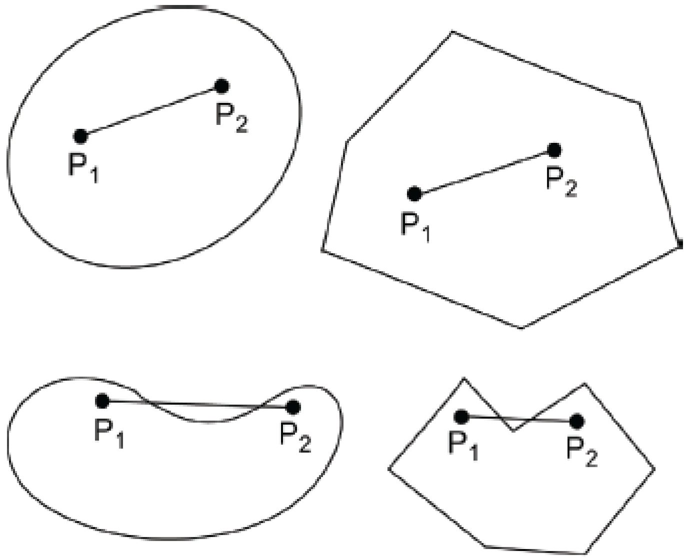
$$f\left(\alpha \mathbf{x}^{(2)} + (1-\alpha)\mathbf{x}^{(1)}\right) \le \alpha f\left(\mathbf{x}^{(2)}\right) + (1-\alpha) f\left(\mathbf{x}^{(1)}\right)$$

  - Check : *iff* Hessian matrix of a function is positive semidefinite or positive definite at all points in the set S
    - Hessian matrix is positive definite → ←(x) strictly convex

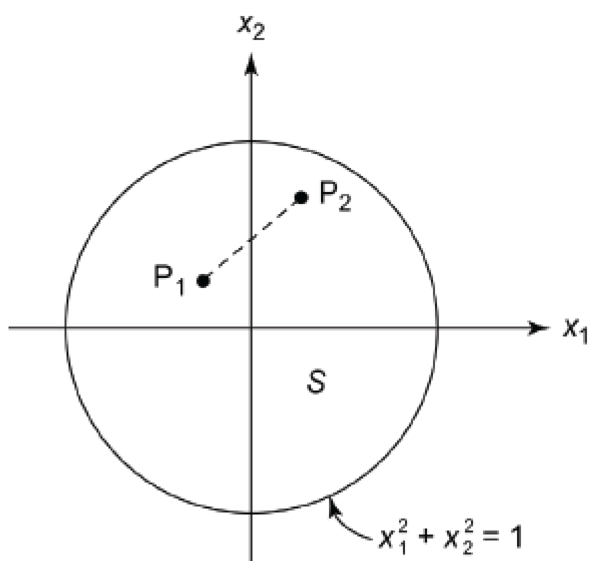
Convexity



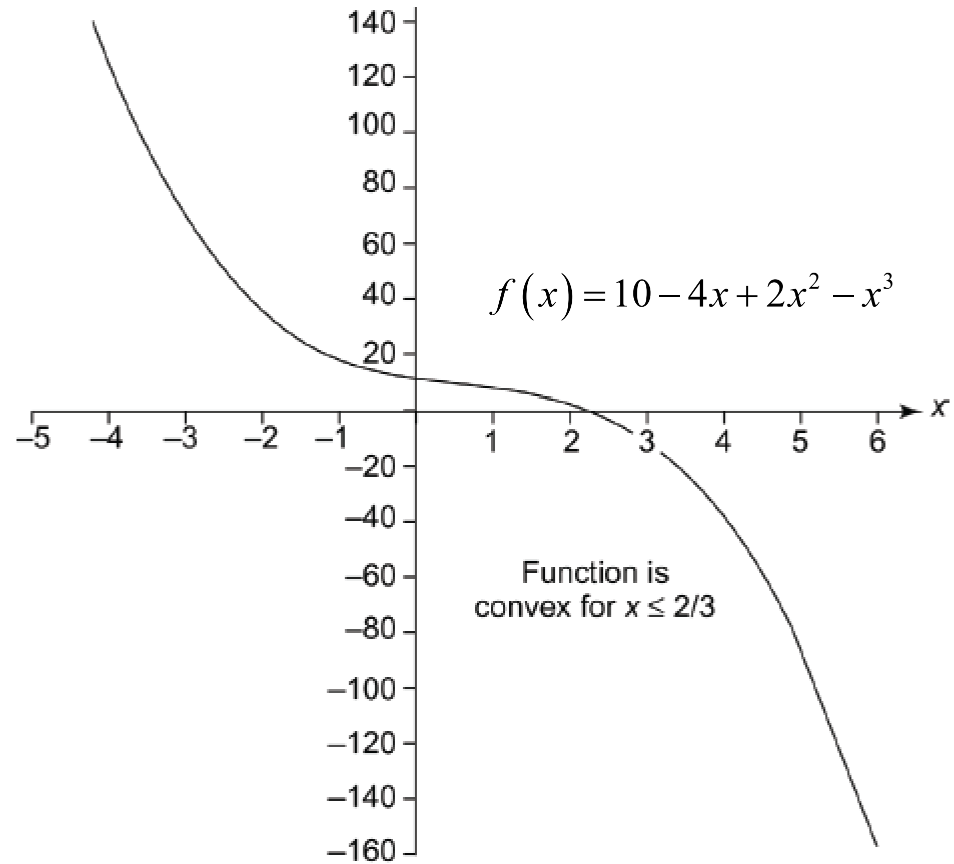
$f(x)$
$\alpha f(x_2) + (1-\alpha) f(x_1)$
$f(x_2)$
$f(x_1)$
$f(x)$
α
x_1
x
x_2
x
$x = \alpha x_2 + (1-\alpha)x_1$



P_1
P_2
P_1
P_2
P_1
P_2
P_1
P_2



x_2
P_2
P_1
x_1
S
$x_1^2 + x_2^2 = 1$



$f(x) = 10 - 4x + 2x^2 - x^3$
Function is convex for $x \leq 2/3$
140
120
100
80
60
40
20
–20
–40
–60
–80
–100
–120
–140
–160
–5
–4
–3
–2
–1
1
2
3
4
5
6
x

Convex Programming Problem

- Constraint set *S* is convex and cost function is also convex over *S*
  - Nonlinear equality constraint set $\rightarrow$ always nonconvex
  - Linear equality/inequality constraint set $\rightarrow$ always convex
- KKT necessary conditions are also sufficient
  - (theorem 4.9)

$$\left\langle S = \left\{ \boldsymbol{x} \middle| h_j(\boldsymbol{x}) = 0;\ j = 1, \ldots, p;\quad g_i(\boldsymbol{x}) \le 0; i = 1, \ldots, m \right\} \text{is a convex set} \right\rangle$$

$$\rightarrow (\times) \leftarrow (\mathrm{o}) \left\langle \text{function } g_i \text{ are convex and } h_j \text{ are linear} \right\rangle$$

  - (theorem 4.10) Any local minimum is also a global minimum
    - Proof ?
    - Convexity check failure $\rightarrow$(x) no global minimum point

Check for Convexity (1)

[1] $f(\boldsymbol{x}) = x_1^{\ 2} + x_2^{\ 2} - 1$

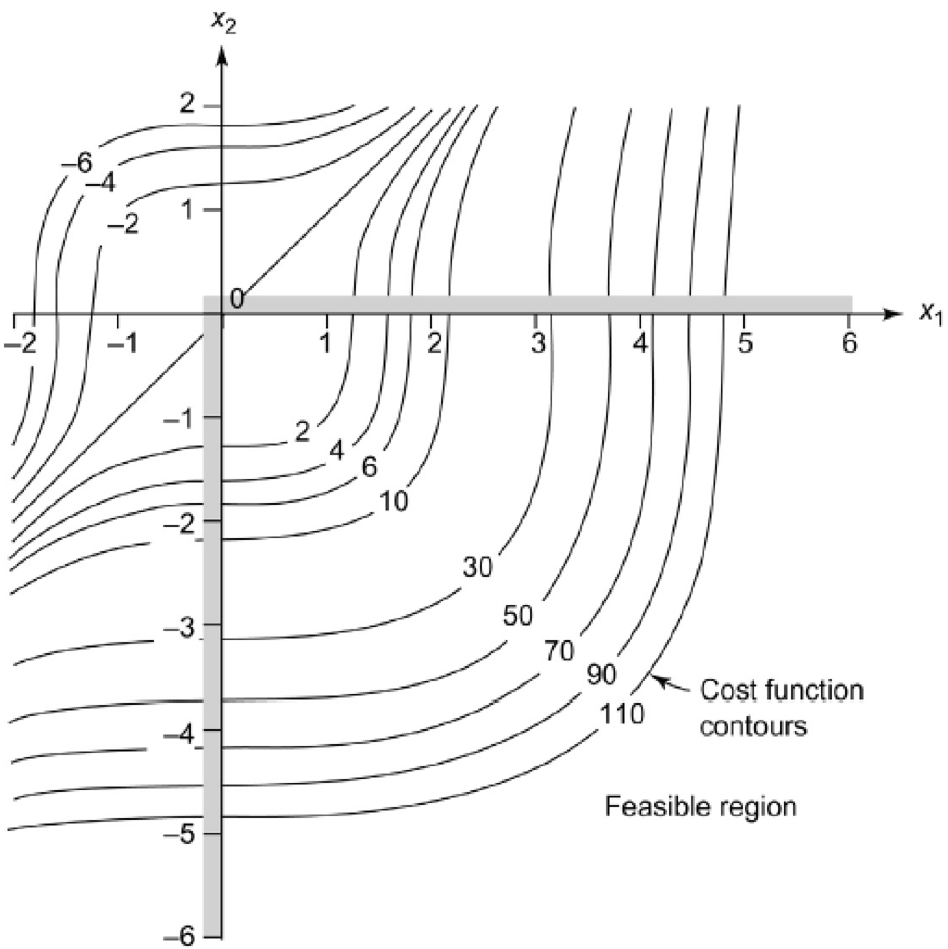
[2] $f(x) = 10 - 4x + 2x^2 - x^3$

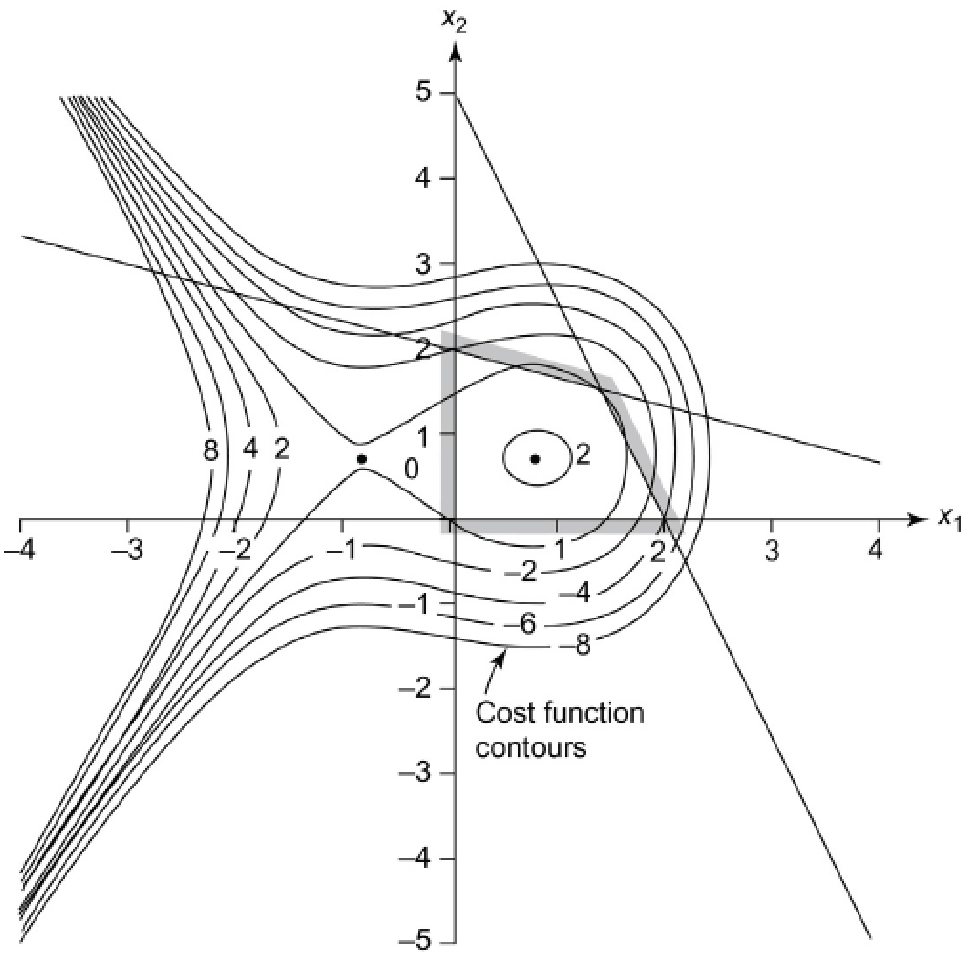
[3] $\begin{cases} \min f(\boldsymbol{x}) = x_1^{\ 3} - x_2^{\ 3} \\ \text{s.t.} \quad x_1 \geq 0, x_2 \leq 0 \end{cases}$

[4] $\begin{cases} \min f(\boldsymbol{x}) = 2x_1 + 3x_2 - x_1^{\ 3} - 2x_2^{\ 2} \\ \text{s.t.} \quad x_1 + 3x_2 \leq 6 \\ \qquad 5x_1 + 2x_2 \leq 10 \\ \qquad x_1, x_2 \geq 0 \end{cases}$

[5] $\begin{cases} \min f(\boldsymbol{x}) = 9x_1^{\ 2} - 18x_1x_2 + 13x_2^{\ 2} - 4 \\ \text{s.t.} \quad x_1^{\ 2} + x_2^{\ 2} + 2x_1 \geq 16 \end{cases}$

Check for Convexity (2)





Transformation of a constraint

- Form of function: convex $\leftrightarrow$ nonconvex
- Convexity of the feasible region: no change

$$g_1 = \frac{a}{x_1 x_2} - b \le 0 \qquad g_2 = a - b x_1 x_2 \le 0$$

$$\nabla^2 g_1 = \frac{2a}{x_1^2 x_2^2}\begin{bmatrix} x_2/x_1 & 0.5 \\ 0.5 & x_1/x_2 \end{bmatrix} \qquad \nabla^2 g_2 = \begin{bmatrix} 0 & -b \\ -b & 0 \end{bmatrix}$$

$$(positive\ \ definite) \qquad (indefinite)$$

- Sufficient Conditions for Convex Programming Problems
  - If *f* (**x**) is a convex cost function defined on a convex feasible set, then the first-order KKT conditions are necessary as well as sufficient for a global minimum

Convex Programming Problem: Summary

Problem must be written in standard form: Minimize $f(\mathbf{x})$ subject to $h_i(\mathbf{x}) = 0$, $g_j(\mathbf{x}) \le 0$

1. Convex set	The geometrical condition, that a line joining two points in the set is to be in the set, is an "*if-and-only-if*" condition for the convexity of the set.
2. Convexity of feasible set S	All of the constraint functions should be convex. *This condition is sufficient but not necessary* that is, functions failing the convexity check may still define convex sets. Nonlinear equality constraints always give nonconvex sets. Linear equalities or inequalities always give convex sets.
3. Convex functions	A function is convex *if and only if* its Hessian is at least *positive semidefinite* everywhere. A function is *strictly convex* if its Hessian is *positive definite* everywhere. However, the *converse is not true*: A strictly convex function may not have a positive definite Hessian everywhere. Thus this condition is *sufficient* but not necessary.
4. Form of constraint function	Changing the form of a constraint function can result in failure of the convexity check for the new constraint or vice versa.
5. Convex programming problem	$f(\mathbf{x})$ is convex over the convex feasible set S. KKT first-order conditions are necessary as well as sufficient for global minimums. Any local minimum point is also a global minimum point.
6. Nonconvex programming problem	If a problem fails a convexity check, it does not imply that there is no global minimum for the problem. It could have only one local minimum in the feasible set S, which would then be a global minimum as well.

Second-order conditions (1)

- Convex problems
  - First-order K-T conditions are necessary as well as sufficient for a global minimum

$$\begin{aligned}&\textit{Minimize}\;\; f(x_1,x_2)=(x_1-1.5)^2+(x_2-1.5)^2\\&\textit{subject to}\;\; g(\mathbf{x})=x_1+x_2-2\le 0\end{aligned}$$

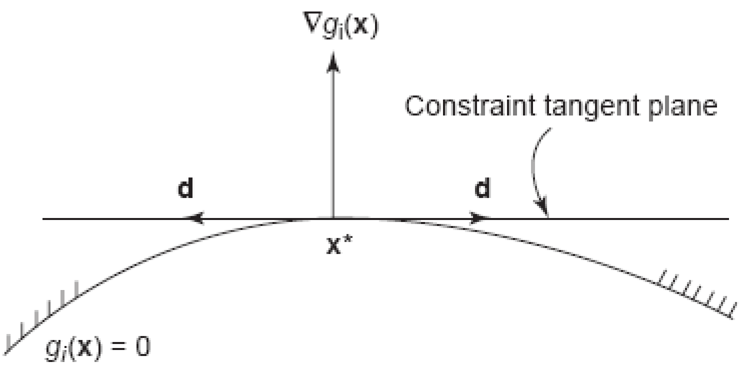
- General problems
  - Let **x*** satisfy the first-order KKT necessary conditions
  - Consider active constraints @**x*** to determine feasible changes ***d***

$$\nabla h_i^T \boldsymbol{d}=0 \quad \text{and} \quad \nabla g_i^T \boldsymbol{d}=0$$

  - If the number of active inequality constraints is equal to the number of independent design variables and all other K-T conditions are satisfied, then the candidate point is a local minimum (***d*** =0)

Second-order conditions (2)

- Necessary condition



for nonzero feasible directions ($\boldsymbol{d} \neq 0$) satisfying

$$\nabla h_i^T \boldsymbol{d} = 0; \quad i = 1, \ldots, p$$

$$\nabla g_i^T \boldsymbol{d} = 0; \quad \text{for all active constraints}$$

$$Q = \boldsymbol{d}^T \nabla^2 \boldsymbol{L}(\boldsymbol{x}^*)\boldsymbol{d} \geq 0 \text{ if } \boldsymbol{x}^* \text{ is a local minimum point}$$

- Sufficient condition

for nonzero feasible directions ($\mathbf{d} \neq 0$) satisfying

$$\nabla h_i^T \mathbf{d} = 0; \quad i = 1, \ldots, p$$

$$\nabla g_i^T \mathbf{d} = 0; \quad i = 1, \ldots, m \text{ for active inequalities with } u_i^* > 0$$

$$\nabla g_i^T \mathbf{d} \leq 0; \quad \text{for constraints with } u_i^* = 0$$

if $Q = \mathbf{d}^T \nabla^2 \mathbf{L}(\mathbf{x}^*)\mathbf{d} > 0$, then $\mathbf{x}^*$ is an isolated local minimum point

Check for Sufficient Conditions

[Example 4.30]

$$\begin{cases} \text{Minimize } f(\mathbf{x}) = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0 \\ \text{subject to } a \le x \le d \quad (0 < a < b < c < d \text{ and } f_0 \text{ are constants}) \end{cases}$$

$\rightarrow x = a$ (local minimum)

[Example 4.31]

$$\begin{cases} \text{Minimize } f(\mathbf{x}) = x_1^2 + x_2^2 - 3x_1x_2 \\ \text{subject to } g = x_1^2 + x_2^2 - 6 \le 0 \end{cases}$$

$\rightarrow (1)\ \mathbf{x}^* = (0,0), u^* = 0 \ \ (2)\ \mathbf{x}^* = (\sqrt{3}, \sqrt{3}), u^* = 0.5 \ \ (3)\ \mathbf{x}^* = (-\sqrt{3}, -\sqrt{3}), u^* = 0.5$

[Example 4.32]

$$\begin{cases} \text{Minimize } f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2 \\ \text{subject to } g_1 = -2x_1 - x_2 + 4 \le 0 \\ \qquad\qquad\quad g_2 = -x_1 - 2x_2 + 4 \le 0 \end{cases}$$

$\rightarrow \mathbf{x}^* = (4/3, 4/3), \mathbf{u}^* = (2/9, 2/9)$

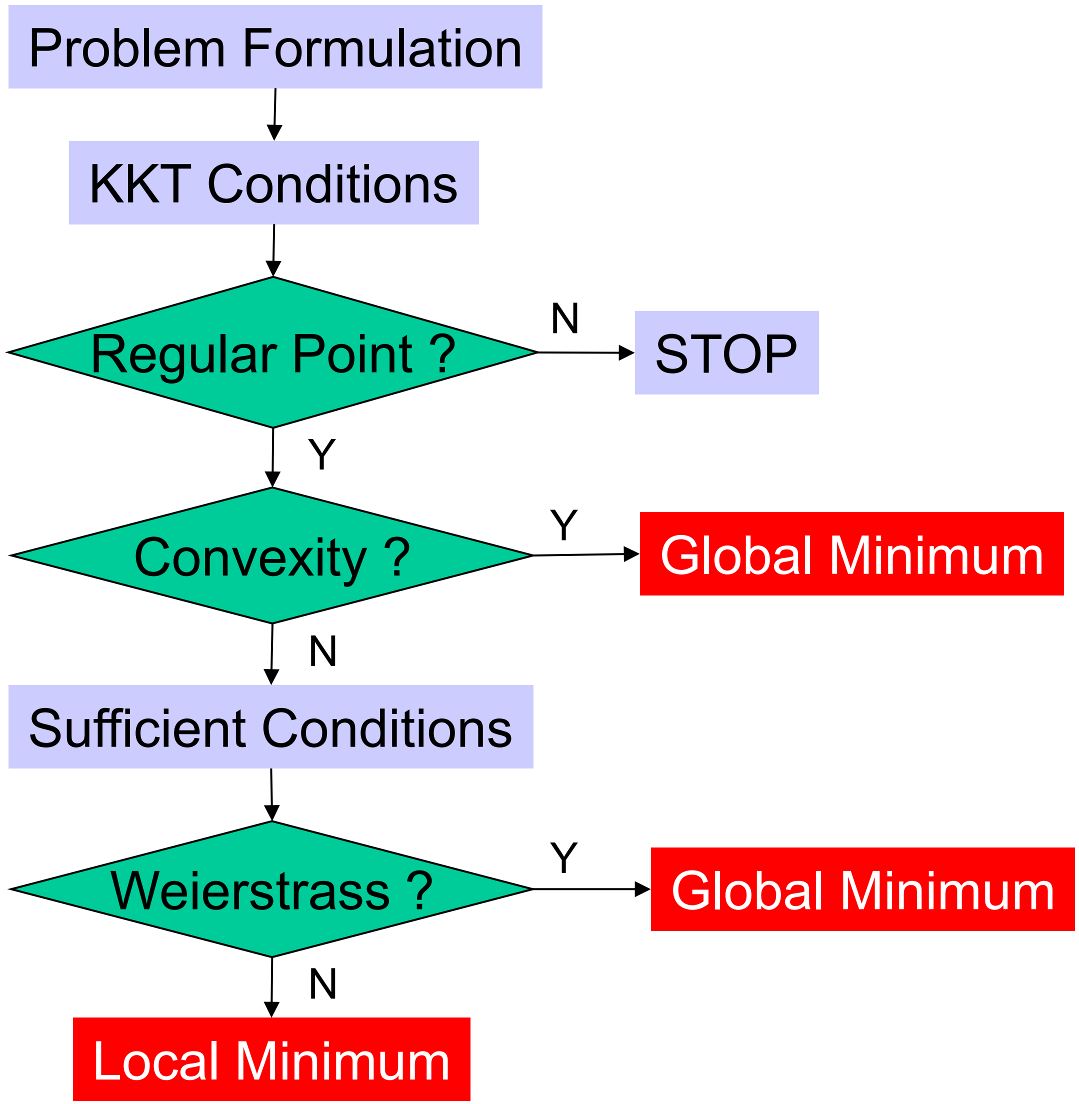
Summary: Optimality Conditions

	Unconstrained	Constrained
Necessary	$\nabla \boldsymbol{f} = 0$	$\nabla \boldsymbol{L} = 0$
Sufficient	$\nabla^2 \boldsymbol{f}$: positive definite	(1) convex problem: global (2) $\nabla^2 \mathbf{L}$: positive definite $\rightarrow$ strong (3) $\left.\begin{array}{l}\nabla \mathbf{h}^T \mathbf{d} = 0 \\ \nabla \mathbf{g}^T \mathbf{d} = 0 \text{(active)}\end{array}\right\} \rightarrow \mathbf{d}^T \nabla^2 \mathbf{L} \mathbf{d} > 0$ $\rightarrow$ isolated local

Procedures (1)

- Problem formulation: DVs, objective, constraints
- Convexity check: global optimum ?
- K-T conditions: solutions
- Sufficiency check
- Sensitivity analysis: changes in the constraint limits

Procedures (2)



Design of a Wall Bracket

$$\begin{cases} \underset{A_1, A_2}{\text{Minimize}} \quad f(A_1, A_2) = l_1 A_1 + l_2 A_2 \\ \text{subject to} \\ \left.\begin{array}{l} \sigma_i \le \sigma_a \\ \dfrac{F_i}{A_i} \le \sigma_a \end{array}\right\} \rightarrow \begin{cases} g_1 = \dfrac{2.0E+06}{A_1} - 16000 \le 0 \rightarrow A_1 \ge 125 \\ g_2 = \dfrac{1.6E+06}{A_2} - 16000 \le 0 \rightarrow A_2 \ge 100 \end{cases} \\ g_3 = -A_1 \le 0 \\ g_4 = -A_2 \le 0 \end{cases}$$

[Convexity]

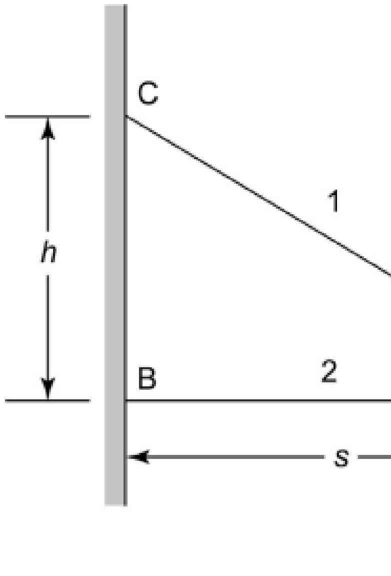
[KKT Necessary Conditions]

$$\begin{cases} L = \sum_{i=1}^{2} A_i l_i + \sum_{i=1}^{2} u_i \left(\dfrac{F_i}{A_i} - \sigma_a + s_i^2 \right) + u_3 \left(-A_1 + s_3^2 \right) + u_4 \left(-A_2 + s_4^2 \right) \\ u_3 = u_4 = 0, \quad \dfrac{\partial L}{\partial A_i} = l_i - u_i \left(\dfrac{F_i}{A_i^2} \right) = 0 \end{cases}$$

$u_1 = 0.391, \ u_2 = 0.25$

$f^* = 10250 cm^3$

[Sensitivity Analysis] $\sigma_a = 16000 \rightarrow 16500 \ N/cm^2$?

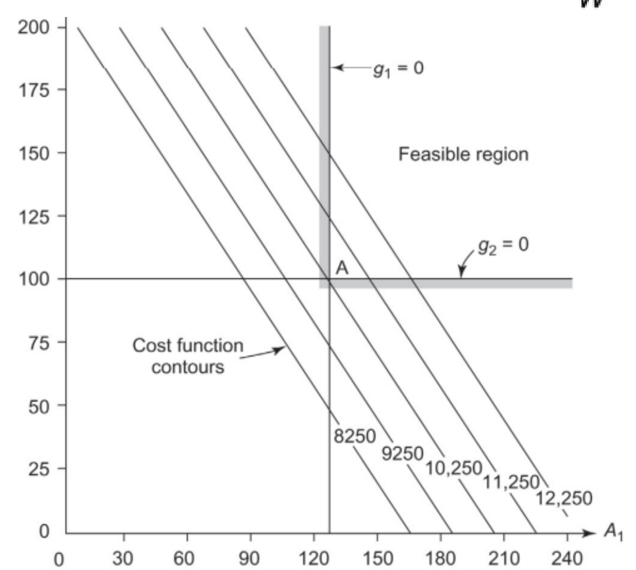


$h = 30$ cm

$s = 40$ cm

$W = 1.2$ MN

$\sigma_a = 16$ kN/cm^2



Design of a Rectangular Beam (1)

$$\underset{b,d}{\text{Minimize}}\ f = bd$$

subject to

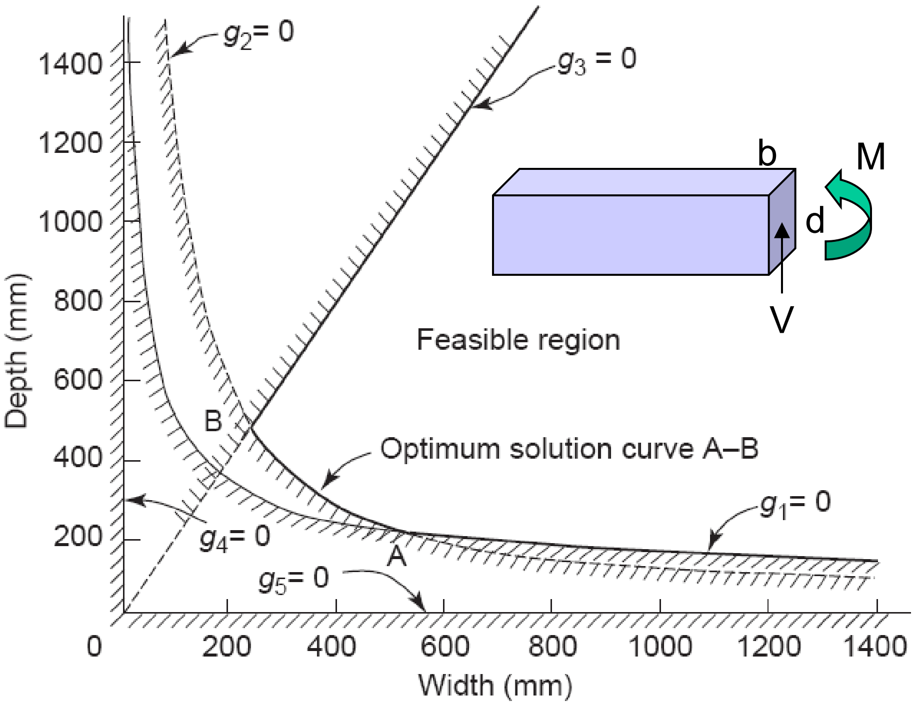
$$\left.\begin{aligned} &\sigma = \frac{6M}{bd^2} \le (\sigma_a)_{bending} \\ &\tau = \frac{3V}{2bd} \le (\tau_a)_{shear} \\ &d \le 2b \\ &b, d \ge 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &g_1 = \frac{2.40E+08}{bd^2} - 10 \le 0 \\ &g_2 = \frac{2.25E+05}{bd} - 2 \le 0 \\ &g_3 = d - 2b \le 0 \\ &g_4 = -b \le 0 \\ &g_5 = -d \le 0 \end{aligned}\right.$$

$$M = 40kN \cdot m$$

$$V = 150kN$$

$$(\sigma_a)_{bending} = 10MPa$$

$$(\tau_a)_{shear} = 2MPa$$



$b = 237mm,\ d = 474mm$ @ point B

$b = 527.3mm,\ d = 213.3mm$ @ point A

Design of a Rectangular Beam (2)

[convexity] no!

$$f = bd \rightarrow \nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{(indefinite)}$$

$$\left.\begin{aligned} g_1 &= \frac{2.40E+08}{bd^2} - 10 \le 0 \rightarrow \nabla^2 g_1 = \frac{2(2.40E+08)}{b^3 d^4}\begin{bmatrix} d^2 & bd \\ bd & 3b^2 \end{bmatrix} \text{(positive definite)} \\ g_2 &= \frac{2.25E+05}{bd} - 2 \le 0 \rightarrow \nabla^2 g_2 = \frac{2(2.40E+08)}{b^3 d^3}\begin{bmatrix} 2d^2 & bd \\ bd & 2b^2 \end{bmatrix} \text{(positive definite)} \end{aligned}\right\} \rightarrow \text{transform?}$$

[KKT necessary conditions]

$$L = bd + u_1\left(\frac{2.40E+08}{bd^2} - 10 + s_1^{\,2}\right) + u_2\left(\frac{2.25E+05}{bd} - 2 + s_2^{\,2}\right) + u_3\left(d - 2b + s_3^{\,2}\right) + u_4\left(-b + s_4^{\,2}\right) + u_5\left(-b + s_5^{\,2}\right)$$

$$\frac{\partial L}{\partial b} = d + u_1\frac{-2.40E+08}{b^2 d^2} + u_2\frac{-2.25E+05}{b^2 d} + u_3(-2) + u_4(-1) = 0$$

$$\frac{\partial L}{\partial d} = b + u_1\frac{-2(2.40E+08)}{bd^3} + u_2\frac{-2.25E+05}{bd^2} + u_3 + u_5(-1) = 0$$

$$\left.\begin{aligned} g_i + s_i^{\,2} &= 0 \\ u_i s_i &= 0 \\ u_i &\ge 0 \end{aligned}\right\} i = 1,\ldots,5$$

$$u_4 = u_5 = 0 \rightarrow 2^3 \text{ cases}$$

Design of a Rectangular Beam (3)

case	u1	u2	u3	b	d	note
1	0	0	0	0	0	
2	0	0	0	0	0	
3	0	5.625×10^4	0			$bd = 1.125 \times 10^5$, $g1 \le 0$, $g3 \le 0$
4		0	0			No solution (inconsistent eqn)
5	0	5.625×10^4	0	237.17	474.34	
6	4402.35	0	-60.57	~~181.71~~	~~363.42~~	
7	0	5.625×10^4	0	527.34	213.33	
8						No solution (3 eqn, 2 unknown)

- Sufficiency conditions
  - not isolated local minimum
- Sensitivity Analysis

Design of a Rectangular Beam (4)

– Global minimum points can be obtained for problems that cannot be classified as convex programming problems. We cannot show global optimality of a point unless we find all of the local minimum points in the closed and bounded feasible set (the Weierstrass Theorem 4.1).
– If second-order sufficiency condition is not satisfied, the only conclusion we can draw is that the candidate point is not an isolated local minimum. It may have many local optima in the neighborhood, and they may all be actually global minimum points.